Uncertain Model Set Calculation From Frequency Domain Data *

Gary J. Balas, Andrew K. Packard, and Peter J. Seiler

Aerospace Engineering and Mechanics
University of Minnesota
balas@umn.edu

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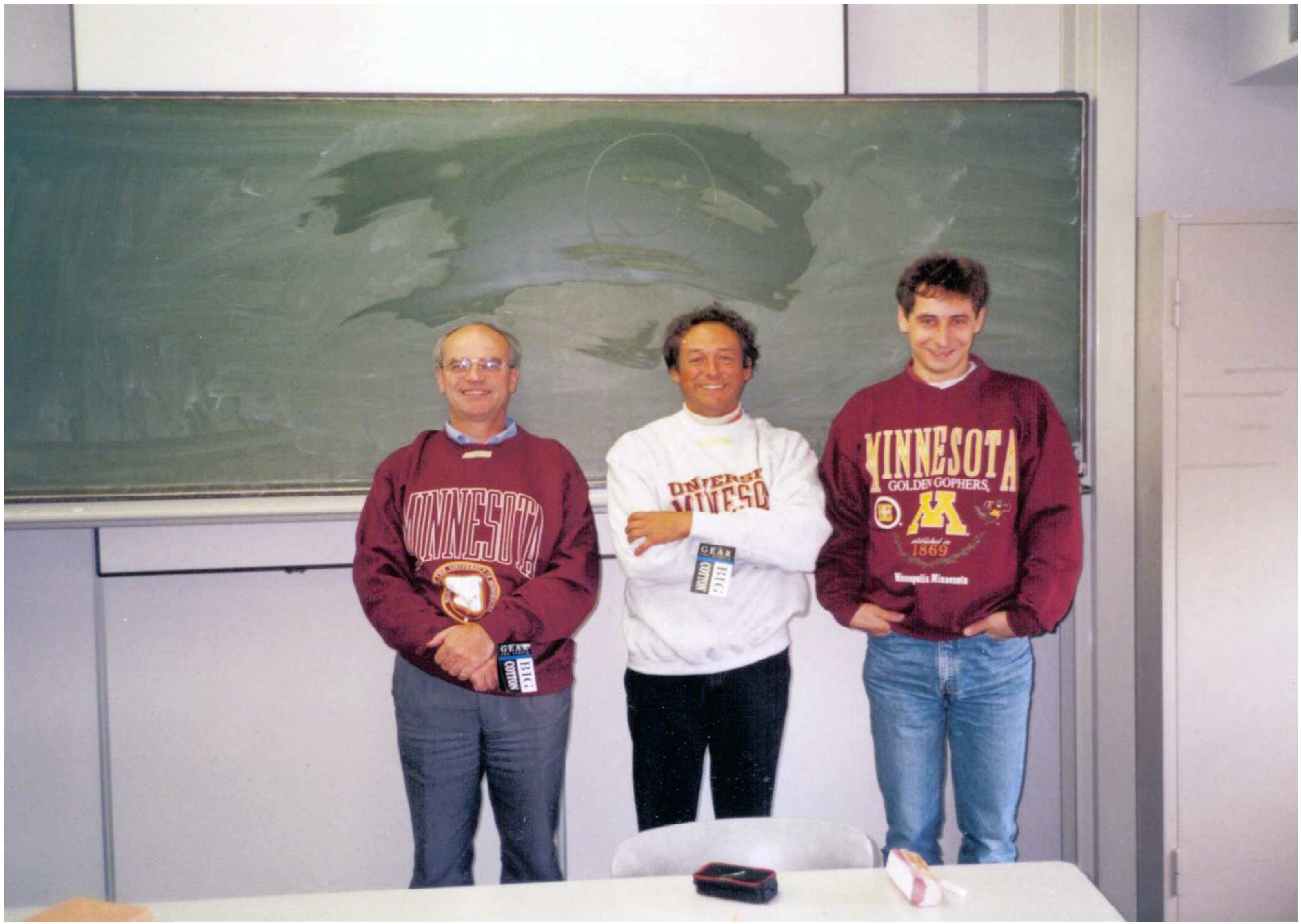


Introduction

Algorithms to optimally cover finite sets of matrices with diagonally-weight unit balls (with Euclidean operator norm) of matrices are considered. The motivation of this work is multi-model robust control design or covering of a finite set of plants with an additive or multiplicative uncertainty model.

- Reformulated as a semi-definite program.
- ‘Containment’ metric.
- Uncertainty model derived from frequency response data via frequency-by-frequency analysis.

Hindi, H., Seong, C-Y, and Boyd, S., “Computing optimal uncertainty models from frequency domain data,” Proceeding 41^{st} IEEE Conf. Decision and Control, vol. 3, 2898–2905 (2002).

Uncertainty Models

$\mathcal{R}$ denotes proper, rational functions, and $\mathcal{R}^{n\times m}$ is the set of $n \times m$ matrices with elements in $\mathcal{R}$. Similarly $\mathcal{S}$ denotes stable (poles in open left-half plane), proper, rational functions. For any ring $\mathcal{K}$, $\mathcal{U}_{\mathcal{K}^{n\times n}}$ denotes the units in $\mathcal{K}^{n\times n}$. For $G \in \mathcal{R}^{n\times m}$, the notation $U_{\#}(G)$ is the number of unstable eigenvalues in a stabilizable and detectable realization of G.

Given $P \in \mathcal{R}^{n\times m}$, and $W_L \in \mathcal{U}_{\mathcal{S}^{n\times n}}, W_R \in \mathcal{U}_{\mathcal{S}^{m\times m}}$

$$\mathcal{A}\left(P, W_L, W_R\right) := \left\{\tilde{P} = P + W_L \Delta W_R : \Delta \in \mathcal{R}^{n\times m}, \|\Delta\|_{\mathcal{L}_\infty} \leq 1, \; U_{\#}(P) = U_{\#}(\tilde{P})\right\}$$

$$\mathcal{M}_{\mathcal{I}}\left(P, W_L, W_R\right) := \left\{\tilde{P} = P(I_m + W_L \Delta W_R) : \Delta \in \mathcal{R}^{n\times m}, \|\Delta\|_{\mathcal{L}_\infty} \leq 1, \; U_{\#}(P) = U_{\#}(\tilde{P})\right\}$$

$$\mathcal{M}_{\mathcal{O}}\left(P, W_L, W_R\right) := \left\{\tilde{P} = (I_n + W_L \Delta W_R)P : \Delta \in \mathcal{R}^{n\times m}, \|\Delta\|_{\mathcal{L}_\infty} \leq 1, \; U_{\#}(P) = U_{\#}(\tilde{P})\right\}$$

Uncertainty Models

Theorem 0.1 *(Additive) Given $P \in \mathcal{R}^{n\times m}$, and $W_L \in \mathcal{U}_{\mathcal{S}^{n\times n}}, W_R \in \mathcal{U}_{\mathcal{S}^{m\times m}}$. Assume P has no poles on the imaginary axis. Then $\tilde{P} \in \mathcal{R}^{n\times m}$ satisfies $\tilde{P} \in \mathcal{A}\left(P, W_L, W_R\right)$ if and only if $U_{\#}(P) = U_{\#}(\tilde{P})$ and*

$$\begin{bmatrix} W_L W_L^* & \tilde{P} - P \\ \left(\tilde{P} - P\right)^* & W_R^* W_R \end{bmatrix} \succeq 0 \qquad (1)$$

for all $\omega \in \mathbf{R}$.

Theorem 0.2 *(Input-multiplicative) Given $P \in \mathcal{R}^{n\times m}$, and $W_L \in \mathcal{U}_{\mathcal{S}^{n\times n}}, W_R \in \mathcal{U}_{\mathcal{S}^{m\times m}}$. Assume P has no poles on the imaginary axis. Then $\tilde{P} \in \mathcal{R}^{n\times m}$ satisfies $\tilde{P} \in \mathcal{M}_{\mathcal{I}}\left(P, W_L, W_R\right)$ if and only if $U_{\#}(P) = U_{\#}(\tilde{P})$ and*

$$\begin{bmatrix} P W_L W_L^* P^* & \tilde{P} - P \\ \left(\tilde{P} - P\right)^* & W_R^* W_R \end{bmatrix} \succeq 0 \qquad (2)$$

for all $\omega \in \mathbf{R}$.

Other over-bound uncertain models can also be represented via LMIs.

Covering a Family of Models

Given a nominal model $P \in \mathcal{R}^{n \times m}$, and a finite collection of models $\{P_i\}_{i=1}^{N}$ (assuming same # of RHP poles), find weights W_L and W_R be chosen so that for all i

$$P_i \in \mathcal{A}\,(P, W_L, W_R) \quad \text{or} \quad P_i \in \mathcal{M}_I\,(P, W_L, W_R) \quad \text{or} \quad P_i \in \mathcal{M}_O\,(P, W_L, W_R)$$

Select W_L and W_R as “small” as possible, (approximately “radius” of the uncertain set).

Define new optimization variables L and R for $W_L W_L^*$ and $W_R^* W_R$ respectively.

Covering a Family of Models

Optimization to determine small, feasible weights is:

1. Chose positive definite matrices Γ_1 and Γ_2
2. Pick a nominal plant model, P
3. Grid the frequency axis using $\{\omega_k\}_{k=1}^{M}$.
4. At each frequency, solve the SDP

$$\min_{L \succ 0, R \succ 0} \mathrm{Tr}\Gamma_1 L + \mathrm{Tr}\Gamma_2 R$$

subject to

$$\begin{bmatrix} L & P_i - P \\ (P_i - P)^* & R \end{bmatrix} \succeq 0 \quad \text{or} \quad \begin{bmatrix} PLP^* & P_i - P \\ (P_i - P)^* & R \end{bmatrix} \succeq 0$$

for all $\{i\}_{i=1}^{N}$, and at all frequencies $\{\omega_k\}_{k=1}^{M}$.

Cover Measures

Given $C \in \mathbf{C}^{n\times m}, D \in \mathbf{C}^{n\times n}, E \in \mathbf{C}^{n\times m}$, define

$$\mathcal{B}_\alpha(C,D,E) := \left\{C + D\Delta E : \Delta \in \mathbf{C}^{n\times n}, \bar{\sigma}(\Delta) \leq \alpha\right\}$$

When $\alpha = 1$, $\mathcal{B}(C,D,E)$.

Given (C_1, D_1, E_1) and (C_2, D_2, E_2), what is the smallest value of α such $\mathcal{B}_\alpha(C_1, D_1, E_1)$ contains $\mathcal{B}(C_2, D_2, E_2)$? Suppose $\mathcal{B}(C_1, D_1, E_1)$ and $\mathcal{B}(C_2, D_2, E_2)$ are two different "covers" of a finite collection of matrices.

- How much does (C_1, D_1, E_1) have to be expanded to include all of $\mathcal{B}(C_2, D_2, E_2)$?
- Conversely, how much does (C_2, D_2, E_2) have to be expanded to include all of $\mathcal{B}(C_1, D_1, E_1)$?

Description which has to be expanded more is a less conservative cover. These are considered as *containment metrics*.

Cover Measures

Theorem 0.3 *Theorem: Suppose* $n \geq m$, $C_i \in \mathbf{C}^{n \times m}$, $D_i \in \mathbf{C}^{n \times n}$, $E_i \in \mathbf{C}^{n \times m}$, *with each* D_i *invertible, and each* E_i *full column rank. Then*

$$\min\{\alpha : \mathcal{B}(C_2, D_2, E_2) \subset \mathcal{B}_\alpha(C_1, D_1, E_1)\} = \max_{\Delta_2 \in \mathbf{C}^{n \times n}, \bar{\sigma}(\Delta_2) \leq 1} \bar{\sigma}\left[M(\Delta_2)\right]$$

where $M(\Delta) = D_1^{-1}\left(C_2 - C_1 + D_2 \Delta E_2\right)\left(E_1^* E_1\right)^{-\frac{1}{2}}$ *Moreover, the maximization can be computed reliably and accurately with structured singular value/SDP methods.*

Special cases:

- Input-multiplicative model must be increased by a factor $\bar{\sigma}\left[(PW_1)^{-1} W_3\right] \bar{\sigma}\left[W_4 W_2^{-1}\right]$ to contain the additive model
- Additive model must be increased by a factor $\bar{\sigma}\left[W_3^{-1}(PW_1)\right] \bar{\sigma}\left[W_2 W_4^{-1}\right]$ to contain the input-multiplicative model.

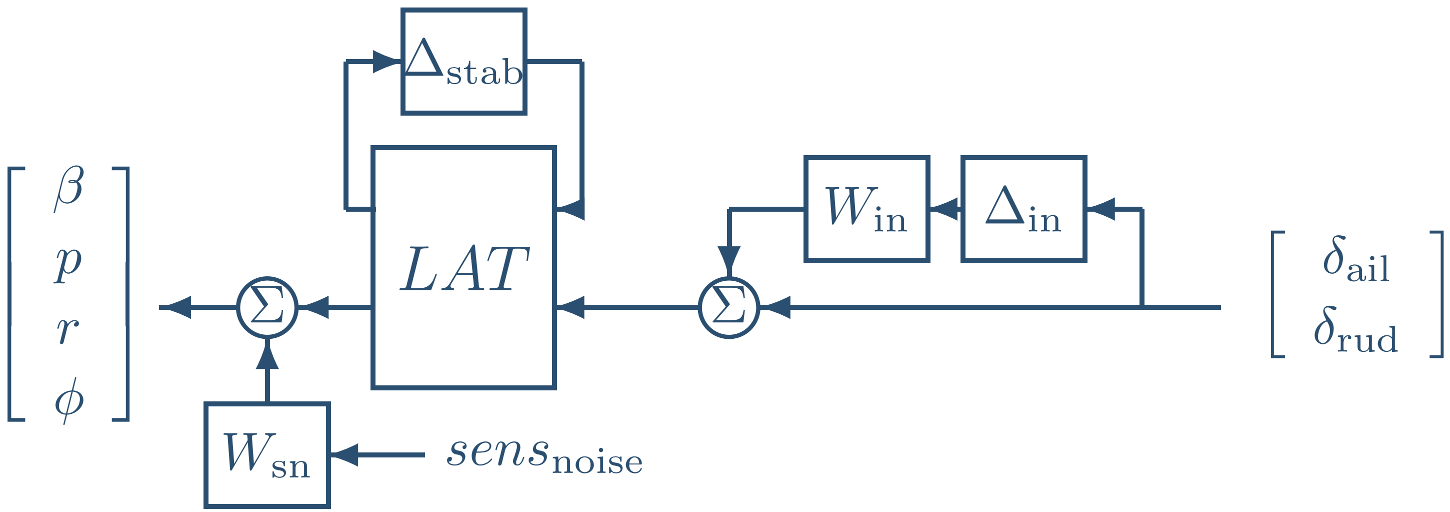
GTM Aircraft: Over-Bound Uncertainty Modeling

Application of over-bounding techniques to generate input multiplicative and additive uncertainty models for the NASA Generic Transport Model (GTM) aircraft.

- 5.5% dynamically-scaled, remotely piloted, twin-turbine swept wing aircraft, NASA Langley Research Center.

Uncertain, Lateral-Directional GTM Model

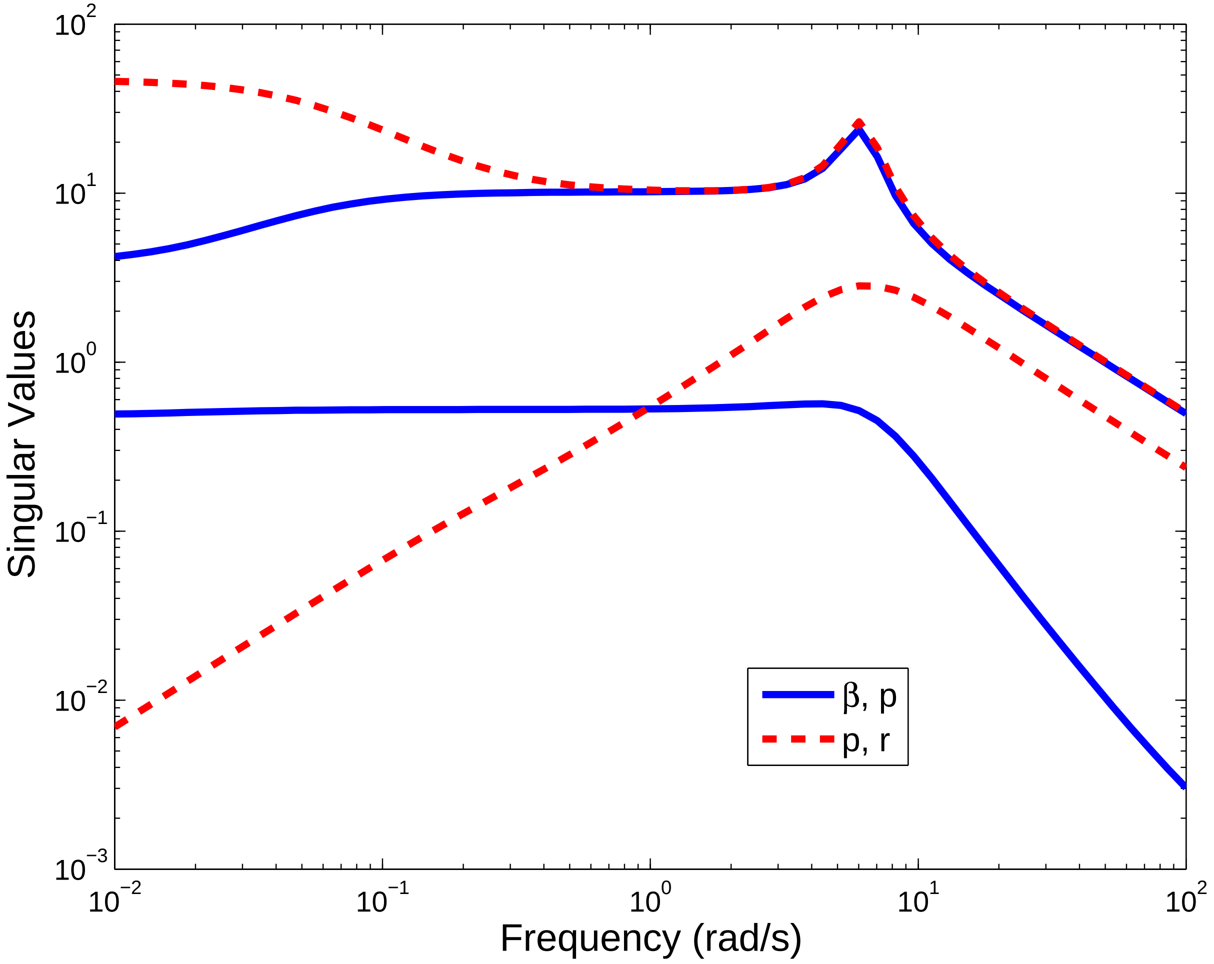
- $\Delta_{stab} \in C_{L_\beta}, C_{Y_\beta}, C_{L_p}, C_{L_r}, C_{N_p}$ (4%), C_{Y_β}, C_{N_r} (8%).
- $W_{\text{ail}} = \frac{4s+4.85}{s+97}$, $W_{\text{rud}} = \frac{4s+9.73}{s+97.3}$, $W_{\text{in}} = diag(W_{\text{ail}}, W_{\text{rud}})$
- $W_{\text{sn}} = 0.05 \cdot I_{3\times3}$

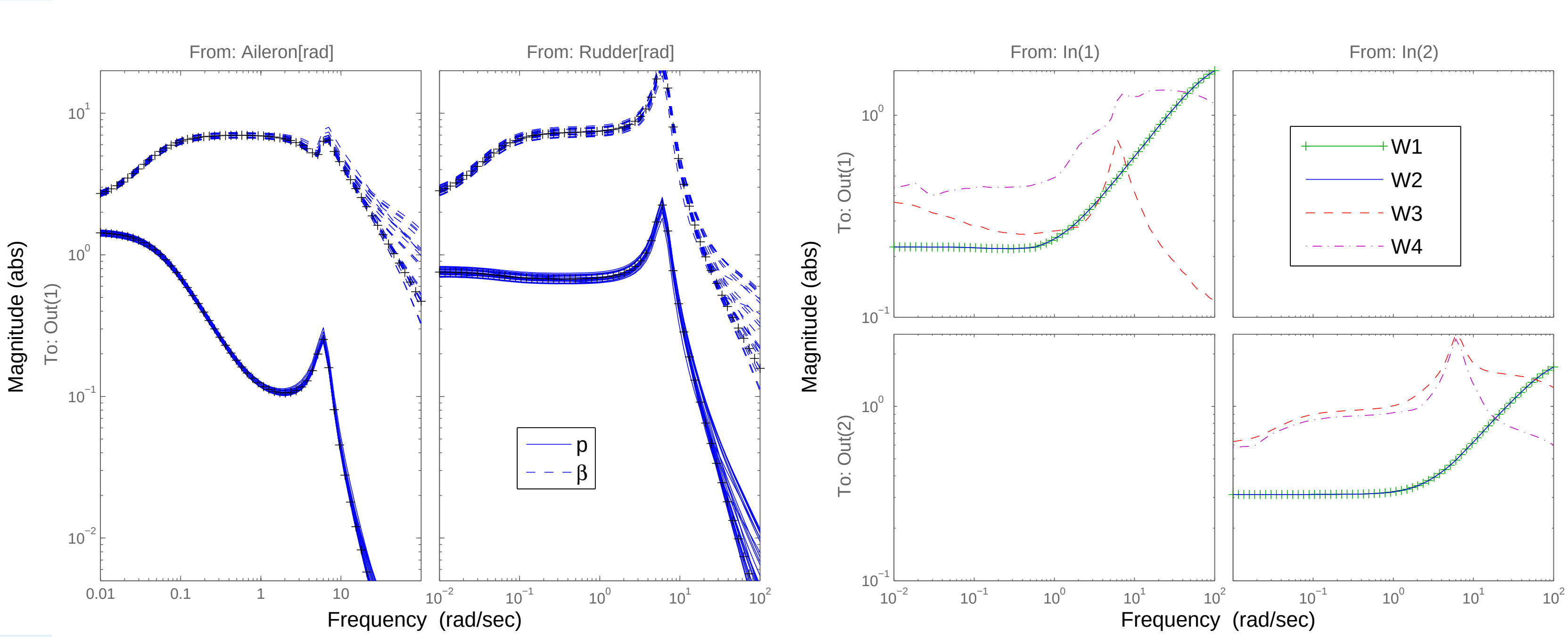


Three set of input/output frequency response data:

1. **I** input uncertainty
2. **IP** input and parametric uncertainty
3. **IPN** input, parametric uncertainty and sensor noise

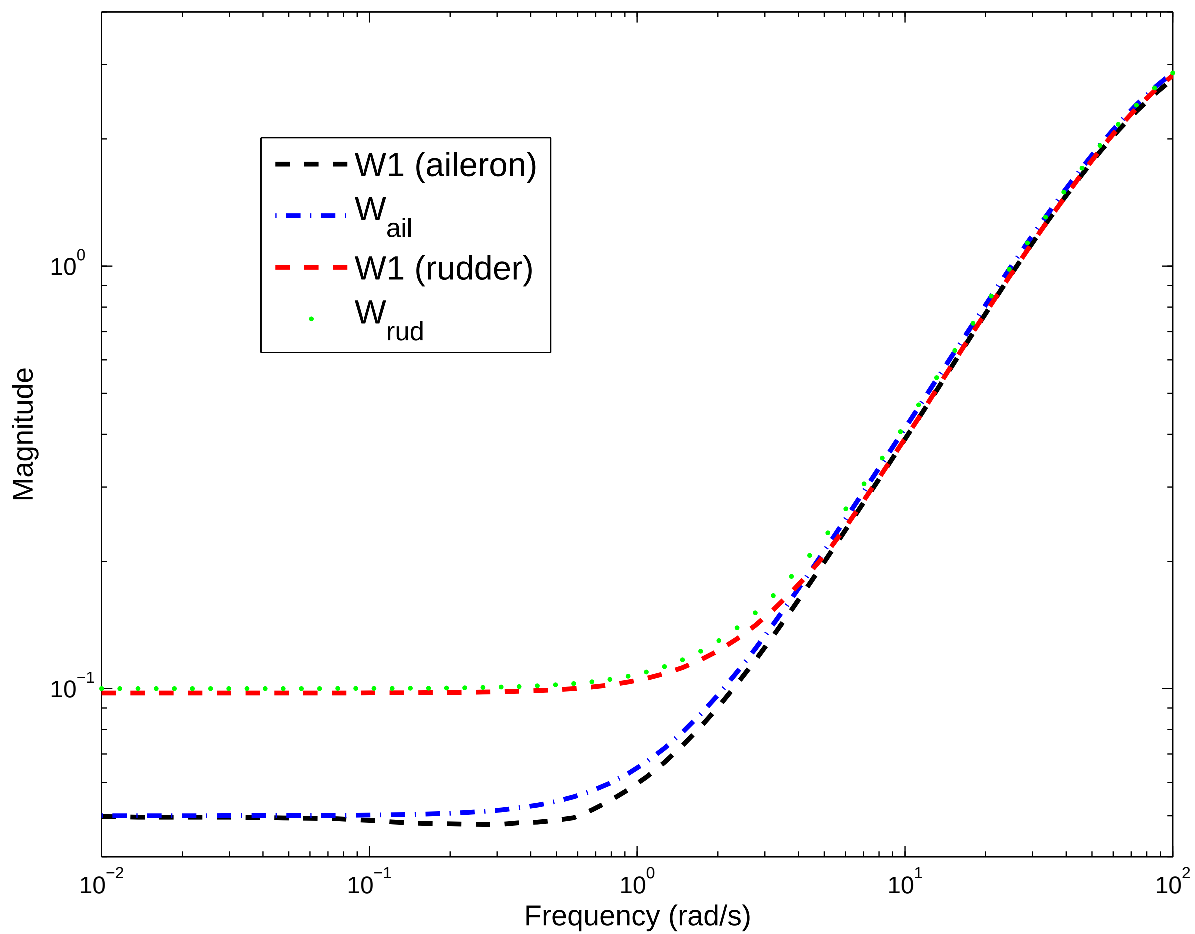
Over-Bounding as a LMI Reasibility Problem



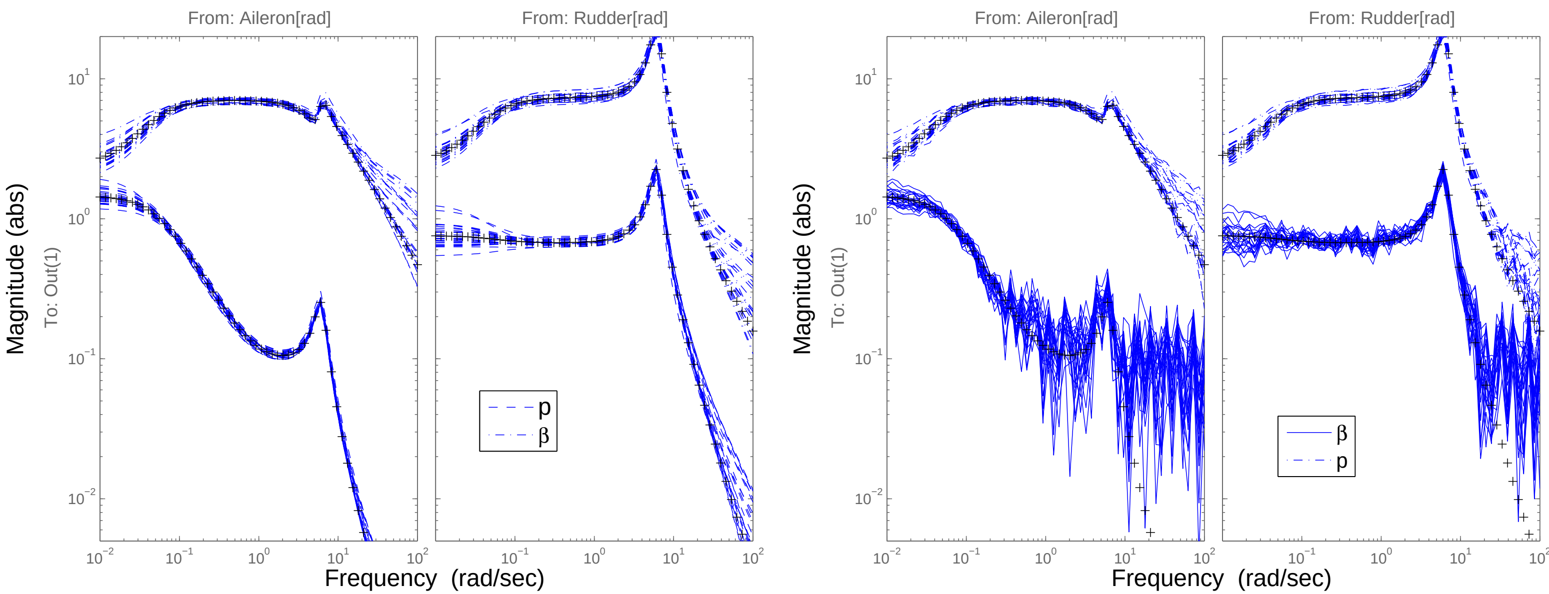


Input multiplicative, $P_{\text{in}} = P(I + W_1 \Delta_{\text{in}} W_2)$, and additive, $P_{\text{add}} = P + W_3 \Delta_{\text{add}} W_4$,

- W_1 and W_2 are identical, $W_1 W_2 \approx W_{\text{ail}}$
- Set $W_1 = I$, $W_2 \approx W_{\text{ail}}$
- W_3 and W_4 vary significantly across frequency

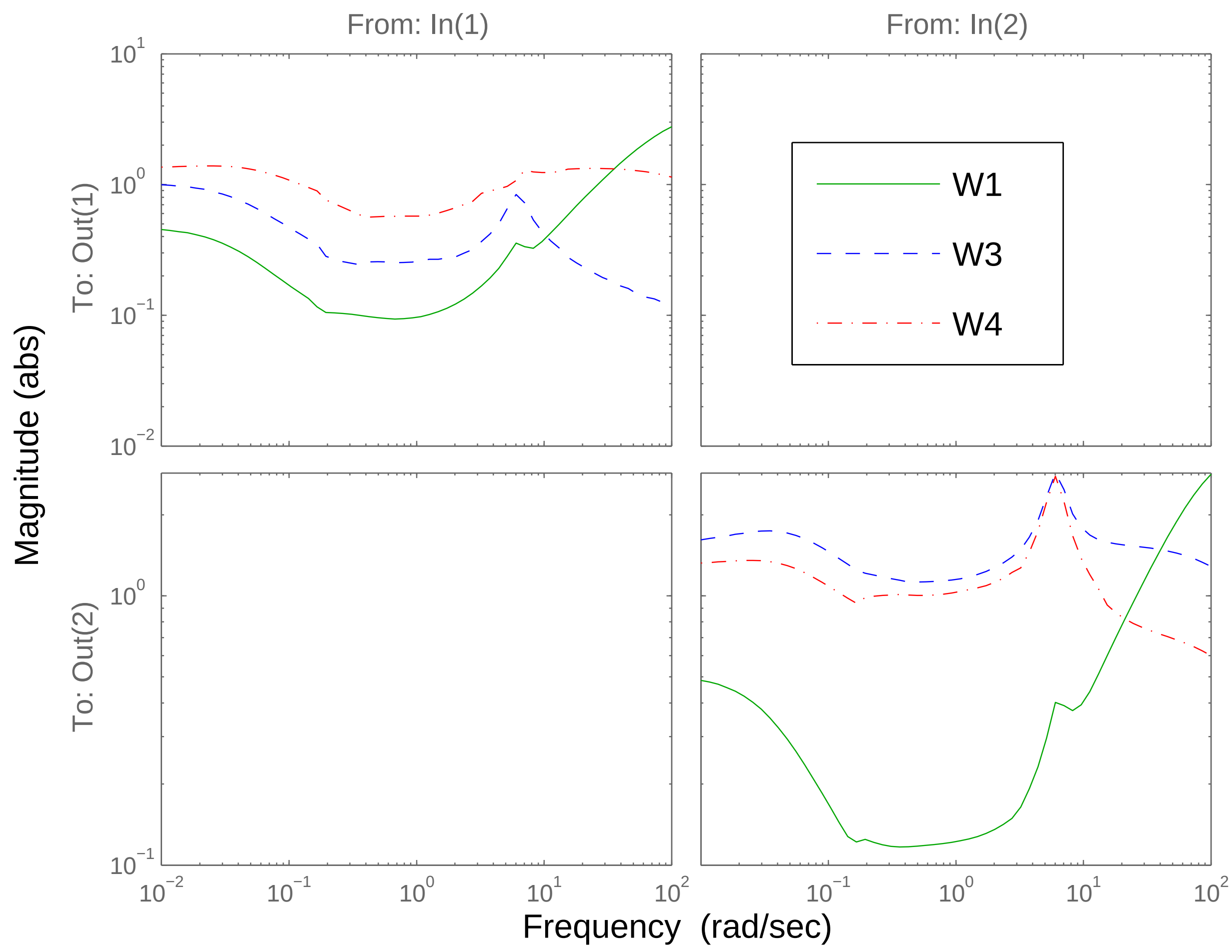


Comparison of Over-Bound and Original Weights

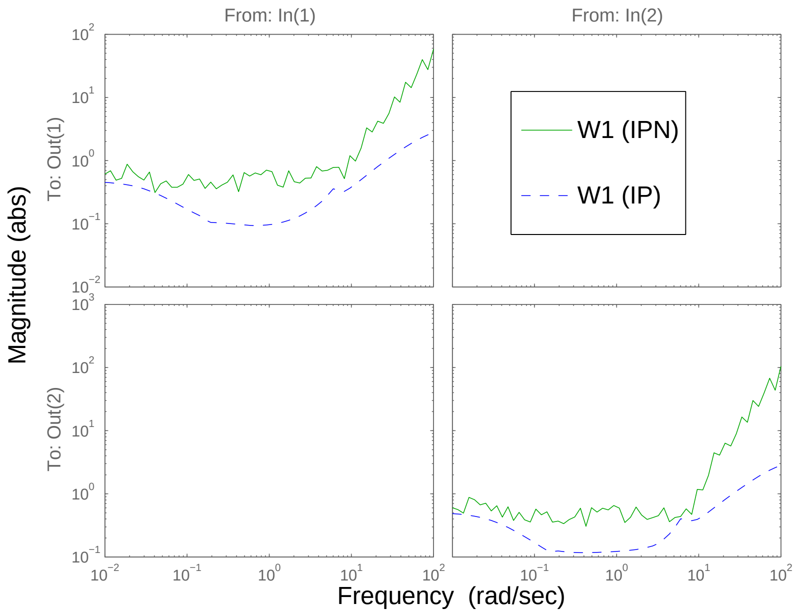


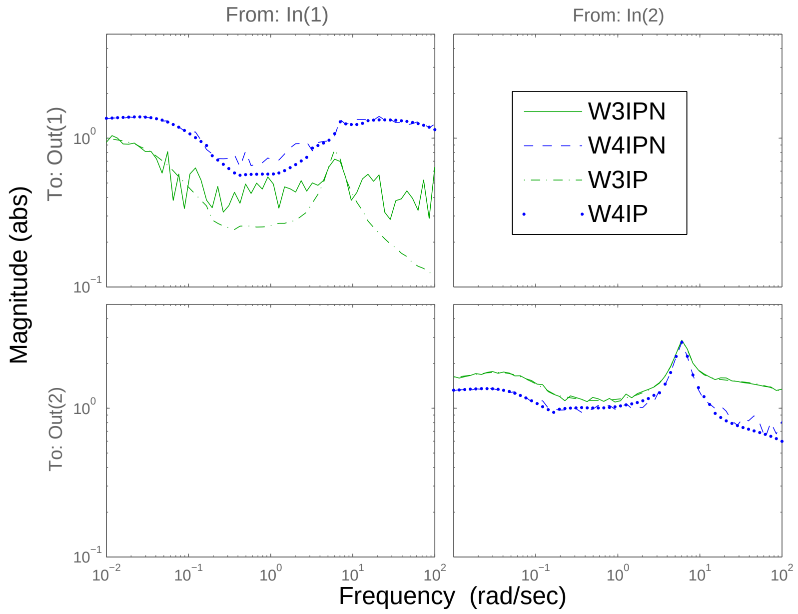
Data Set IP (left) and Data Set IPN (right)

Data Set IP



IP: Input Multiplicative and Additive Uncertainty Over-Bounds





Input Multiplicative (left) and Additive Uncertainty (right)
Over-Bound of Data Set IPN

Containment Metric

Compare conservatism of these relative model sets for IP:

- $P_{\text{in}} = P_{\text{nom}}(I + W_1 \Delta_{\text{in}} W_2)$
- $P_{\text{add}} = P_{\text{nom}} + W_3 \Delta_{\text{add}} W_4$

Containment metric for additive uncertainty over-bound weights relative to the input multiplicative over-bound model, $P_{\text{add}}(W_3, W_4) \in P_{\text{in}}(W_1, W_2)$ is at each frequency ω.

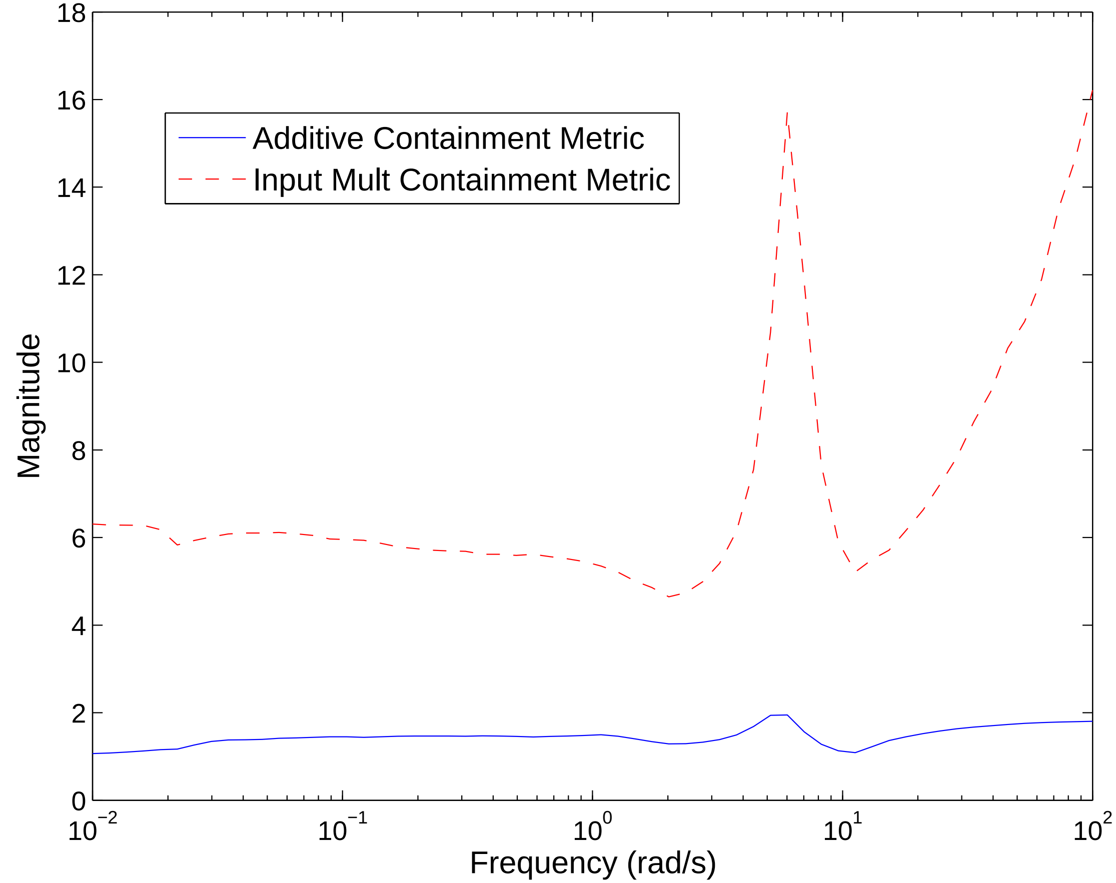
$$|\Delta_{\text{add}}| = \bar{\sigma}(W_3^{-1} P_{\text{nom}} W_1)\bar{\sigma}(W_2 W_4^{-1}), \qquad \text{at} \ \ \omega$$

Input multiplicative over-bound weights relative to additive uncertainty over-bound weights

$$|\Delta_{\text{in}}| = \bar{\sigma}((P_{\text{nom}} W_1)^{-1} W_3)\bar{\sigma}(W_4 W_2^{-1}), \qquad \text{at} \ \ \omega$$

Containment Metrics

- Containment metric is 1 if the model set, $P_{\mathrm{add}}(W_3, W_4)$ for example, contains all the models in $P_{\mathrm{in}}(W_1, W_2)$.
- Model description with the larger containment metric is the *least* conservative description.



Summary

- Algorithms were developed to optimally cover finite sets of matrices with diagonally-weighted unit balls (with Euclidean operator norm) of matrices.

- Metrics were derived to to assess the relative size of the each model set relative to a specific model structure.

- Application to lateral-directional axis of a radio-controlled aircraft.