Randomized methods for stochastic constrained control
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Luca Deori and Simone Garatti @Politecnico di Milano
John Lygeros @ETH Zurich
Discrete time stochastic linear system:

\[ x_{t+1} = Ax_t + Bu_t + w_t \]

Objective:

Design a state-feedback policy minimizing a finite horizon cost function subject to state/input constraints.

Cost function:

\[ J = E \left[ \sum_{t=0}^{M} x_t^T Q_t x_t + \sum_{t=0}^{M-1} u_t^T R_t u_t \right], Q_t > 0, R_t > 0 \]

Input/state constraints:

\[ f_t(x_t, u_t) \leq 0, t = 0, 1, ..., M \]
Constrained optimal control problem

Discrete time stochastic linear system:

\[ x_{t+1} = Ax_t + Bu_t + w_t \]

Cost function:

\[ J = E \left[ \sum_{t=0}^{M} x_t^T Q x_t + \sum_{t=0}^{M-1} u_t^T R u_t \right] \]

Input/state constraints:

\[ \|C x_t\|_\infty \leq \bar{y}, \quad t = 0,1,...,M \quad \text{[safety constraint]} \]

\[ \|u_t\|_\infty \leq \bar{u}, \quad t = 0,1,...,M-1 \quad \text{[limit on control effort]} \]

Finite horizon control problem:

Comments:

- Easy to solve without constraints, e.g., LQG if noise is Gaussian
- Difficult in presence of constraints (no closed-form solution)
Constrained optimal control problem

Comments:

- Easy to solve without constraints, e.g., LQG if noise is Gaussian
- Difficult in presence of constraints (no closed-form solution)

Head for a sub-optimal solution, resting on some convenient parameterization of the state-feedback control policy

Parameterization of the control policy

\[ x_{t+1} = Ax_t + Bu_t + w_t \]

\[ u_t = g_t(x_t, x_{t-1}, \ldots, x_0) \]

State feedback control policy
Parameterization of the control policy

\[ x_{t+1} = Ax_t + Bu_t + w_t \]

\[ u_t = \sum_{j=0}^{t-1} \theta_{t,j} \varphi(w_j) + \gamma_t \]

Policy affine in the reconstructed (and possibly saturated) noise

Both input and state depend linearly on the parameters \( \theta_{t,j} \) and \( \gamma_t \)
Parameterization of the control policy

Control: affine function of $\phi(w)$

$$u_t = \sum_{j=0}^{t-1} \theta_{t,j} \phi(w_j) + \gamma_t$$

$$w_t = x_{t+1} - Ax_t - Bu_t$$

• Input and state depend **linearly** on the parameters $\theta_{t,j}$ and $\gamma_t$
• If $\phi =$ identity, equivalent to a parameterization affine in the state
Compact matrix form

\[ u_t = \sum_{j=0}^{t-1} \theta_{t,j} \varphi(w_j) + \gamma_t \]

\[ U = \Gamma + \Theta \varphi(W) \]

- Open-loop term
- Feedback term: strictly lower triangular for causality

\[ \Gamma = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{M-1} \end{bmatrix} \]

\[ \Theta = \begin{bmatrix} u_0 & 0 & \cdots & 0 \\ u_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{M-1} & 0 & \cdots & 0 \end{bmatrix} \]

Constrained optimal control problem

Discrete time stochastic linear system:

\[ x_{t+1} = Ax_t + Bu_t + w_t \]

Finite horizon control problem:

Cost function:

\[ J = E \left[ \sum_{t=0}^{M} x_t^T Q_t x_t + \sum_{t=0}^{M-1} u_t^T R_t u_t \right] \]

Input/state constraints:

\[ f_t(x_t, u_t) \leq 0, t = 0, 1, \ldots, M \]
Let

\[
X := \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_M \end{bmatrix} \quad U := \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M-1} \end{bmatrix} \quad W := \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{M-1} \end{bmatrix}
\]

Then, all quantities involved in the control problem along the look-ahead time horizon \([0,M]\) can be rewritten in matrix form.

Control policy: \(U = \Gamma + \Theta \varphi(W)\)

State dynamics: \(X = Ax_0 + BU + DW\)

Cost function: \(J(\Gamma, \Theta) = E[X^TQX + U^TRU]\)

Input/state constraints: \(F(X, U) \leq 0\)

where \(A, B, D, Q\) and \(R\) are appropriately defined matrices, and \(F(\ldots)\) is the collection of the \(f_i(\ldots)\) functions.
Control problem formulation

\[ x_{t+1} = Ax_t + Bu_t + w_t \]

\[ u_t = \sum_{j=0}^{t-1} \theta_j \phi(w_j) + \gamma_t \]

Robust approach: constraints must be satisfied for every and each disturbance realization

\[ \Rightarrow \text{hard constraints} \]
Control problem formulation

\[ x_{t+1} = Ax_t + Bu_t + w_t \]

\[ u_t = \sum_{j=0}^{t-1} \theta_{t,j} \varphi(w_j) + \gamma_t \]

\[ \Pr = \varepsilon \]

Chance-constrained approach: constraints must be satisfied for most disturbance realizations except for a set of probability \( \leq \varepsilon \) → soft constraints

Input constraint: robust approach

Robust optimization problem:

\[ \min_{\Gamma, \Theta} J(\Gamma, \Theta) \]

s.t. \( \|U(w)\|_\infty \leq \overline{U} \quad \forall w \)
Input constraint: robust approach

Robust optimization problem:
\[
\begin{align*}
\min_{\Gamma, \Theta} & \ J(\Gamma, \Theta) \\
\text{s.t.} & \ ||U(w)||_\infty \leq \bar{u} \ \forall w
\end{align*}
\]

\[
U = \Gamma + \Theta \varphi(W)
\]

\[
\begin{align*}
\min_{\Gamma, \Theta} & \ J(\Gamma, \Theta) \\
\text{s.t.} & \ |\gamma_j| + ||\Theta||_1 \varphi \leq \bar{u}, \ j = 1, 2, \ldots, mM
\end{align*}
\]

finite convex optimization program

Input constraint: chance-constrained approach

Chance-constrained optimization problem:
\[
\begin{align*}
\min_{\Gamma, \Theta} & \ J(\Gamma, \Theta) \\
\text{s.t.} & \ P(||U(w)||_\infty \leq \bar{u}) \geq 1 - \varepsilon
\end{align*}
\]
**Input constraint: chance-constrained approach**

Chance-constrained optimization problem:

\[
\min_{\Gamma, \Theta} J(\Gamma, \Theta)
\]

s.t. \( P(\|U(w)\|_\infty \leq \overline{w}) \geq 1 - \varepsilon \)

Generally hard to solve because probabilistic constraint is non-convex

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**Randomized solution**

Use the “scenario” approach to reduce the chance-constrained optimization to a finite convex optimization problem.

This involves:

- extracting \( N \) noise realizations ("scenarios")

- solving the corresponding scenario chance-constrained program, where the \( N \) extracted noise realizations are treated as if they were the only admissible ones
Scenario approach to chance-constrained optimization

\[ \min_{\eta} J(\eta) \text{ subject to:} \]
\[ P(f(\eta, \delta) \leq 0) \geq 1 - \varepsilon \]

\[ \delta \in \Delta \text{ uncertainty parameter} \]

\[ \eta \in \mathbb{R}^{d} \text{ optimization variable} \]

Scenario approach

- extract \( N \) uncertainty instances \( \delta^{(1)}, \ldots, \delta^{(N)} \) and consider only the corresponding constraints

- choose \( 0 \leq \alpha < \varepsilon \) and solve:

\[ \min_{\eta} J(\eta) \text{ subject to:} \]
\[ f(\eta, \delta^{(i)}) \leq 0 \text{ for } N - \lfloor \alpha N \rfloor \text{ indices in } \{1, \ldots, N \} \]

where a fraction \( \alpha \) of the \( N \) constraints is removed so as to improve the cost
Scenario approach to chance-constrained optimization

\[ \Delta \eta_1 \eta_2 \]

Scenario approach to chance-constrained optimization

\[ \Delta \]

\[ f(\eta, \delta^{(i)}) \leq 0 \]
Scenario approach to chance-constrained optimization

\[ f(\eta, \delta^{(1)}) \leq 0 \]

Solution with \( N \) constraints

\[ \eta^{*}_{2,N} \]

\[ \eta^{*}_{1,N} \]

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Scenario approach to chance-constrained optimization

\[ f(\eta, \delta^{(1)}) \leq 0 \]

Solution with \( N-1 \) constraints

\[ \eta^{*}_{2,N-1} \]

\[ \eta^{*}_{1,N-1} \]

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Choose one of the active constraints based on some algorithm (greedy, random) and remove it

Solution with N-1 constraints

Scenario approach to chance-constrained optimization

Solution with N-1 constraints
What about the other, unseen, uncertainty instances?

**Assumptions**
\[
J(\eta) = c^T \eta \\
\delta(\eta, \delta) \quad \text{convex functions of } \eta, \forall \delta \in \Delta
\]

**Theorem**
Fix \( \beta \in (0,1) \)
If \( N \) is such that
\[
(\lceil anN \rceil + d - 1)^{\lceil anN \rceil + \ell - 1} \sum_{i=0}^{\lceil anN \rceil} \binom{\lceil anN \rceil}{i} \varepsilon^i (1 - \varepsilon)^{\ell - i} \leq \beta
\]
then, with confidence not less than \( 1 - \beta \) the scenario solution \( \eta_{N-\lceil anN \rceil} \) is feasible for the original chance-constrained problem.

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**Assumptions**
\[
J(\eta) = c^T \eta \\
\delta(\eta, \delta) \quad \text{convex functions of } \eta, \forall \delta \in \Delta
\]

**Remark**
Linearity of \( J(\eta) \) can be relaxed to convexity by adding an auxiliary optimization variable \( k \) and considering:
\[
\min_{\eta, k} k \quad \text{subject to:} \\
P(J(\eta) \leq k \land f(\eta, \delta) \leq 0) \geq 1 - \varepsilon
\]
**Scenario approach to chance-constrained optimization**

**Role of $\beta$**

- $\eta^*_{N-[\alpha N]}$ is certainly feasible for the original C-C problem
  - $\beta \to 0$ : $N \to \infty$

$N$ increases with the logarithm of $\beta$.

$$N \geq \frac{1}{\varepsilon} \left( d + \log \left( \frac{1}{\beta} \right) + \sqrt{2d \log \left( \frac{1}{\beta} \right)} \right) \quad (\alpha = 0, \text{ no constraint removal})$$

$\beta = 10^{-10}$ feasibility guaranteed with a confidence that is 1, in practice!

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**Role of $\alpha$**

- Expected violation level for the scenario solution
  - $\gamma^*_{N-[\alpha N]} \to \gamma$ solution to the original C-C problem
  - $N \to \infty \quad \text{as } O \left( \frac{1}{\varepsilon - \alpha} \right)$

User-chosen parameter that compromises quality of the approximation and computational effort
Chance-constrained optimization problem:
\[
\min_{\Gamma, \Theta} J(\Gamma, \Theta) \\
\text{s.t. } P(\|U(w)\|_\infty \leq \overline{u}) \geq 1 - \epsilon
\]

Scenario solution (without constraint removal)

Extract \( N \) realizations of the disturbance \( w^{(i)} \) and solve
\[
\min_{\Gamma, \Theta} J(\Gamma, \Theta) \\
\text{s.t. } \|U(w^{(i)})\|_\infty \leq \overline{u}, \ i = 1, 2, \ldots, N
\]

Finite convex optimization program!
Input constraint: chance-constrained approach

Chance-constrained optimization problem:
\[
\min_{\Gamma, \Theta} f(\Gamma, \Theta) \\
\text{s.t. } P(||U(w)||_\infty \leq \overline{u}) \geq 1 - \epsilon
\]

Scenario solution (without constraint removal)

Extract \(N\) realizations of the disturbance \(w(i)\) and solve
\[
\min_{\Gamma, \Theta} f(\Gamma, \Theta) \\
\text{s.t. } ||U(w(i))||_\infty \leq \overline{u}, \ i = 1, 2, \ldots, N
\]

Theorem
\(\epsilon \in (0, 1), \ \beta \in (0, 1)\)
If \(N \geq \frac{1}{\epsilon} \left(d + \log \left(\frac{1}{\beta}\right) + \sqrt{2d \log \left(\frac{1}{\beta}\right)}\right)\), then
\(P(||U(w)||_\infty \leq \overline{u}) \geq 1 - \epsilon\) with high confidence.
Input constraint: chance-constrained approach

Chance-constrained optimization problem:
\[ \min_{\Gamma, \Theta} J(\Gamma, \Theta) \]
\[ \text{s.t. } P(\|U(w)\|_\infty \leq \bar{u}) \geq 1 - \varepsilon \]

Scenario solution (without constraint removal)

Extract \( N \) realizations of the disturbance \( w^{(i)} \) and solve
\[ \min_{\Gamma, \Theta} J(\Gamma, \Theta) \]
\[ \text{s.t. } \|U(w^{(i)})\|_\infty \leq \bar{u}, \ i = 1, 2, \ldots, N \]

Theorem
\( \epsilon \in (0, 1), \beta \in (0, 1) \)
If \( N \geq \frac{1}{\epsilon} \left( \bar{u} \log \left( \frac{1}{\beta} \right) + \sqrt{2d\log \left( \frac{1}{\beta} \right)} \right) \), then
\[ P(\|U(w)\|_\infty \leq \bar{u}) \geq 1 - \varepsilon \] with high confidence

Control policy parameterization

\( u_t = \sum_{j=0}^{T-1} \theta_{t,j} \varphi(w_j) + y_t \)

\[ U = \Gamma + \Theta \varphi(W) \]

open-loop term
feedback term: strictly lower triangular for causality

\( \Gamma = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{M-1} \end{bmatrix} \)

\( \Theta = \begin{bmatrix} w_0 & 0_{M \times M} & \cdots & 0_{M \times M} \\ w_1 & \ddots & \cdots & 0_{M \times M} \\ \vdots & \ddots & \ddots & \vdots \\ w_{M-1} & \cdots & \cdots & 0_{M \times M} \end{bmatrix} \)

\( U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M-1} \end{bmatrix} \)

\( W = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{M-1} \end{bmatrix} \)
Control policy parameterization

\[ U = \Gamma \]

\[ \Gamma = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{M-1} \end{bmatrix} \]

\[ U = \Gamma + \Theta \varphi(W) \]

\[ \Theta = \begin{bmatrix} 0_{m \times n_e} & 0_{m \times n_e} & \cdots & 0_{m \times n_e} \\ 0_{m \times n_e} & 0_{m \times n_e} & \cdots & 0_{m \times n_e} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{m \times n_e} & 0_{m \times n_e} & \cdots & 0_{m \times n_e} \end{bmatrix} \]

Constraint on the state variables

Robust constraint

\[ \|\mathcal{X}(w)\|_\infty \leq \bar{y}, \ \forall w \]
Constraint on the state variables

Disturbance with unbounded support

Robust constraint

\[ \|X(w)\|_{\infty} \leq \bar{y}, \ \forall w \]

Unfeasible

\[ X = Ax_0 + BU + DW \]
Constraint on the state variables

Robust constraint

\[ \| \mathcal{C}(w) \|_\infty \leq \bar{y}, \ \forall w \]

Unfeasible

\[ X = Ax_0 + BU + DW \]

Probabilistic constraint

\[ P(\| \mathcal{C}(w) \|_\infty \leq \bar{y}) \geq 1 - \varepsilon \]

Disturbance with unbounded support

Constraint on the state variables

Robust constraint

\[ \| \mathcal{C}(w) \|_\infty \leq \bar{y}, \ \forall w \]

Unfeasible

\[ X = Ax_0 + BU + DW \]

Probabilistic constraint

\[ P(\| \mathcal{C}(w) \|_\infty \leq \bar{y}) \geq 1 - \varepsilon \]

Feasible/unfeasible depending on \( \bar{y} \)
**Constraint on the state variables (cont’d)**

\[ \|CX(w)\|_\infty \leq h \]

the bound on the state \( h \) is a decision variable that can be set to a value compatible with the system dynamics, input constraints, and disturbance characteristics

**How to account for the two contrasting objectives?**

---

**Penalized cost**

New cost function

\[ J_\mu = J + \mu h \]

State constraint: **probabilistic**

Input constraint:

- **robust**
  \[
  \min_{\Gamma, \Theta} J(\Gamma, \Theta) + \mu h \\
  \text{s.t. } \frac{\|U(w)\|_\infty}{\|CX(w)\|_\infty} \leq \bar{u}, \forall w \\
  P\left(\|CX(w)\|_\infty \leq h\right) \geq 1 - \epsilon
  \]

- **probabilistic**
  \[
  \min_{\Gamma, \Theta} J(\Gamma, \Theta) + \mu h \\
  \text{s.t. } P\left(\frac{\|U(w)\|_\infty}{\|CX(w)\|_\infty} \leq \bar{u} \wedge \|CX(w)\|_\infty \leq h\right) \geq 1 - \epsilon
  \]
Penalized cost

New cost function \( J_\mu = J + \mu h \)

State constraint: probabilistic

Input constraint:

\[
\begin{align*}
\text{robust} & : \\
\min_{\Gamma, \Theta} & J(\Gamma, \Theta) + \mu h \\
\text{s.t.} & \left\{ \| U(w) \|_\infty \leq \bar{u}, \forall w \right\} \\
& \left\{ \mathbb{P} \left[ \| CX(w) \|_\infty \leq h \right] \geq 1 - \varepsilon \right\}
\end{align*}
\]

\[
\begin{align*}
\text{probabilistic} & : \\
\min_{\Gamma, \Theta} & J(\Gamma, \Theta) + \mu h \\
\text{s.t.} & \left\{ \mathbb{P} \left( \bigwedge \left[ \| U(w) \|_\infty \leq \bar{u} \right] \right) \geq 1 - \varepsilon \right\}
\end{align*}
\]

Both problems can be approximately solved via the scenario approach.

\[ \mu = \text{tuning parameter to trade-off between } J \text{ (performance) and } h \text{ (safety)} \]

- Chance-constrained feasibility guaranteed (scenario theory)
- \( \mu \) has no precise meaning. How can we choose a proper value for \( \mu \)?
Two-stage method

Input constraint: robust

\[
\begin{align*}
\min_{\Gamma, \Theta} & \left\{ J(\Gamma, \Theta) \right\} \\
\text{s.t.} & \quad \|U(w)\|_{\infty} \leq \bar{u}, \quad \forall w \\
J^* &= \text{optimal cost}
\end{align*}
\]

\[
\min_{\Gamma, \Theta, h} \left\{ \min_{\Gamma, \Theta} \left\{ J(\Gamma, \Theta) \right\} \right\} \\
\text{s.t.} & \quad \left\{ \begin{array}{l}
\|U(w)\|_{\infty} \leq \bar{u}, \quad \forall w \\
\mathbb{P}(\|h(X(w))\|_{\infty} \leq h) \geq 1 - \varepsilon \\
J(\Gamma, \Theta) \leq J^* + \alpha
\end{array} \right.
\]

Problem 1
- minimize \( J \)
- input constrain only

Problem 2
- minimize \( h \)
- state/input constraint
- constraint on the performance degradation

Input constraint: probabilistic

\[
\begin{align*}
\min_{\Gamma, \Theta} & \left\{ J(\Gamma, \Theta) \right\} \\
\text{s.t.} & \quad \mathbb{P}(\|U(w)\|_{\infty} \leq \bar{u}) \geq 1 - \varepsilon \\
J^* &= \text{optimal cost}
\end{align*}
\]

\[
\min_{\Gamma, \Theta, h} \left\{ \min_{\Gamma, \Theta} \left\{ J(\Gamma, \Theta) \right\} \right\} \\
\text{s.t.} & \quad \left\{ \begin{array}{l}
\mathbb{P}(\|U(w)\|_{\infty} \leq \bar{u}) \geq 1 - \varepsilon \\
\mathbb{P}(\|h(X(w))\|_{\infty} \leq h) \geq 1 - \varepsilon \\
J(\Gamma, \Theta) \leq J^* + \alpha
\end{array} \right.
\]
Two-stage method

Input constraint: robust

\[
\min_{\Gamma, \Theta} J(\Gamma, \Theta) \\
\text{s.t. } \|U(w)\|_\infty \leq \bar{u} \quad \forall w \\
J^* = \text{optimal cost}
\]

\[
\min_{\Gamma, \Theta, h} h \\
\text{s.t. } \left\{ \begin{align*}
\|U(w)\|_\infty & \leq \bar{u}, \forall w \\
P(\|U(w)\|_\infty \leq h) & \geq 1 - \varepsilon \\
J & \leq J^* + \alpha
\end{align*} \right.
\]

Input constraint: probabilistic

\[
\min_{\Gamma, \Theta} J(\Gamma, \Theta) \\
\text{s.t. } P(\|U(w)\|_\infty \leq \bar{u}) \geq 1 - \varepsilon \\
J^* = \text{optimal cost}
\]

\[
\min_{\Gamma, \Theta, h} h \\
\text{s.t. } \left\{ \begin{align*}
P(\|U(w)\|_\infty \leq \bar{u}) & \geq 1 - \varepsilon \\
P(\|C_X(w)\|_\infty \leq h) & \geq 1 - \varepsilon \\
J(\Gamma, \Theta) & \leq J^* + \alpha
\end{align*} \right.
\]

Two-stage method

\[\alpha = \text{tuning parameter to trade-off between } J \text{ (performance) and } h \text{ (safety)}\]

- Chance-constrained feasibility guaranteed when robust input constraint
- \(\alpha\) has a precise meaning (maximum allowed performance degradation)
Properties

- Monotonicity
  - $J$ is an increasing function of $\mu \ [of \ \alpha]$
  - $h$ is a decreasing function of $\mu \ [of \ \alpha]$
  ($\mu$ and $\alpha$ are indeed tuning parameter)

- Continuity
  - $J$ is a continuous function of $\mu \ [of \ \alpha]$
  - $h$ is a continuous function of $\mu \ [of \ \alpha]$
  (exhaustive exploration of all possible trade-offs between $J$ and $h$).

- Equivalence
  - for every $\mu$ there exists an $\alpha$ returning the same values for $J$ and $h$
  - for every $\alpha$ there exists a $\mu$ returning the same values for $J$ and $h$

A numerical example

\[ Q = \begin{bmatrix} I_4 & 0_4 \\ 0_4 & 0_4 \end{bmatrix} \]
\[ R = 10^{-6} I_3 \]

\[ \begin{align*}
    \dot{x} &= \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \\
    x &= \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}
\end{align*} \]
### Input saturation constraint

#### Input saturation at ±5

<table>
<thead>
<tr>
<th></th>
<th>No control (u=0)</th>
<th>LQG control without input saturation</th>
<th>LQG control with input saturation</th>
<th>Robust Approach</th>
<th>Scenario Approach with input saturation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J )</td>
<td>82.33</td>
<td>30.76</td>
<td>45.79</td>
<td>43.62</td>
<td>37.08</td>
</tr>
</tbody>
</table>

Constraint violation = 23%

Scenario parameters: \( \varepsilon = 0.1, \beta = 10^{-5} \)

Constraint violation less than 5%

### Receding Horizon implementation

**Norm of the mass displacements**

- Blue = Scenario Approach with input saturation
- Cyan = Robust method
- Red = LQG control with input saturation

Prediction window = 5  Control window = 5
### State constraint: penalized cost

Spring deformation constraint: \( \|CX(w)\|_\infty \leq h \)

\[
CX_t = \begin{bmatrix}
  d_{1,t} \\
  d_{2,t} - d_{1,t} \\
  d_{3,t} - d_{2,t} \\
  d_{4,t} - d_{3,t}
\end{bmatrix}
\]

<table>
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<tr>
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<th>No control ((u=0))</th>
<th>LQG control with input saturation</th>
<th>Scenario with input saturation (\mu=0)</th>
<th>Scenario with input saturation (\mu=1)</th>
<th>Scenario with input saturation (\mu=10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J )</td>
<td>82.33</td>
<td>45.79</td>
<td>37.49</td>
<td>38.15</td>
<td>43.37</td>
</tr>
<tr>
<td>( h )</td>
<td>13.20</td>
<td>10.58</td>
<td>9.05</td>
<td>7.15</td>
<td>5.78</td>
</tr>
</tbody>
</table>

Constraint violation less than 5%

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### State constraint: two-stage approach

Spring deformation constraint: \( \|CX(w)\|_\infty \leq h \)

\[
CX_t = \begin{bmatrix}
  d_{1,t} \\
  d_{2,t} - d_{1,t} \\
  d_{3,t} - d_{2,t} \\
  d_{4,t} - d_{3,t}
\end{bmatrix}
\]

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<th>LQG control with input saturation</th>
<th>Scenario with input saturation (\alpha=0)</th>
<th>Scenario with input saturation (\alpha=0.05)</th>
<th>Scenario with input saturation (\alpha=0.1)</th>
</tr>
</thead>
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<tr>
<td>( J )</td>
<td>82.33</td>
<td>45.79</td>
<td>37.49</td>
<td>39.50</td>
<td>41.41</td>
</tr>
<tr>
<td>( h )</td>
<td>13.20</td>
<td>10.58</td>
<td>9.05</td>
<td>6.50</td>
<td>6.08</td>
</tr>
</tbody>
</table>

Constraint violation less than 5%
Final remarks

- computational approach to stochastic optimal control with input/state constraints
- trade-off between performance and satisfaction of the state constraint
- no assumption on the noise characteristics (no i.i.d. or Gaussianity assumptions)
- straightforward extension to the case of time-varying and also uncertain linear systems (e.g. Markov Jump linear systems)
  \[ x_{t+1} = A(\delta_t)x_t + B(\delta_t)u_t + w_t \]
- extension to a nonlinear setting is possible using a statistical learning theoretical approach (computational issues arise)

Application to Model Predictive Control

MPC approach:
- measure the state
- solve M-step finite optimization problem
- apply the first control action
- repeat at the next step
Application to Model Predictive Control

MPC approach:
- measure the state
- solve M-step finite optimization problem
- apply the first control action
- repeat at the next step

- mean-square stability can be proven for asymptotically stable systems if robust input saturation constraints are present
- input constraint is satisfied over infinite time (first control action is $u_0 = y_0$)
- probabilistic guarantees on state constraint over infinite time is an open issue

References

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