Control Systems Design, SC4026

Lecture 4

- Controllability (a.k.a. Reachability) and Observability
- Algebraic Tests (Kalman rank condition & Hautus test)
- A few Examples
- Duality
- Lack of Controllability and of Observability

Controllability

- Consider
$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n$$

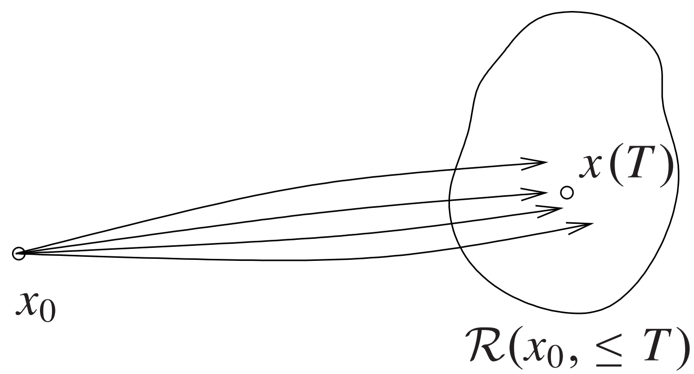
- with known solution
$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- **Definition:** System is controllable if for any $x_0 \in \mathbb{R}^n$, $x_t \in \mathbb{R}^n$, there exists input $u(\tau)$, with $\tau \in [0, t]$, $0 < t < \infty$, such that $x(t) = x_t$

Controllability and Reachability

(Notice distinction in nomenclature, as in AM book, between reachability & controllability)

- Pick point x_0, time $T \geq 0$. Define time-dependent set $\mathcal{R}(x_0, \leq T)$ (reachability set)



- Note on the *computation* of reachability sets

Algebraic Conditions for Controllability

- Controllability depends on the form $e^{At}B$, i.e. on matrices A, B:

$$e^{At}B = [I + At + A^2 \frac{t^2}{2!} + \ldots]B = [B|AB|A^2B|A^3B|\ldots] \begin{bmatrix} 1 \\ t \\ \frac{t^2}{2!} \\ \vdots \end{bmatrix}$$

- **Theorem:** (A, B) controllable if and only if

$$\text{rank } [B|AB|A^2B|\ldots|A^{n-1}B] = n \, (= \text{number of rows})$$

- **Lemma** (Cayley-Hamilton):
  Any square matrix satisfies its own characteristic equation

- Proof (of Theorem): consider characteristic polynomial of A

  $\det[sI-A] = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = 0$, for an $n \times n$ matrix A

- Using above Lemma,

$$A^n = -a_1 A^{n-1} - a_2 A^{n-2} - \ldots - a_{n-1}A - a_n I$$

$$\Downarrow$$

$$\text{Col}(A^n B) \subseteq \text{Col}[B|AB|A^2B|A^3B|\ldots|A^{n-1}B]$$

$$\text{Col}[B|AB|\ldots] \subseteq \text{Col}\ [B|AB|\ldots|A^{n-1}B]$$

$$\Downarrow$$

$$\text{rank}[B|AB|A^2B|A^3B|\ldots] = \text{rank}[B|AB|A^2B|A^3B|\ldots|A^{n-1}B] \quad \square$$

- The test

$$\text{rank } [B|AB|A^2B|\dots|A^{n-1}B] = n$$

is known as Kalman rank condition

- Extensions to time-varying, non-linear case through notion of state-transition matrix (also mentioned in Lec. 2) – see bibliography for more details

- In MATLAB, use command `ctrb` over `ss` structure to obtain controllability matrix

Controllability: An Example

- Consider: $A = \begin{bmatrix} -5 & -4 & 4 \\ 1 & 0 & -2 \\ -1 & -1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

- Compute: $[B|AB|A^2B|\ldots|A^{n-1}B] = \begin{bmatrix} 3 & -7 & 19 \\ -1 & 1 & -1 \\ 1 & -3 & 9 \end{bmatrix}$

- Perform elementary column operations $(2^o + 1^o, 3^o - 1^o)$:

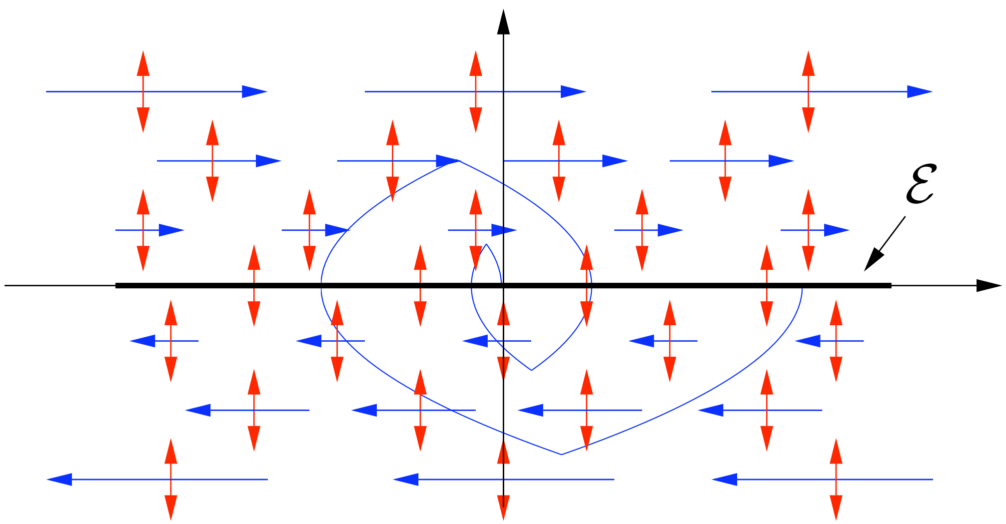
$$\text{rank} \begin{bmatrix} 3 & -7 & 19 \\ -1 & 1 & -1 \\ 1 & -3 & 9 \end{bmatrix} = \text{rank} \begin{bmatrix} 3 & -4 & 16 \\ -1 & 0 & 0 \\ 1 & -2 & 8 \end{bmatrix} = 2 < n = 3$$

$\Rightarrow (A, B)$ <u>not controllable</u>

Controllability: a Second Example

- Consider the following simple model *(double integrator)*

$$\begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = u \end{cases}$$



Controllability: a Third Example

- Consider the following simple model

$$\begin{cases} \dot{x}_1 & = x_1 + \delta x_2 + u \\ \dot{x}_2 & = x_2 + \delta u \end{cases}$$

$$A = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ \delta \end{bmatrix}$$

- controllable for $\delta \neq 0$

Controllability: a Fourth Example

- Inverted pendulum on cart (Segway)

$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & mg/M & 0 & 0 \\ 0 & (M+m)g/Ml & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/M \\ 1/Ml \end{bmatrix} u$$

- Here $n = 4$. Controllability matrix is $[B|AB|A^2B|A^3B]$:

$$\begin{bmatrix} 0 & 1/M & 0 & mg/M^2l \\ 0 & 1/Ml & 0 & (M+m)g/(Ml)^2 \\ 1/M & 0 & mg/M^2l & 0 \\ 1/Ml & 0 & (M+m)g/(Ml)^2 & 0 \end{bmatrix}$$

- Controllability matrix has full rank (i.e., equal to 4)

$$\Downarrow$$

(A, B) is controllable

$\vdots$

(perhaps this is one good reason to buy a Segway . . .)[1]

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[1] Notice though that we have only shown that we can control the *linearized* model . . .

Observability

- Consider autonomous (no control) system:

$$\frac{d}{dt}x(t) = Ax(t), \quad x(0) = x_0 \in \mathbb{R}^n; \qquad y(t) = Cx(t)$$

- with solution

$$y(t) = Ce^{At}x_0$$

- **Definition:** System is <u>observable</u> if any $x_0 \in \mathbb{R}^n$ can be derived from observation $y(\tau)$ within the interval $\tau \in [0, t]$, $t > 0$

- Observability follows from Ce^{At}, hence it depends on matrices A, C:

$$Ce^{At} = C\left[I + At + A^2\frac{t^2}{2!} + \ldots\right] = \left[1|t|\frac{t^2}{2!}|\ldots\right]\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}$$

- **Theorem:** (A, C) observable if and only if

$$\text{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \, (= \text{number of columns})$$

- In MATLAB, use `obsv` over `ss` structure to obtain obs. matrix

Duality Controllability/Observability

- Controllability, observability are dual concepts:

$$(A, B) \ \underline{\text{controllable}} \Leftrightarrow (A^T, B^T) \ \underline{\text{observable}}$$

$$(A, C) \ \underline{\text{observable}} \Leftrightarrow (A^T, C^T) \ \underline{\text{controllable}}$$

- The above fact relates to the following propositions:
  - For every property that holds for controllability there exists a dual property in terms of observability
  - State feedback and observer design problems are closely related (as we shall see later in class)
  - Both controllability and observability are invariant under similarity transformations (will elaborate in exercise session)

Controllability & Observability: Alternative Test by Hautus

- Rationale: $\lambda I - A$ has rank n for all λ not equal to an eigenvalue of A $\Rightarrow$ rank check needs only to be evaluated when λ is equal to an eigenvalue of A

- Finding uncontrollable or unobservable *eigenvalues* can be done using:

- **Hautus controllability condition:** (A, B) is controllable iff

$$\text{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n, \text{ for all } \lambda \in \mathbb{C}$$

- **Hautus observability condition:** (A, C) is observable iff

$$\text{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n, \text{ for all } \lambda \in \mathbb{C}$$

Controllability: Example of Hautus Test

- Consider (A, B) used above: $A = \begin{bmatrix} -5 & -4 & 4 \\ 1 & 0 & -2 \\ -1 & -1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

- Matrix A has eigenvalues in the set $\{-1, -2, -3\}$

- Perform following computations:

  – $\text{rank} \left[\ (\lambda I - A)|_{\lambda=-1} \ \middle| \ B \ \right] = \text{rank} \left[\begin{array}{ccc|c} 4 & 4 & -4 & 3 \\ -1 & -1 & 2 & -1 \\ 1 & 1 & 0 & 1 \end{array} \right] = 3$

$$\Downarrow$$

eigenvalue -1 is controllable

– rank $\left[\ (\lambda I - A)|_{\lambda=-2}\ \middle|\ B\ \right]$ = rank $\left[\begin{array}{rrr|r} 3 & 4 & -4 & 3 \\ -1 & -2 & 2 & -1 \\ 1 & 1 & -1 & 1 \end{array}\right] = 2$

$$\Downarrow$$

eigenvalue -2 is uncontrollable

– rank $\left[\ (\lambda I - A)|_{\lambda=-3}\ \middle|\ B\ \right]$ = rank $\left[\begin{array}{rrr|r} 2 & 4 & -4 & 3 \\ -1 & -3 & 2 & -1 \\ 1 & 1 & -2 & 1 \end{array}\right] = 3$

$$\Downarrow$$

eigenvalue -3 is controllable

A few Reasons for the Emergence of <u>un</u>controllability and of <u>un</u>observability

UC 1 - physical uncontrollability - uncoupled variables are not affected by input

UC 2 - parallel interconnection controlled by single input

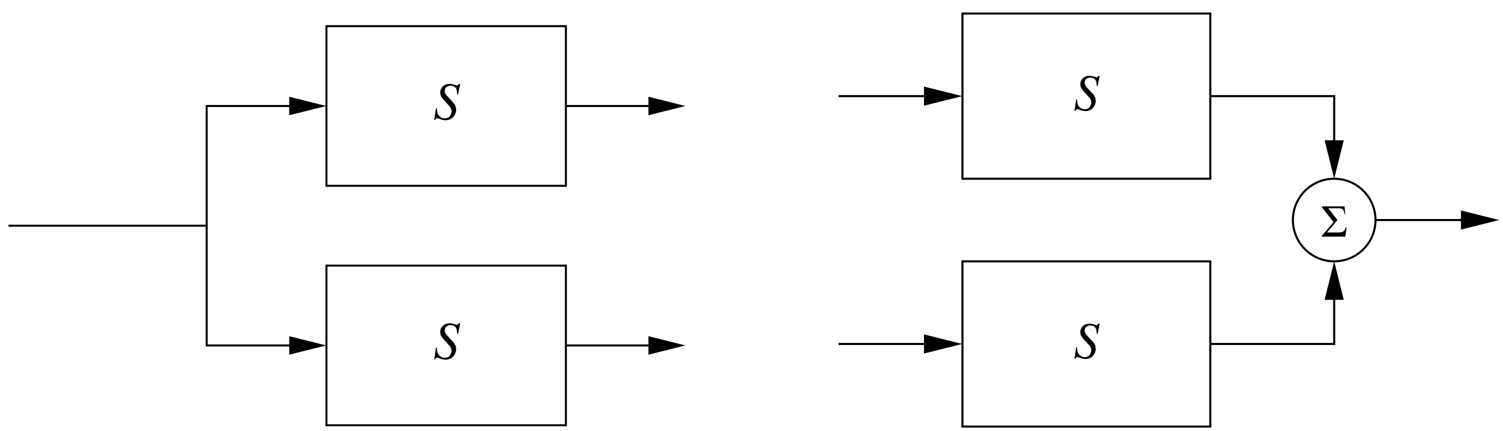
UO 1 - directly unmeasured variables are not fed back to measured ones

UO 2 - single variables cannot be extracted from global observation function

- This is in particular true for SISO systems

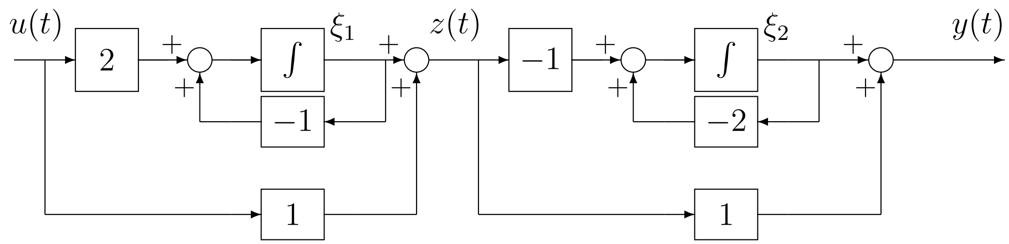
A few Examples of uncontrollable and unobservable Models

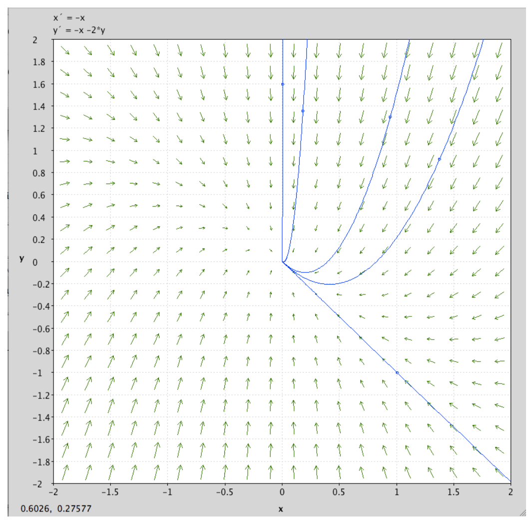
1. Consider identical systems in parallel:



The first is not controllable, whereas the second is not observable

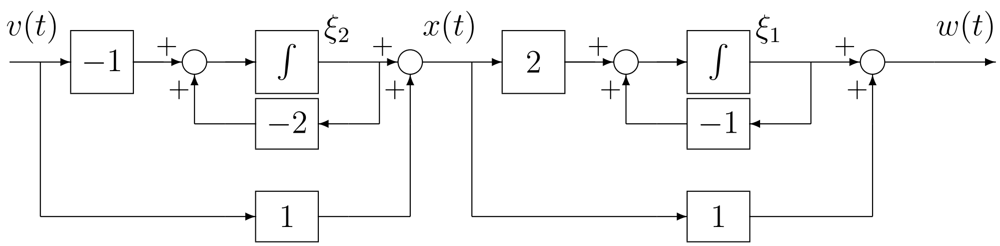
2. Consider system in series, described via the following block diagram:

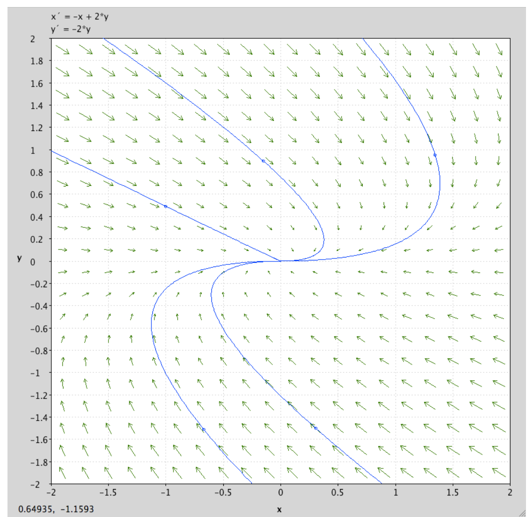




It is controllable, though not observable (check with Hautus test)

3. Consider system in series, described via the following block diagram:





It is observable, though not controllable

4. Consider model $\dot{x} = ax + bu$. Introduce variables $y = cx$. Then $\dot{y} = ay + bcu$. The controllability matrix of the parallel composition of the two models is

$$\begin{pmatrix} b & ab \\ bc & abc \end{pmatrix},$$

which has rank 1. Thus, it is not controllable.