# Fuzzy Gain Scheduling: Controller and Observer Design Based on Lyapunov Method and Convex Optimization

Petr Korba, Robert Babuška, Henk B. Verbruggen, and Paul M. Frank

***Abstract*—This paper addresses model-based fuzzy control. A constructive and automated method for the design of a gain-scheduling controller is presented. Based on a given Takagi–Sugeno fuzzy model of the plant, the controller is designed such that stability and prescribed performance of the closed loop are guaranteed. These properties are valid in a wide working range around an equilibrium without restrictions to slowly varying trajectories. The synthesis is based on linear matrix inequalities and convex optimization techniques. If required, a fuzzy state estimator and an extended controller can be included, providing a zero steady-state error in the presence of disturbances and modeling errors. The proposed method has been applied to a control of a laboratory liquid-level process. Hence, the performance has been evaluated in simulations as well as in real-time control.**

***Index Terms*—Fuzzy gain-scheduling, linear matrix inequality (LMI), model-based fuzzy control, performance, quasi-linear parameter varying systems, stability, Takagi–Sugeno (TS) fuzzy models.**

## I. Introduction

LATELY, many publications on *model-based fuzzy control* (MBFC) have appeared; see, for instance, [1]–[4], among many others. MBFC can be seen as a modern class of fuzzy control techniques, which are fundamentally different from *heuristics-based fuzzy control* (HBFC). In the latter method, it is implicitly assumed that there is no model of the process to be controlled and the controller design is therefore based on the knowledge of an experienced operator and his/her linguistic description of the given problem, expressed by means of fuzzy rules. In contrast, MBFC is always based on a model of the plant under control. Such a model can be obtained from measured data (black-box modeling), from first principles (white-box modeling), or by combining the two approaches. An advantage of HBFC is that no mathematical model of the process is needed to design a controller. This method, however, has two significant drawbacks.

- Systematic and formally tractable design and tuning techniques are lacking.
- Basic properties such as stability, performance or robustness of the closed-loop system can only be investigated via extensive tests or simulations.

These drawbacks are one reason for the recent interest in MBFC techniques. In the literature related to MBFC, the following control schemes have been reported.

- Control via inversion of the fuzzy model—either in feedforward control through direct dynamic compensation, or, in the case of a considerable model-plant mismatch, in feedback control within the internal model control scheme [1], [4].
- Fuzzy model-based predictive control—used when it is not possible to invert the model (e.g., the model has a nonminimum phase behavior) and in the presence of constraints [3].
- Fuzzy gain-scheduling methods—typically based on slowly varying scheduling variables that capture nonlinearities and parameter dependencies; the global control law is obtained by means of fuzzy logic as an interpolation between a number of locally valid linear controllers [1], [5], [6].

The common drawback of the inverse and predictive control methods is the lack of techniques for analyzing the stability, robustness and performance properties of the closed loop system. A well-known limitation of classical gain-scheduling methods is the fact that only slowly varying trajectories are admissible for the controller to work as desired in the closed-loop.

In this paper, a fuzzy controller and observer design for the Takagi–Sugeno (TS) type of fuzzy model [7] is proposed. Moreover, the goal was to develop an automated design procedure. Note that it is quite easy to devise a numerical scheme to assess controller performance but much harder to rate a design method. Design methods can be assessed as follows.

- How are the requirements translated into a set of design parameters (e.g., how many parameters are needed)?
- How does the adjustment of each design parameter affect the closed-loop system? (e.g., do the parameters have understandable physical meaning)?
- Are the design parameters decoupled (does each parameter adjust the respective requirement or a tradeoff between competing requirements)?

A design method that rates high in each of these criteria is said to be automated [8]. Here, a constructive and automated design algorithm based on the Lyapunov direct method and convex optimization techniques is presented. Moreover, it

Manuscript received October 11, 1999; revised June 10, 2002 and August 12, 2002.

P. Korba is with ABB Switzerland, Ltd., CH 5405, Baden-Daettwil, Switzerland (e-mail: petr.korba@ch.abb.com).

R. Babuška and H. B. Verbruggen are with the Delft Center for Systems and Control, Delft University of Technology, 2628 CD Delft, the Netherlands (e-mail: r.babuska@dcsc.tudelft.nl).

P. M. Frank is with the Faculty of Electrical Engineering, Department of Measurement and Control, University of Duisburg, 47048 Duisburg, Germany.

Digital Object Identifier 10.1109/TFUZZ.2003.812680

guarantees stability and performance requirements for the closed loop system. Not only *state-feedback*, but also *output feedback* controllers with prescribed performance and tracking control problems are considered. In this paper, performance specifications for TS fuzzy control systems are introduced via location of eigenvalues of the underlying local linear time-invariant (LTI) subsystems given by the rule consequents, and an extended fuzzy gain-scheduler based on parallel distributed compensation (PDC), known, e.g., from [9], is developed. The proposed techniques are validated by means of a laboratory experiment; a second-order liquid level control system where only one state is measured.

The paper is organized as follows. Standard notions, such as the TS fuzzy model with a PDC controller and the resulting closed-loop description, are recalled in Section II. A brief review of the stabilizing fuzzy control synthesis techniques is given in Section III. New enhancements of these techniques are introduced in Section IV. They include an extended fuzzy scheduler, fuzzy state estimator and performance specifications in terms of linear matrix inequalities (LMIs). Section V is devoted to a laboratory application of the described method to a two-tank system: the design procedure, simulations and real experimental results are presented. Section VI concludes this paper. In Appendix, we show that, starting from the formalism of Lyapunov function stability, it is quite straightforward to prove that with the presented design method bounded-input–bounded-output (BIBO) stability is guaranteed as well.

## II. Fuzzy Model, Controller, and Closed-Loop System

### A. TS Fuzzy Model

The controller design procedure is based on the representation of a given nonlinear plant in terms of the fuzzy model given by (1). The antecedent part of each rule $\boldsymbol{R}_i$ contains fuzzy linguistic descriptions $M_{ji}$ of the scheduling variables $\delta_j(t)$ and the consequent part contains a local linear model of the nonlinear system

$$\begin{aligned}\boldsymbol{R}_i\text{: } &\textbf{IF } \delta_1 \text{ is } M_{1i} \textbf{ and } \dots \textbf{ and } \delta_j(t) \text{ is } M_{ji}, \textbf{ then}\\ &\dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\\ &\boldsymbol{y}_i(t) = \boldsymbol{C}_i\boldsymbol{x}(t) + \boldsymbol{D}_i\boldsymbol{u}(t).\end{aligned} \tag{1}$$

The entire fuzzy model of the plant (1) is obtained by fuzzy blending of the consequent submodels. For a given pair of vectors $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$, the final output of the fuzzy system is inferred as a weighted sum of the contributing submodels

$$\dot{\boldsymbol{x}}(t) = \frac{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))[\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)]}{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))} \tag{2}$$

$$\boldsymbol{y}(t) = \frac{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))[\boldsymbol{C}_i\boldsymbol{x}(t) + \boldsymbol{D}_i\boldsymbol{u}(t)]}{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))} \tag{3}$$

with $w_i(\boldsymbol{\delta}(t)) = \text{aggop}[M_{1i}(\delta_1(t)), \dots, M_{ji}(\delta_j(t))]$ where $w_i(\boldsymbol{\delta}(t)) \geq 0$ is the degree of fulfillment of rule $i$, $\text{aggop}(\cdot)$ is the aggregation operator (for instance, the product or the minimum), $\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t)) > 0$ for all $i = 1, 2, \dots, r$. With $0 \leq h_i(\boldsymbol{\delta}(t)) = (w_i(\boldsymbol{\delta}(t))/\sum_{j=1}^{r} w_j(\boldsymbol{\delta}(t))) \leq 1$, (2) and (3) can be written as

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))[\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)] \tag{4}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))[\boldsymbol{C}_i\boldsymbol{x}(t) + \boldsymbol{D}_i\boldsymbol{u}(t)]. \tag{5}$$

The TS fuzzy model can also be regarded as a *quasilinear system*, i.e., a system linear in both $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$ whose matrices $\boldsymbol{A}(\cdot), \dots, \boldsymbol{D}(\cdot)$ are not constant, but varying:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(\boldsymbol{\delta}(t))\boldsymbol{x}(t) + \boldsymbol{B}(\boldsymbol{\delta}(t))\boldsymbol{u}(t) \tag{6}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}(\boldsymbol{\delta}(t))\boldsymbol{x}(t) + \boldsymbol{D}(\boldsymbol{\delta}(t))\boldsymbol{u}(t). \tag{7}$$

From (4) and (5), one can see that for all possible values of $\boldsymbol{\delta}(t)$, which are assumed to be known online, these matrices are bounded within a polytope whose vertices are the matrices of the individual rules:

$$\begin{bmatrix}\boldsymbol{A}(\boldsymbol{\delta}(t)) & \boldsymbol{B}(\boldsymbol{\delta}(t))\\ \boldsymbol{C}(\boldsymbol{\delta}(t)) & \boldsymbol{D}(\boldsymbol{\delta}(t))\end{bmatrix} \in \mathbf{Co}\left\{\begin{bmatrix}\boldsymbol{A}_i & \boldsymbol{B}_i\\ \boldsymbol{C}_i & \boldsymbol{D}_i\end{bmatrix} : i = 1, 2, \dots, r\right\} \tag{8}$$

where

$$\mathbf{Co}\,\{\boldsymbol{S}_i : i = 1, 2, \dots, r\} = \left\{\sum_{i=1}^{r} h_i(t)\boldsymbol{S}_i : \sum_{i=1}^{r} h_i(t) = 1, h_i(t) \geq 0\right\}$$

and

$$\boldsymbol{S}_i = \begin{bmatrix}\boldsymbol{A}_i & \boldsymbol{B}_i\\ \boldsymbol{C}_i & \boldsymbol{D}_i\end{bmatrix}.$$

For the sake of simplicity, the direct transmission matrices $D_i$ are considered to be zero here. This can be assumed without any restrictions to real systems because they have dynamic parts between their inputs and outputs. Note, the presented design method is particularly intended for control of nonlinear systems. Hence, the scheduling variables are usually a function of the state; i.e., $\boldsymbol{\delta}(t) = \boldsymbol{\delta}(\boldsymbol{x}(t))$ and (6) yields $\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}$, etc.

*Definition II.1* **Wide Working Range** *(WWR):* The WWR be defined around one equilibrium point by the antecedent part of the fuzzy rule base. It corresponds to all admissible values of the scheduling vector $\boldsymbol{\delta}(t)$ and it can be regarded as an extension of the term *single operating point* known from the theory of linear systems (represented in the TS fuzzy model structure by a single fuzzy-rule). For the analysis and synthesis of TS fuzzy systems, the relevant properties are considered throughout the corresponding WWR.

### B. Fuzzy Controller

The controller design by means of the described method begins with the determination of the linear submodels in some operating regions of interest of the nonlinear system to be controlled; for more details see, e.g., [10]. Then, convex optimization techniques are used to design local controllers within a fuzzy gain-scheduling scheme with the desired overall behavior. In this way, a wide-range stabilization and control problems can be solved. In the continuous-time case, the simplest TS fuzzy control rule being considered here, has the form:

$$\boldsymbol{R}_i\text{: }\mathbf{IF}\ \delta_1(t)\text{ is }M_{1i}\ \mathbf{and}\ \ldots\ \mathbf{and}\ \delta_j(t)\text{ is }M_{ji},\quad \mathbf{then}\ \boldsymbol{u}_i(t) = -\boldsymbol{F}_i\boldsymbol{x}(t) + \boldsymbol{V}_i\boldsymbol{r}(t) \tag{9}$$

where $\boldsymbol{r}(t)$ is a stepwise reference signal. The controller's output is inferred as the weighted mean

$$\boldsymbol{u}(t) = \frac{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))[-\boldsymbol{F}_i\boldsymbol{x}(t) + \boldsymbol{V}_i\boldsymbol{r}(t)]}{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))} \tag{10}$$

which yields

$$\begin{aligned}\boldsymbol{u}(t) &= \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))[-\boldsymbol{F}_i\boldsymbol{x}(t) + \boldsymbol{V}_i\boldsymbol{r}(t)]\\ &= -\boldsymbol{F}(\boldsymbol{\delta}(t))\boldsymbol{x}(t) + \boldsymbol{V}(\boldsymbol{\delta}(t))\boldsymbol{r}(t).\end{aligned} \tag{11}$$

If the scheduling vector $\boldsymbol{\delta}(t)$ is a function of the state vector $\boldsymbol{x}(t)$, $\boldsymbol{u}(t)$ represents a nonlinear gain-scheduled control law.

The goal of the controller design is to determine the constant matrices $\boldsymbol{F}_i$ and $\boldsymbol{V}_i$ such that the desired dynamics of the closed-loop system and some desired steady-state input–output behavior are obtained. Designing the state-feedback gains $\boldsymbol{F}_i$ requires dealing with the system dynamics and hence ensuring stability. This problem is solved by means of LMIs. For the TS fuzzy controller (9), the best values for the static feed-forward gains $\boldsymbol{V}_i$ are given by

$$\boldsymbol{V}_i = (\boldsymbol{C}_i(-\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}_i)^{-1}\boldsymbol{B}_i)^{-1}. \tag{12}$$

These ensure for each closed-loop subsystem a unit steady-state gain. However, a reasonable requirement for the controller (9) based on (12) is rather to satisfy $\boldsymbol{x}(t) \to \mathbf{0}$ when $t \to \infty$. This implies the *stabilization problem* of the control system where $\boldsymbol{r}(t) = 0$. In other words, this TS fuzzy controller cannot usually be used satisfactorily in *tracking control problems* where a given reference trajectory $\boldsymbol{r}(t) \neq 0$ is to be followed. The reasons for this are the ever-present mismatch between the fuzzy model and the real plant and also the dynamic of the reference signal resulting in steady-state errors.

An extended fuzzy scheduler (EFS), which tackles the problem for tracking of stepwise constant reference signals via an additional feedback with an integral action, is introduced in Section IV-A. In the EFS control scheme, the $\boldsymbol{V}_i$'s become integration constants that must be automatically calculated within the LMI-based design as a part of the entire closed-loop controller.

For other types of tracking signals (e.g., ramps), the dynamic prefilter used as an extension of the open-loop system is to be changed correspondingly (e.g., double-integrator) to filter out the deviation between the actual output and the desired reference signal. This only leads to a corresponding change of the extended system matrix having for the integrative prefilter the form (35). Using this approach based on substitutions like (27)–(29), one can work with the same sets of LMIs to deal with both system stabilization and tracking.

Recall that the reference signals are not constrained to be slowly varying. However, the control system is assumed to exhibit its behavior only in the considered WWR according to Definition II.1.

## III. Stability: LMI Techniques for Analysis and Synthesis

### A. Closed-Loop System

The closed-loop system consisting of the fuzzy model and the fuzzy controller is obtained by substituting the controller (11) to the state equation of the fuzzy model (4). The closed-loop system is given by

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t)) \times [[\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{F}_j]\,\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{V}_j\boldsymbol{r}(t)]\,. \tag{13}$$

It is assumed throughout this paper that the weight of each rule in the fuzzy controller is equal to that of the corresponding rule in the fuzzy model—we call this the *shared rules* principle. This assumption is easy to satisfy since all weighting factors of the controller can be simply taken over from the known fuzzy model. Then, (13) can be rewritten as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) = &\sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))h_i(\boldsymbol{\delta}(t))\boldsymbol{G}_{ii}\boldsymbol{x}(t)\\ &+ 2\sum_{i}^{r}\sum_{j>i}^{r} h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t))\frac{\boldsymbol{G}_{ij}+\boldsymbol{G}_{ji}}{2}\boldsymbol{x}(t)\\ &+ \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t))\boldsymbol{B}_i\boldsymbol{V}_j\boldsymbol{r}(t)\end{aligned} \tag{14}$$

with

$$\boldsymbol{G}_{ij} = \boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{F}_j. \tag{15}$$

For the particular case of common matrices $\boldsymbol{B}_i$, i.e., $\boldsymbol{B}_i = \mathbf{B}$ for all submodels $i = 1, 2, \ldots, r$, and for the shared rules, the following simplified description of the entire closed-loop system can be derived:

$$\dot{x} = \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))\,[(\boldsymbol{A}_i - \boldsymbol{B}\boldsymbol{F}_i)x(t) + \boldsymbol{B}\boldsymbol{V}_i\boldsymbol{r}(t)]. \tag{16}$$

The terms known from standard PDC controllers [11] are given by (15). They are responsible for the stability of the control system—matrices $\boldsymbol{F}_j$ are calculated via LMIs such that an appropriate quadratic Lyapunov function can be found. The remaining terms given by the products $\boldsymbol{B}_i\boldsymbol{V}_j$ do not affect the dynamics; they are in the feed-forward channel. They represent the steady-state gain of the control loop with $\boldsymbol{V}_j$ simple calculated as shown in (12) so that the unity steady-state gain is ensured for the dynamic fuzzy system (3) to follow the reference signal $\boldsymbol{r}(t)$ as closely as possible.

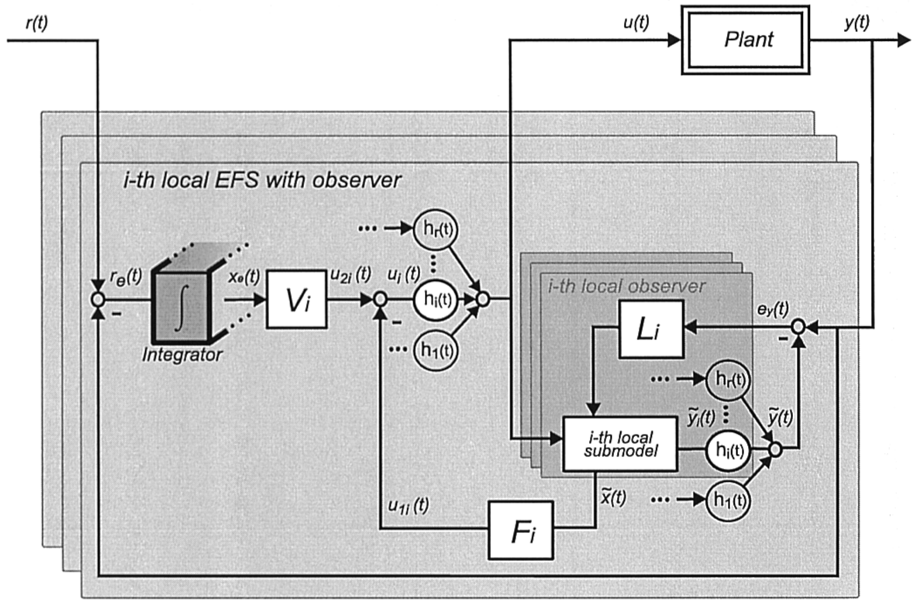



Fig. 1. Closed-loop system with a dynamic output feed-back fuzzy controller and an additional integral action.

### B. Stability Conditions for Closed-Loop TS Fuzzy System

The considered systems are characterized by matrices $\boldsymbol{A}(\cdot), \ldots, \boldsymbol{D}(\cdot)$, $\boldsymbol{F}(\cdot)$ and $\boldsymbol{V}(\cdot)$ bounded in polytopes like (8). There is no need to distinguish between external and internal nature of the time-varying parameters $\boldsymbol{\delta}(\cdot)$ (internal stands for $\boldsymbol{\delta}(\boldsymbol{x})$ where $\boldsymbol{\delta}$ depends on some state variables $\boldsymbol{x}$ making the problem nonlinear) used for scheduling if the control system exhibits its trajectories within the considered polytopic differential inclusion (PDI) which corresponds to the WWR, for details see e.g [12]. The main idea is based on the fact that every trajectory of the considered system is also a trajectory of the PDI for which some properties can be guaranteed if a common Lyapunov function is found. Then, every trajectory of our (possibly nonlinear) system has these properties as well.

Furthermore, starting from the formalism of Lyapunov function stability, it is quite straightforward to prove that using the presented design method for the considered class of control systems, BIBO stability is also guaranteed; see the Appendix.

*Theorem III.1:* The equilibrium of the continuous-time closed-loop fuzzy control system described by (14) is asymptotically stable within its corresponding WWR, if there exists a common positive definite matrix $\boldsymbol{P}$ such that

$$\mathcal{L}_P(\boldsymbol{G}_{ij}) < 0, \qquad i = j \tag{17}$$

$$\mathcal{L}_P(\boldsymbol{G}_{ij}) \leq 0, \qquad i < j \tag{18}$$

for all $i, j = 1, 2, \ldots, r$ except for the pairs $(i, j)$ that imply $\forall\, t > 0 : h_i(\boldsymbol{\delta}(t)) h_j(\boldsymbol{\delta}(t)) = 0$. The linear operator $\mathcal{L}_P(\boldsymbol{G}_{ij})$ is defined for $\boldsymbol{G}_{ij} = \boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{F}_j$ by

$$\mathcal{L}_P(\boldsymbol{G}_{ij}) = \left(\frac{\boldsymbol{G}_{ij} + \boldsymbol{G}_{ji}}{2}\right)^T \boldsymbol{P} + \boldsymbol{P}\left(\frac{\boldsymbol{G}_{ij} + \boldsymbol{G}_{ji}}{2}\right). \tag{19}$$

*Proof:* See [11].

The problem of checking the stability of a fuzzy closed-loop system is to find a matrix $\boldsymbol{P}$ that satisfies (17) and (18). Based on this stability test, more involved algorithms for constructive design of different fuzzy controllers, are presented.

Considering common $\boldsymbol{B}$, i.e., $\boldsymbol{B}_i = \boldsymbol{B}$, for all $i = 1, 2, \ldots, r$, the stability conditions of Theorem III.1 can be simplified as follows.

*Remark III.1:* If $\boldsymbol{B}_i = \boldsymbol{B}$ for all $i = 1, 2, \ldots, r$ then the equilibrium of the fuzzy control system (14) is stable if there exists a common positive definite matrix $\boldsymbol{P}$ satisfying (17).

This remark directly follows from Theorem III.1 if (16) instead of (14) is considered. In general, the problem of finding a common matrix $\boldsymbol{P}$ for the problems of common $\boldsymbol{B}$ is simpler than the general case of different $B_i$'s.

It has long been considered difficult to find a positive–definite matrix $\boldsymbol{P}$ that satisfies the conditions of the above given theorem and characterizes that way the common Lyapunov function. A trial-and-error procedure was first used [13]. In [14], a procedure for analytical construction of a common $\boldsymbol{P}$ is given for second-order fuzzy systems (the dimension of the state of local submodels is two). Here, as in [11], the common $\boldsymbol{P}$ problem for fuzzy controller design is solved numerically, i.e., the stability conditions of the theorems are expressed in LMIs [12]. The LMIs can then be solved to find a $\boldsymbol{P}$ or to determine that no such $\boldsymbol{P}$ exists. These is either the *convex feasibility* problem or the *general eigenvalue* problem. Numerically, these problems can be solved in polynomial time by means of powerful tools that became available lately. This work can be seen as an extension of [9] and [11]. Introducing additional constraints on the locations of the eigenvalues of the underlying subsystems, some other useful performance criteria can be satisfied, e.g., the suppression of overshoots. Furthermore, in many applications, not only stabilization, but also tracking problems must be considered. Hence, an extended controller is proposed with an integrator in the feed-forward channel for tracking of stepwise reference signals.

### C. LMI Techniques for Synthesis

Theorem III.1 can be used only as a stability test. For the design of stabilizing fuzzy controllers, it must be slightly modified to be linear in all optimization variables to be calculated by an LMI-solver [11].

*Theorem III.2:* The equilibrium of the continuous-time closed-loop fuzzy control system described by (14) is asymptotically stabilizable within its corresponding WWR, if there are a common positive definite matrix $\boldsymbol{Q}$ and a set of matrices $\boldsymbol{K}_j$ for $j = 1, 2, \ldots, r$ such that

$$\mathcal{L}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i) < 0, \qquad i = j \tag{20}$$

$$\mathcal{L}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i) \leq 0, \qquad i < j \tag{21}$$

for all $i, j = 1, 2, \ldots, r$ except for the pairs $(i, j)$ that imply $\forall\, t > 0 : h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t)) = 0$. The linear operator $\mathcal{L}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i)$ is defined for any matrix variables $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{K}_j \in \mathbb{R}^{m\times n}$ as shown in (22) at the bottom of the page. The desired fuzzy state-feedback gain matrices $\boldsymbol{F}_j$ are then given by $\boldsymbol{F}_j = \boldsymbol{K}_j\boldsymbol{Q}^{-1}$, $j = 1, 2, \ldots, r$. The common matrix $\boldsymbol{P}$ can be obtained as $\boldsymbol{P} = \boldsymbol{Q}^{-1}$.

*Proof:* It follows from $\mathcal{L}_P(\boldsymbol{G}_{ij})$ for $\boldsymbol{G}_{ij} = \boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{F}_j$, $\boldsymbol{F}_j = \boldsymbol{K}_j\boldsymbol{Q}^{-1}, \boldsymbol{Q} = \boldsymbol{P}^{-1}$.

Unlike (17) and (18), (20) and (21) are LMIs with respect to variables $\boldsymbol{Q}, \boldsymbol{K}_j$. It is easy to find $\boldsymbol{Q} > 0$ and the corresponding $\boldsymbol{K}_j$ or to determine that no such $\boldsymbol{Q}, \boldsymbol{K}_j$ exist. LMI-based techniques can be used for systematic analysis and also for the design of TS fuzzy control systems. Section IV presents some extensions of the above basic algorithms.

## IV. Enhancements

### A. Extended Fuzzy Scheduler

Based on the fuzzy scheduler described in Section II-B, which is for $\boldsymbol{r} = 0$ also known as PDC [11], [13], an EFS can be derived. The purpose of introducing the presented controller is to ensure zero steady-state tracking error for stepwise reference signals; also in the presence of disturbances or model uncertainties. Its principle is based on the well-known procedure of introducing an integral action in the forward channel. This synthesis problem has been recast here as an LMI problem for the fuzzy gain-scheduling design. A new state variable $x_e$ is introduced to integrate the tracking error, see Fig. 1 as well. It is defined as:

$$\dot{\boldsymbol{x}}_e(t) = \boldsymbol{r}(t) - \boldsymbol{y}(t) = \boldsymbol{r}(t) - \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))\boldsymbol{C}_i\boldsymbol{x}(t) \tag{23}$$

where $y(t)$ is given by (3). Then, the entire extended fuzzy system can be described as:

$$\dot{\boldsymbol{x}}_w(t) = \boldsymbol{A}_w(\cdot)\boldsymbol{x}_w(t) + \boldsymbol{B}_w(\cdot)\boldsymbol{u}(t) + \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}\boldsymbol{r}(t) \tag{24}$$

$$\boldsymbol{y}_w(t) = \boldsymbol{C}_w(\cdot)\boldsymbol{x}_w(t) \tag{25}$$

where

$$\boldsymbol{x}_w(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_e(t) \end{bmatrix} \tag{26}$$

$$\boldsymbol{A}_w(\cdot) = \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t)) \begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{0} \\ -\boldsymbol{C}_i & \boldsymbol{0} \end{bmatrix} = \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))\boldsymbol{A}_w^i \tag{27}$$

$$\boldsymbol{B}_w(\cdot) = \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t)) \begin{bmatrix} \boldsymbol{B}_i \\ \boldsymbol{0} \end{bmatrix} = \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))\boldsymbol{B}_w^i \tag{28}$$

$$\boldsymbol{C}_w(\cdot) = \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t)) \left[\boldsymbol{C}_i \quad \boldsymbol{0}\right] = \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))\boldsymbol{C}_w^i \tag{29}$$

and $\boldsymbol{u}(t)$ is the control signal generated by the EFS controller. The EFS controller consists of two parts: $\boldsymbol{u}_1$ is based on the state variables of the controlled system and $\boldsymbol{u}_2$ is based on the additional state $x_e(t)$:

$$\begin{aligned} \boldsymbol{u}_1(t) &= \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))\boldsymbol{F}_i\boldsymbol{x}(t) \\ \boldsymbol{u}_2(t) &= \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))\boldsymbol{V}_i x_e(t). \end{aligned} \tag{30}$$

The entire gain-scheduled control law is then given by

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}_2(x_e) - \boldsymbol{u}_1(\boldsymbol{x}) \\ &= \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t)) \left[-\boldsymbol{F}_i \ \boldsymbol{V}_i\right] \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_e(t) \end{bmatrix} \\ &= \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))\boldsymbol{F}_w^i\boldsymbol{x}_w(t) = \boldsymbol{F}_w(\cdot)\boldsymbol{x}_w(t) \end{aligned} \tag{31}$$

$$\text{with } \boldsymbol{F}_w(\cdot) = \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t)) \left[-\boldsymbol{F}_i \ \boldsymbol{V}_i\right] \tag{32}$$

as the extended-state feedback-gain matrix.

Substituting (31) into (24), the closed-loop behavior of the extended fuzzy system is then given by

$$\dot{\boldsymbol{x}}_w(t) = \tilde{\boldsymbol{A}}_w(\cdot)\boldsymbol{x}_w(t) + \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}\boldsymbol{r}(t) \tag{33}$$

with the time-varying system matrix $\tilde{\boldsymbol{A}}_w(\cdot)$ satisfying

$$\tilde{\boldsymbol{A}}_w(t) \in \mathcal{P} := \mathbf{Co}\left\{\tilde{\boldsymbol{A}}_w^{ij},\ i = 1, 2, \ldots, r\right\} \qquad \forall\, t \tag{34}$$

where the matrices $\tilde{\boldsymbol{A}}_w^{ij}$ are formed as follows:

$$\begin{aligned} \tilde{\boldsymbol{A}}_w^{ij} &= \boldsymbol{A}_w^i + \boldsymbol{B}_w^i\boldsymbol{F}_w^j \\ &= \begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{0} \\ -\boldsymbol{C}_i & \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_i \\ \boldsymbol{0} \end{bmatrix} \left[-\boldsymbol{F}_j \ \boldsymbol{V}_j\right] \\ &= \begin{bmatrix} \boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{F}_j & \boldsymbol{B}_i\boldsymbol{V}_j \\ -\boldsymbol{C}_i & \boldsymbol{0} \end{bmatrix}. \end{aligned} \tag{35}$$

$$\mathcal{L}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i) = \begin{cases} \boldsymbol{A}_i\boldsymbol{Q} - \boldsymbol{B}_i\boldsymbol{K}_i + \boldsymbol{Q}\boldsymbol{A}_i^T - \boldsymbol{K}_i^T\boldsymbol{B}_i^T, & i = j \\ \boldsymbol{A}_i\boldsymbol{Q} - \boldsymbol{B}_i\boldsymbol{K}_j + \boldsymbol{A}_j\boldsymbol{Q} - \boldsymbol{B}_j\boldsymbol{K}_i + \boldsymbol{Q}\boldsymbol{A}_i^T - \boldsymbol{K}_j^T\boldsymbol{B}_i^T + \boldsymbol{Q}\boldsymbol{A}_j^T - \boldsymbol{K}_i^T\boldsymbol{B}_j^T, & i < j. \end{cases} \tag{22}$$

Equation (35) gives the transformations $(\boldsymbol{A}_w^i, \boldsymbol{B}_w^i, \boldsymbol{F}_w^j)$ for the synthesis problem of the controller with the integral action in the forward channel (denoted here as EFS) into the standard problem ( $\boldsymbol{A}_i, \boldsymbol{B}_i, \boldsymbol{F}_j$). Therefore, (33) can replace the corresponding (14) in all theorems to design an EFS instead of PDC.

When solving real-world control problems, where model-plant mismatches inevitably appear, the EFS scheme (33) has proven to be superior to the PDC scheme, which is equipped just with a simple feed-forward channel (14).

## B. Fuzzy State Estimator

In Sections II–III, different *state-feedback controllers* have been employed in the rule consequents. Hence, all states of the plant have been implicitly assumed to be online available. However, in real processes, this is not always the case. To overcome this problem, a fuzzy observer can be used both for FS and EFS. Based on the plant's inputs and outputs, the observer estimates the states. The augmented fuzzy system that contains the observer and the controller is regarded as a *dynamic output-feedback fuzzy controller*. The demand for fuzzy observers is thus well motivated. Observers are known to satisfy the requirement that $\boldsymbol{e}(t) \to 0$ when $t \to \infty$, where $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ means the deviation between the plant's state vector $\boldsymbol{x}(t)$ and the state vector $\tilde{\boldsymbol{x}}(t)$ estimated by the observer. This requirement can be satisfied by a fuzzy observer based on the same model of the plant as the controller with an additional time-varying state-injection matrix $\boldsymbol{L}(\cdot)$

$$\begin{aligned} R_i\text{:}\quad & \textbf{IF } \delta_1(t) \text{ is } M_{1i} \textbf{ and } \ldots \textbf{ and } \delta_j(t) \text{ is } M_{ji}, \textbf{ then} \\ & \dot{\tilde{\boldsymbol{x}}}_i(t) = \boldsymbol{A}_i\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) - \boldsymbol{L}_i[\boldsymbol{y}(t) - \tilde{\boldsymbol{y}}(t)] \\ & \tilde{\boldsymbol{y}}_i(t) = \boldsymbol{C}_i\tilde{\boldsymbol{x}}(t) \end{aligned} \tag{36}$$

where $i = 1, 2, \ldots, r$. For further considerations, the aforementioned fuzzy system can be expressed as

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}}(t) &= \frac{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))[\boldsymbol{A}_i\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) - \boldsymbol{L}_i(\boldsymbol{y}(t) - \tilde{\boldsymbol{y}}(t))]}{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))} \\ &= \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))[\boldsymbol{A}_i\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) - \boldsymbol{L}_i(\boldsymbol{y}(t) - \tilde{\boldsymbol{y}}(t))] \end{aligned} \tag{37}$$

$$\begin{aligned} \tilde{\boldsymbol{y}}(t) &= \frac{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))\boldsymbol{C}_i\tilde{\boldsymbol{x}}(t)}{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))} \\ &= \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))\boldsymbol{C}_i\tilde{\boldsymbol{x}}(t). \end{aligned} \tag{38}$$

The weights $w_i$ generally depend either on the measured scheduling vector $\boldsymbol{\delta}$ only, or on the scheduling vector $\tilde{\boldsymbol{\delta}}$ estimated by the observer itself or on some of its components. However, the weights of the contributing local observers are assumed to be the same as the weights used for the fuzzy model (the shared-rules principle). Note that the analysis of the augmented fuzzy system is straightforward only if the real states and the estimated ones can be assumed to reside in the same fuzzy region. If they reside in different regions, the problem is much more difficult—the discrepancy becomes unstructured. The separation principle holds only if the scheduling variables do not depend on the estimated state [15]. The above fact is a difficult problem and there is no clear solution yet. For the sake of simplicity, it is assumed that $\tilde{\boldsymbol{\delta}}(t) = \boldsymbol{\delta}(t)$ for $\forall\, t$. In other words, the state-estimation is required to converge fast enough such that $\boldsymbol{x}$ can be replaced by $\tilde{\boldsymbol{x}}$ in the control loop. This fast convergence can be achieved by a suitable choice of the state-injection matrix $\boldsymbol{L}(\cdot)$, which is responsible, similarly to the controller design, not only for a convergence, but rather for a convergence with some minimal decay rate. This decay rate should be slightly faster than the desired performance of the control loop.

Bearing in mind the previous assumptions, the stability analysis of the augmented fuzzy system containing the fuzzy observer (36) and an *estimated-state based extended fuzzy scheduler* (39) (see Fig. 1) becomes straightforward

$$\begin{aligned} \boldsymbol{u}(t) &= \boldsymbol{u}_2(\boldsymbol{x}_e) - \boldsymbol{u}_1(\tilde{\boldsymbol{x}}) \\ &= \frac{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))\,[-\boldsymbol{F}_i\boldsymbol{V}_i]}{\sum_{i=1}^{r} w_i(\boldsymbol{\delta}(t))} \begin{bmatrix} \tilde{\boldsymbol{x}}(t) \\ \boldsymbol{x}_e(t) \end{bmatrix} \\ &= \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))\,[-\boldsymbol{F}_i\boldsymbol{V}_i] \begin{bmatrix} \tilde{\boldsymbol{x}}(t) \\ \boldsymbol{x}_e(t) \end{bmatrix}. \end{aligned} \tag{39}$$

Substituting (39) into (36) and using the notations (26) and (35), the following equations describing the augmented system are obtained:

$$\begin{aligned} \dot{\boldsymbol{x}}_w(t) = &\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t)) \\ &\times \left[\left(\boldsymbol{A}_w^i + \boldsymbol{B}_w^i\boldsymbol{F}_w^j\right)\boldsymbol{x}_w(t) + \boldsymbol{B}_i\boldsymbol{F}_j\boldsymbol{e}(t)\right] \\ &+ \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}\boldsymbol{r}(t) \end{aligned} \tag{40}$$

$$\dot{\boldsymbol{e}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t))\,[(\boldsymbol{A}_i + \boldsymbol{L}_i\boldsymbol{C}_j]\,\boldsymbol{e}(t). \tag{41}$$

By combining these equations into one, we get

$$\begin{aligned} \dot{\boldsymbol{x}}_a(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t))\boldsymbol{G}_{ij}\boldsymbol{x}_a(t) + \begin{bmatrix} \boldsymbol{0} \\ 1 \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{r}(t) \\ &= \sum_{i=1}^{r} h_i(\boldsymbol{\delta}(t))h_i(\boldsymbol{\delta}(t))\boldsymbol{G}_{ii}\boldsymbol{x}_a(t) \\ &\quad + 2\sum_{i}^{r}\sum_{j>i}^{r} h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t))\frac{\boldsymbol{G}_{ij} + \boldsymbol{G}_{ji}}{2}\boldsymbol{x}_a(t) \\ &\quad + \begin{bmatrix} \boldsymbol{0} \\ 1 \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{r}(t) \end{aligned} \tag{42}$$

$$\text{with } \boldsymbol{x}_a(t) = \begin{bmatrix} \boldsymbol{x}_w(t) \\ \boldsymbol{e}(t) \end{bmatrix} \text{ and} \tag{43}$$

$$\boldsymbol{G}_{ij} = \left[\begin{array}{c|c} \boldsymbol{A}_w^i + \boldsymbol{B}_w^i\boldsymbol{F}_w^j & \begin{matrix} \boldsymbol{B}_i\boldsymbol{F}_j \\ \boldsymbol{0} \end{matrix} \\ \hline \boldsymbol{0} & \boldsymbol{A}_i + \boldsymbol{L}_i\boldsymbol{C}_j \end{array}\right]. \tag{44}$$

Note the form of the $\boldsymbol{G}_{ij}$ matrix in (44) showing that under the considered assumptions the separation property holds. In other words, the controller given by $\boldsymbol{F}_w^i = [-\boldsymbol{F}_i\, \boldsymbol{V}_i]$ and the observer given by $\boldsymbol{L}_i$ can be designed separately.

The stability theorems for the augmented system and for the convergence of the observer can be derived by means of the Lyapunov direct method and a quadratic function that can be solved by an LMI tool in a way similar to the FS and EFS fuzzy controllers.

*Theorem IV.1:* The equilibrium of the continuous-time augmented fuzzy system described by (42) is asymptotically stable if there exists a common positive–definite matrix $\boldsymbol{P}$ such that

$$\mathcal{L}_P(\boldsymbol{G}_{ij}) < 0, \qquad i = j \tag{45}$$

$$\mathcal{L}_P(\boldsymbol{G}_{ij}) \leq 0, \qquad i < j \tag{46}$$

for all $i, j = 1, 2, \ldots, r$ except for the pairs $(i, j)$ that imply $\forall\, t : h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t)) = 0$. The linear operator $\mathcal{L}_P$ is defined for any matrix $\boldsymbol{P}$ by

$$\mathcal{L}_P(\boldsymbol{G}_{ij}) = \left(\frac{\boldsymbol{G}_{ij} + \boldsymbol{G}_{ji}}{2}\right)\boldsymbol{P} + \boldsymbol{P}\left(\frac{\boldsymbol{G}_{ij} + \boldsymbol{G}_{ji}}{2}\right)^T. \tag{47}$$

*Proof:* Follows directly from Theorem III.1.

*Remark IV.1:* If $\boldsymbol{B}_i = \boldsymbol{B}$ and $\boldsymbol{C}_i = \boldsymbol{C}$, for all $i = 1, 2, \ldots, r$, then the equilibrium of the augmented continuous-time fuzzy system (42) is asymptotically stable if there exists a common positive–definite matrix $\boldsymbol{P}$ satisfying $\mathcal{L}_P(\boldsymbol{G}_{ii}) < 0$.

The previous remark follows directly from Theorem IV.1. In general, the problem of finding a common matrix $\boldsymbol{P}$ for the problems of common $\boldsymbol{B}$ and $\boldsymbol{C}$ is less conservative than for the general case of different $\boldsymbol{B}_i$'s and $\boldsymbol{C}_i$'s.

The stability assessment problem of the augmented fuzzy systems is to find a matrix $\boldsymbol{P}$ that satisfies (45) and (46). Keeping in mind the assumptions, the problem of the fuzzy observer can be solved separately from the controller design problem using instead of (44) just $\boldsymbol{G}_{ij} = \boldsymbol{A}_i + \boldsymbol{L}_i\boldsymbol{C}_j$. Then, Theorem IV.1 expresses the conditions for the asymptotic convergence of the time-varying model-based fuzzy observer characterized by its gain

$$\boldsymbol{L}(\cdot) \in \mathcal{P} := \mathbf{Co}\,\{\boldsymbol{L}_1, \ldots, \boldsymbol{L}_r\}\,. \tag{48}$$

Note that when using the transformations

$$\left.\begin{array}{l}\boldsymbol{A}_i^{obs} := \boldsymbol{A}_i^T \\ \boldsymbol{B}_i^{obs} := \boldsymbol{C}_i^T\end{array}\right\} \Longrightarrow \boldsymbol{L}_i := -\boldsymbol{F}_i^T \tag{49}$$

the search for the common matrix $\boldsymbol{P}$ satisfying the stability conditions of Theorem III.1 for the fuzzy model and the corresponding fuzzy controller given by their parameter triplets $\{(\boldsymbol{A}_i, \boldsymbol{B}_i) \rightarrow \boldsymbol{F}_i\}$ becomes equivalent to the search for a common matrix $\boldsymbol{P}$ in the case of a fuzzy observer based on the fuzzy model characterized by parameter triplets $\left\{(\boldsymbol{A}_i^{obs}, \boldsymbol{B}_i^{obs}) \rightarrow \boldsymbol{L}_i\right\}$. Similarly, Theorems III.2 and IV.2 can be used for the fuzzy observer synthesis.

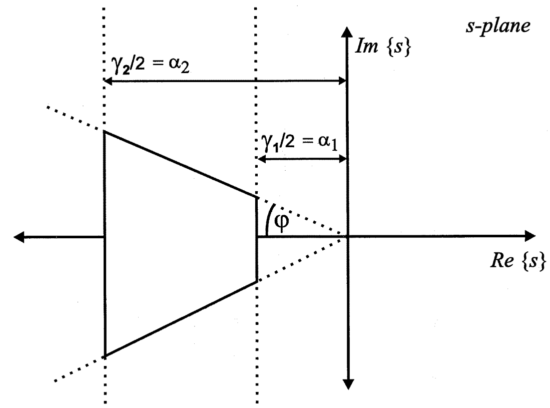



Fig. 2. Proposed regional eigenvalue constraints.

### *C. Performance*

In the synthesis of controllers and observers, in addition to the stability requirements some performance of the closed-loop system is to be considered. The synthesis based on a quadratic Lyapunov function enables representing certain performance specifications, such as decay rates or constraints on the control input, in the form of LMIs. The basic idea can be found in [12]. The performance specifications are introduced via exponential stability of the control system.

Another useful requirement such as suppressing overshoots (damping) can be derived via so-called LMI regions. LMI regions, although based on the definition of eigenvalues defined for LTI systems [16], can also find some practical use for fuzzy systems. Similarities have been found between an LMI region and a performance criterion based on the exponential stability combined with a quadratic Lyapunov function [10]. Such a multiobjective approach has proven to be useful in practice when coping with some implementation constraints and desired performance specifications for the closed-loop dynamics. In this respect, this approach is superior to other known synthesis techniques where the desired control performance is achieved by a trial and error tuning which not only involves a great deal of time, but eventually neither the stability nor the performance of the entire closed-loop fuzzy system are guaranteed.

*Exponential Stability—Decay Rates:*

*Proposition IV.1:* The condition that

$$\dot{V}(\boldsymbol{x}(t)) \leq \gamma V(\boldsymbol{x}(t)) \tag{50}$$

for all trajectories of $\boldsymbol{x}(t)$ of an unforced continuous-time closed-loop TS fuzzy system given by (14) is equivalent to

$$\mathcal{L}_P(\boldsymbol{G}_{ij}, \alpha) < 0, \qquad i = j \tag{51}$$

$$\mathcal{L}_P(\boldsymbol{G}_{ij}, \alpha) \leq 0, \qquad i < j \tag{52}$$

for all $i, j = 1, 2, \ldots, r$ except for the pairs $(i, j)$ that imply $\forall\, t : h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t)) = 0$, where $2\alpha = \gamma < 0$, $\boldsymbol{P} > 0$ and

$\boldsymbol{G}_{ij} = \boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{F}_j$. The linear operator $\mathcal{L}_P$ is defined, similarly to Section III, for any matrix $\boldsymbol{P}$ and any constant $\alpha$ by

$$\mathcal{L}_P(\boldsymbol{G}_{ij}, \alpha) = \left(\frac{\boldsymbol{G}_{ij} + \boldsymbol{G}_{ji}}{2}\right)^T \boldsymbol{P} + \boldsymbol{P}\left(\frac{\boldsymbol{G}_{ij} + \boldsymbol{G}_{ji}}{2}\right) - 2\alpha\boldsymbol{P}. \tag{53}$$

Based on the aforementioned feasibility problem where $\alpha < 0$ is assumed to be a given negative constant, the largest lower bound on the decay rate corresponding to the minimal $\alpha$ from (53) can be found in a constructive manner for a quadratic Lyapunov function by solving the generalized eigenvalue problem (GEVP) in $\boldsymbol{P}$ and $\alpha$. Hence, define a linear operator $\mathcal{L}_{P,\alpha}(G_{ij})$ by means of (53) where $\alpha$ is now assumed to be a scalar variable. Then, the problem of finding the maximal decay rate in the case of exponential stability analysis can be formulated as minimizing $\alpha$ subject to $\boldsymbol{P} > 0$ such that

$$\begin{aligned} \mathcal{L}_{P,\alpha}(\boldsymbol{G}_{ii}) &< 0, \qquad i = j \\ \mathcal{L}_{P,\alpha}(\boldsymbol{G}_{ij}) &\leq 0, \qquad i < j. \end{aligned}$$

*Decay rates in the synthesis of TS fuzzy controllers:* The approach based on the decay rates of the exponential stability and LMI techniques can be used for the synthesis of TS fuzzy controllers with prespecified closed-loop damping. As in the case of simple stability analysis shown in Section III, the conditions that guarantee the desired decay rate $\gamma$ must be based on linear operators with respect to all their variables. Then, the generalized eigenvalue problem can be solved by existing LMI solvers with respect to the minimization of $\gamma$ subject to those LMIs.

*Theorem IV.2:* The equilibrium of the continuous-time fuzzy control systems described by (14) or (33) is asymptotically stabilizable with closed-loop damping $\gamma$, if there exist a common positive–definite matrix $\boldsymbol{Q}$ and a set of matrices $\boldsymbol{K}_j$ for $j = 1, 2, \ldots, r$ such that

$$\mathcal{L}_{Q,K_j,\alpha}(\boldsymbol{A}_i, \boldsymbol{B}_i) < 0, \qquad i = j \tag{54}$$

$$\mathcal{L}_{Q,K_j,\alpha}(\boldsymbol{A}_i, \boldsymbol{B}_i) \leq 0, \qquad i < j \tag{55}$$

for all $i, j = 1, 2, \ldots, r$ except for the pairs $(i, j)$ that imply $\forall\, t : h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t)) = 0$.

The linear operator $\mathcal{L}_{Q,K_j,\alpha}(\boldsymbol{A}_i, \boldsymbol{B}_i)$ is defined for any matrix variables $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, $\boldsymbol{K}_j \in \mathbb{R}^{m\times n}$ and the scalar variable $\alpha$ as shown in (56) at the bottom of the page. The desired fuzzy state-feedback gain matrices $\boldsymbol{F}_j$ are then given by $\boldsymbol{F}_j = \boldsymbol{K}_j\boldsymbol{Q}^{-1}$, $j = 1, 2, \ldots, r$. The common matrix $\boldsymbol{P}$ can be obtained as $\boldsymbol{P} = \boldsymbol{Q}^{-1}$, the decay rate is $\gamma = 2\alpha$.

*Proof:* The proof follows from $\mathcal{L}_{P,\alpha}(\boldsymbol{G}_{ii})$ for $\boldsymbol{G}_{ij} = \boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{F}_j$, $\boldsymbol{F}_j = \boldsymbol{K}_j\boldsymbol{Q}^{-1}$ with $\boldsymbol{Q} = \boldsymbol{P}^{-1}$ based on Proposition IV.1.

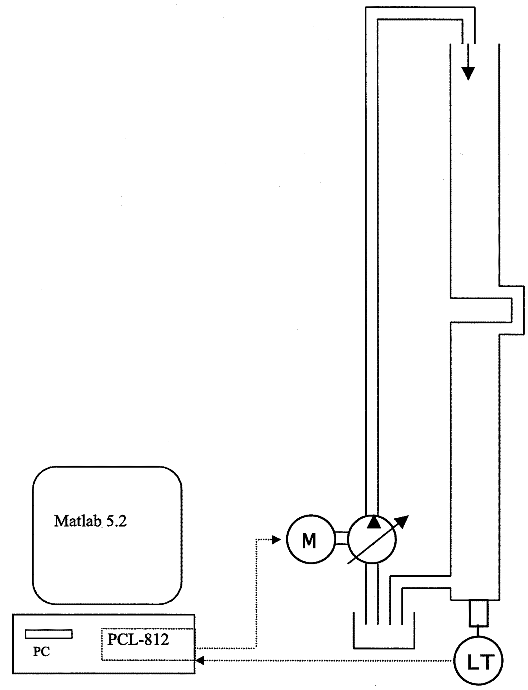



Fig. 3. Two-tank laboratory process.

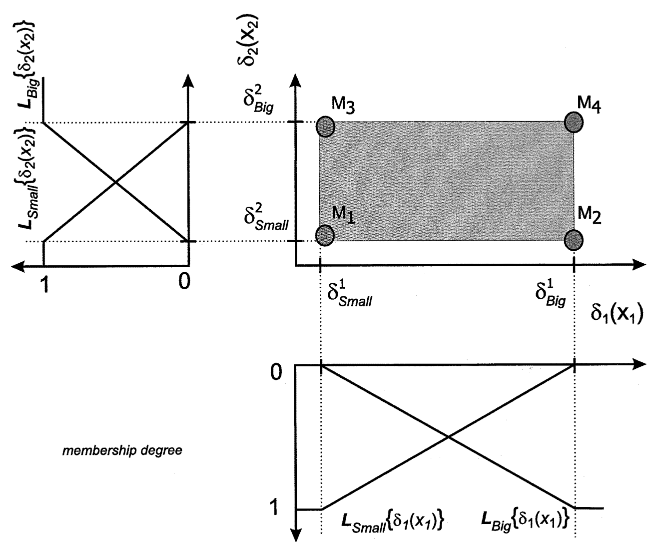



Fig. 4. Four locally valid submodels of the two-tank system.

The LMI-based methods enable us to systematically design linear TS fuzzy control systems with desired performance in terms of closed-loop damping, i.e., with a desired decay rate.

*Regional Eigenvalue Constraints for Synthesis of TS Fuzzy Controllers:* In this section, performance specifications for the control system are introduced via a suitably parameterized

$$\mathcal{L}_{Q,K_j,\alpha}(\boldsymbol{A}_i, \boldsymbol{B}_i) = \begin{cases} \boldsymbol{A}_i\boldsymbol{Q} - \boldsymbol{B}_i\boldsymbol{K}_i + \boldsymbol{Q}\boldsymbol{A}_i^T - \boldsymbol{K}_i^T\boldsymbol{B}_i^T - 2\alpha\boldsymbol{Q}, & i = j \\ \boldsymbol{A}_i\boldsymbol{Q} - \boldsymbol{B}_i\boldsymbol{K}_j + \boldsymbol{A}_j\boldsymbol{Q} - \boldsymbol{B}_j\boldsymbol{K}_i + \boldsymbol{Q}\boldsymbol{A}_i^T - \boldsymbol{K}_j^T\boldsymbol{B}_i^T + \boldsymbol{Q}\boldsymbol{A}_j^T - \boldsymbol{K}_i^T\boldsymbol{B}_j^T - 4\alpha\boldsymbol{Q}, & i < j \end{cases}. \tag{56}$$

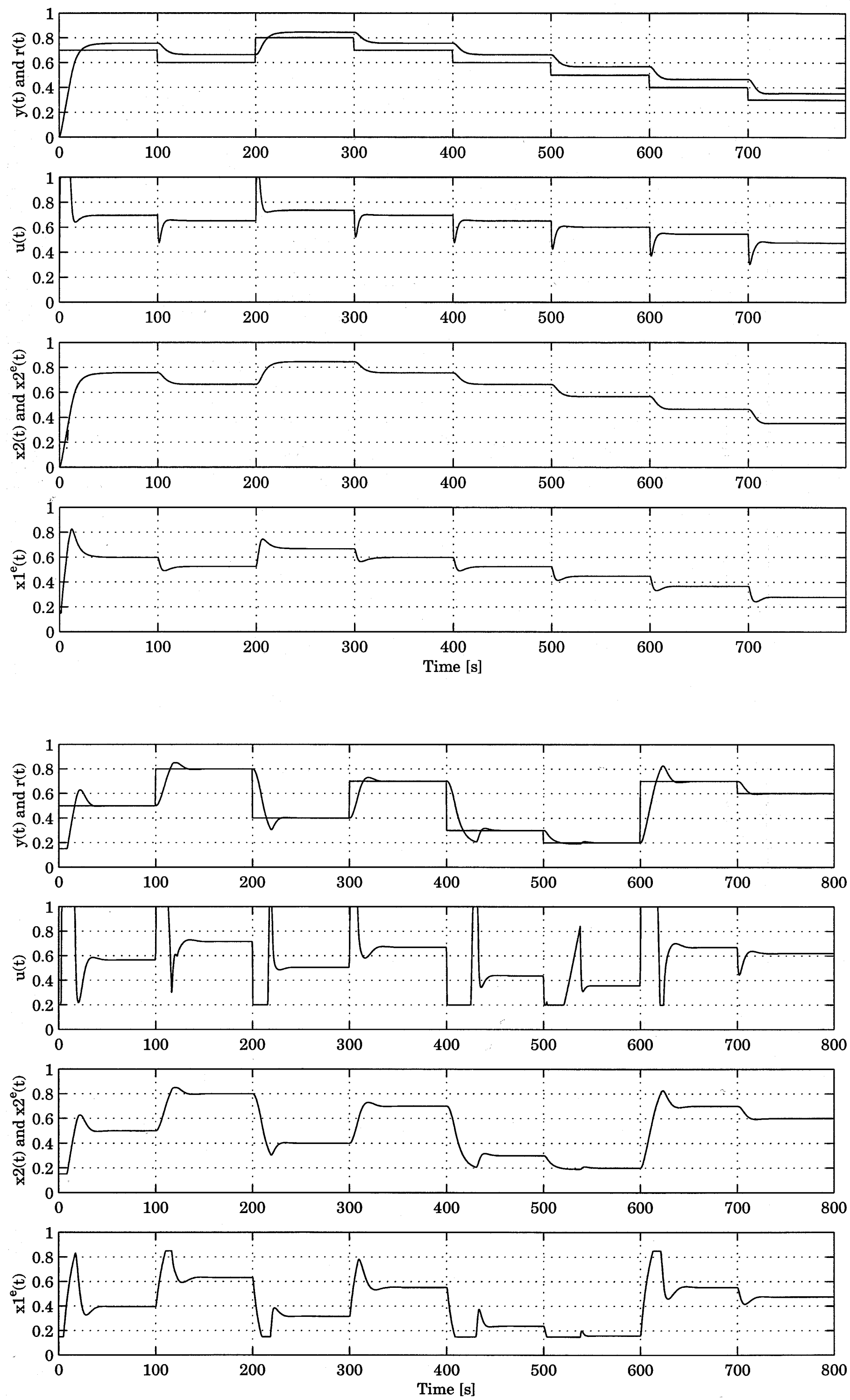



Fig. 5. Simulation examples for LMI-based design parameters $[\alpha_1, \alpha_2, \varphi] = [-0.01, -1.00, 60°]$. (Top) Using simple TS fuzzy scheduler based on PDC results in steady-state errors. (Bottom) Extended fuzzy scheduler; steady-state error disappears, but there is a strong inclination to oscillatory behavior in the closed-loop system.

location of eigenvalues of the underlying locally valid LTI subsystems given by the fuzzy rule-consequents (corresponding to (14), (33) or (42) for frozen $\boldsymbol{\delta}$). A suitable location of eigenvalues has been found in [10], as depicted in Fig. 2. It can be characterized by a small number of parameters $(\alpha_1, \alpha_2, \varphi)$ which represent the tuning knobs for the nonlinear controller (or observer) design. Furthermore, these parameters have a clear physical meaning: lower and upper bounds on the speed of response and the level of suppressing overshoots. These considerations enable some insight and a more intuitive and

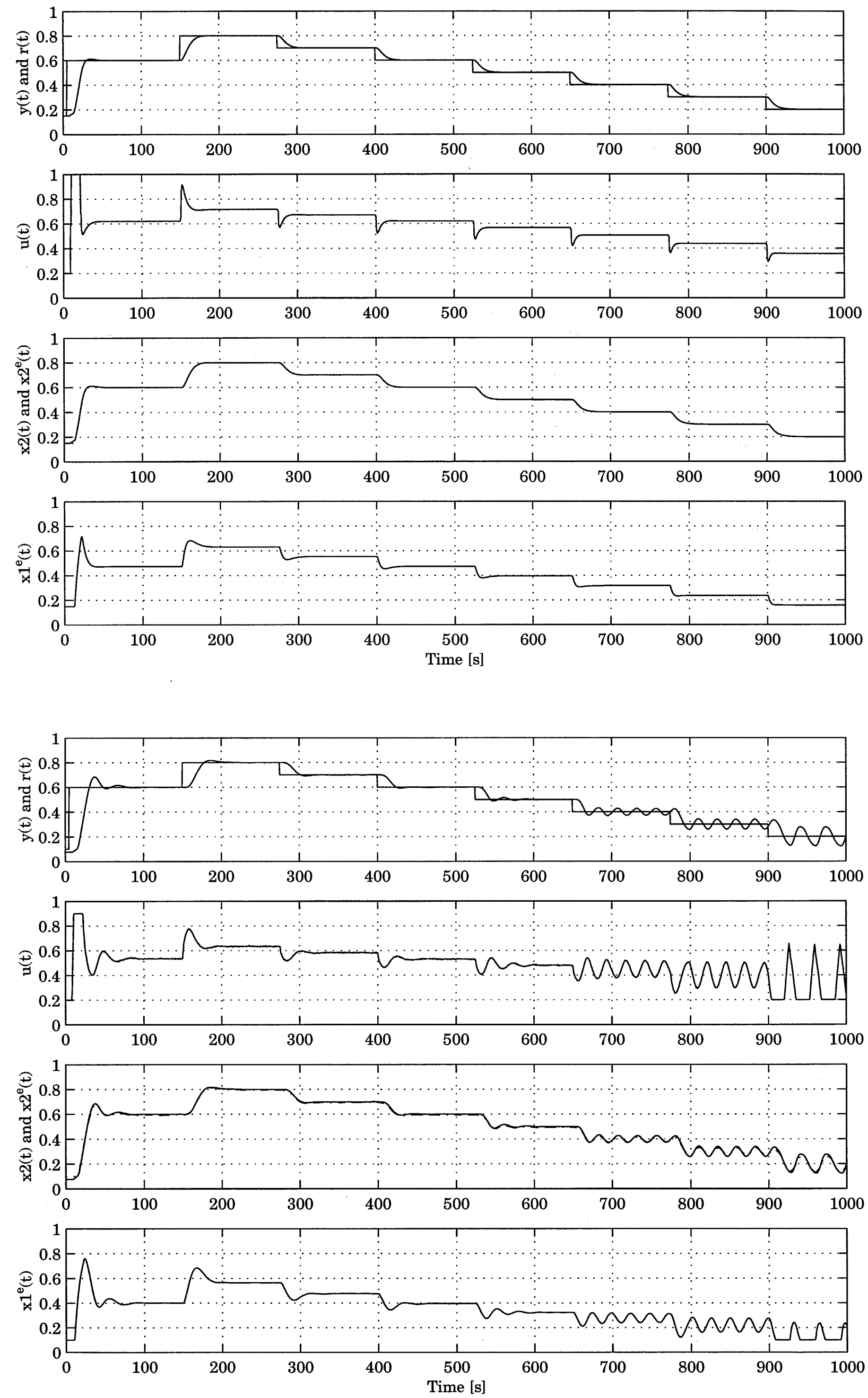



Fig. 6. Simulation example (top) and corresponding real-world experiment (bottom) with the EFS controller $[\alpha_1, \alpha_2, \varphi] = [-0.01, -1.00, 25°]$.

an easy-to-automate design procedure for nonlinear controllers and observers.

Considering the closed-loop system synthesis based on Theorem IV.2, the condition (54) can be replaced by the LMI triplets (57). This enables a design of stable TS fuzzy systems such that all their subsystems have the eigenvalues located within an intersection of a vertical strip given by $(\alpha_1; \alpha_2)$ and a sector characterized by $\varphi$.

For the sector characterization, the plane transformation known from linear algebra is employed

$$\begin{aligned} \begin{bmatrix} \mathcal{L}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i)\sin(\varphi) & \underline{\mathcal{L}}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i)\cos(\varphi) \\ -\underline{\mathcal{L}}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i)\cos(\varphi) & \mathcal{L}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i)\sin(\varphi) \end{bmatrix} &< 0 \\ \mathcal{L}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i, \alpha_1) &< 0 \\ \mathcal{L}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i, \alpha_2) &> 0 \end{aligned} \tag{57}$$

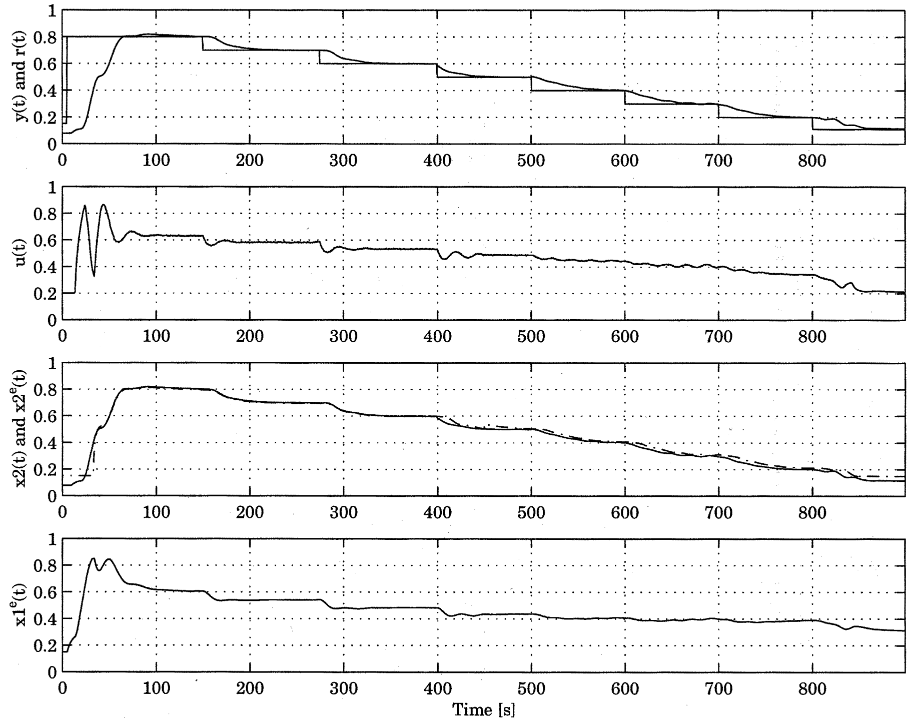



Fig. 7. Real-world experiment when applying the EFS controller with decreased locally dependent performance requirements for each subsystem

$$[\alpha_1, \alpha_2, \varphi] = \begin{bmatrix} -0.10 & -1.00 & 25^\circ \\ -0.01 & -0.70 & 25^\circ \\ -0.01 & -0.70 & 25^\circ \\ -0.01 & -0.60 & 25^\circ \end{bmatrix}.$$

with (58), as shown at the bottom of the next page. The $\mathcal{L}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i, \alpha_i)$ operator is defined in (56). Conditions (57) must hold for $\boldsymbol{P} > 0$, for all $i, j = 1, 2, \ldots, r$ except for pairs $(i, j)$ that imply $\forall\ t > 0 : h_i(\boldsymbol{\delta}(t))h_j(\boldsymbol{\delta}(t)) = 0$. Then, it can be shown that also the requirement for the desired closed-loop decay rate $-2\alpha_2 < -\gamma < -2\alpha_1$ is satisfied and the local closed-loop eigenvalues are located in the above region.

In the theory of LTI systems, the choice of the $\varphi$-sector can clearly be understood in terms of overshoots. Its effect in case of quasi linear parameter varying systems, such as TS fuzzy systems, is rather intuitive and is well demonstrated in the example reported in Section V.

For many practical reasons, the above problem can also be reformulated as a GEVP problem to obtain the fastest decay rates $\gamma_{\max} = 2\alpha_{\max}$ subject to a given upper bound constraint $|\gamma_{\max}| < |\gamma_{up}| = |2\alpha_2|$ and a prescribed sector $\varphi$, if the middle LMI in all triplets (57) $\mathcal{L}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i, \alpha_1) < 0$ is replaced by $\mathcal{L}_{Q,K_j,\gamma_{\max}}(\boldsymbol{A}_i, \boldsymbol{B}_i) < 0$.

A practical hint is that using the presented design technique, a tradeoff between controller performance and its complexity can be tuned just modifying the linear operator $\mathcal{L}_{Q,K_j}(\ldots, [\alpha_1, \alpha_2]) \cdot f(\varphi)$ in the aforementioned theorems as follows.

- Replacing $\boldsymbol{K}_j$ by $\boldsymbol{K}$ means designing a robust linear controller instead of a gain scheduler, if the optimization problem is feasible.
- Replacing $(\alpha_1, \alpha_2, \varphi)$ by different $(\alpha_{1i}, \alpha_{2i}, \varphi_i)$ forces locally dependent performance of the overall gain scheduler (local tuning).
- A tricky modification is the replacement of $\boldsymbol{Q}$ by $\boldsymbol{Q}_j$. Herewith, theoretical guarantees valid globally throughout the entire WWR are given up. However, the method becomes nonconservative, i.e., a feasible solution can always be found. The properties of the obtained gain-scheduler are guaranteed locally. The functionality must be verified in simulations.

## V. An Example

The presented TS fuzzy scheduler and the extended fuzzy scheduler with the corresponding observer have been tested in

$$\underline{\mathcal{L}}_{Q,K_j}(\boldsymbol{A}_i, \boldsymbol{B}_i) = \begin{cases} \boldsymbol{A}_i\boldsymbol{Q} - \boldsymbol{B}_i\boldsymbol{K}_i - \boldsymbol{Q}\boldsymbol{A}_i^T + \boldsymbol{K}_i^T\boldsymbol{B}_i^T, & i = j \\ \boldsymbol{A}_i\boldsymbol{Q} - \boldsymbol{B}_i\boldsymbol{K}_j + \boldsymbol{A}_j\boldsymbol{Q} - \boldsymbol{B}_j\boldsymbol{K}_i - \boldsymbol{Q}\boldsymbol{A}_i^T + \boldsymbol{K}_j^T\boldsymbol{B}_i^T - \boldsymbol{Q}\boldsymbol{A}_j^T + \boldsymbol{K}_i^T\boldsymbol{B}_j^T, & i < j. \end{cases} \tag{58}$$

simulations and real-time experiments with a two-tank laboratory system. The system consists of two cascaded tanks depicted in Fig. 3. Water is supplied into the upper tank through a controlled peristaltic pump. A pressure transmitter attached to the bottom of the lower tank measures the level of the liquid in this tank. The process is connected to a personal computer through a data acquisition board. The sampling time is 1.0 s. The goal is to fill the lower tank to a desired level as fast as possible, however, without any overshoot of the given setpoint.

### A. TS Fuzzy Model

The fuzzy model is defined by means of four linear submodels and it has a common input and output matrix:

$$\begin{aligned}
\boldsymbol{A}_1 &= \begin{bmatrix} -0.1627 & 0 \\ 0.1627 & -0.1446 \end{bmatrix} \\
\boldsymbol{A}_2 &= \begin{bmatrix} -0.3873 & 0 \\ 0.3873 & 0.1446 \end{bmatrix} \\
\boldsymbol{A}_3 &= \begin{bmatrix} -0.1627 & 0 \\ 0.1627 & 0.3443 \end{bmatrix} \\
\boldsymbol{A}_4 &= \begin{bmatrix} -0.3873 & 0 \\ 0.3873 & 0.3443 \end{bmatrix} \\
\boldsymbol{B}_i &= \begin{bmatrix} 0.1667 \\ 0 \end{bmatrix} \\
\boldsymbol{C}_i &= \begin{bmatrix} 0 \; 1 \end{bmatrix} \text{ for } i = 1, \ldots, 4
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\boldsymbol{R}_1\text{: } \mathbf{If}\ \delta_1(t) \text{ is Big } \mathbf{and}\ \delta_2(t) \text{ is Big } \mathbf{then} \\
&\quad \dot{\boldsymbol{x}}_1(t) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{B}_1 u(t) \quad y_1(t) = \boldsymbol{C}_1\boldsymbol{x}(t) \\
&\boldsymbol{R}_2\text{: } \mathbf{If}\ \delta_1(t) \text{ is Small } \mathbf{and}\ \delta_2(t) \text{ is Small } \mathbf{then} \\
&\quad \dot{\boldsymbol{x}}_2(t) = \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{B}_2 u(t) \quad y_2(t) = \boldsymbol{C}_2\boldsymbol{x}(t) \\
&\boldsymbol{R}_3\text{: } \mathbf{If}\ \delta_1(t) \text{ is Big } \mathbf{and}\ \delta_2(t) \text{ is Small } \mathbf{then} \\
&\quad \dot{\boldsymbol{x}}_3(t) = \boldsymbol{A}_3\boldsymbol{x}(t) + \boldsymbol{B}_3 u(t) \quad y_3(t) = \boldsymbol{C}_3\boldsymbol{x}(t) \\
&\boldsymbol{R}_4\text{: } \mathbf{If}\ \delta_1(t) \text{ is Small } \mathbf{and}\ \delta_4(t) \text{ is Big } \mathbf{then} \\
&\quad \dot{\boldsymbol{x}}_4(t) = \boldsymbol{A}_4\boldsymbol{x}(t) + \boldsymbol{B}_4 u(t) \quad y_4(t) = \boldsymbol{C}_4\boldsymbol{x}(t)
\end{aligned}$$

$$\text{where } \delta_1(t) = \frac{1}{\sqrt{x_1(t)}} \quad \delta_2(t) = \frac{1}{\sqrt{x_2(t)}}. \tag{60}$$

The operating space of this model is depicted in Fig. 4.

The parameters of in (59) have been obtained from a nonlinear white-box model

$$\begin{aligned}
\dot{\boldsymbol{x}}_1(t) &= -p_2\sqrt{x_1(t)} + p_1 u(t) \\
\dot{\boldsymbol{x}}_2(t) &= p_2\sqrt{x_1(t)} - p_3\sqrt{x_2(t)} \\
y(t) &= x_2(t).
\end{aligned} \tag{61}$$

After converting this model into a quasi-linear parameter-varying system (62) by means of linearizing substitutions (60), matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_i$ and $\boldsymbol{C}_i$ have been obtained for the known physically given extreme values of $\boldsymbol{x} \in [\boldsymbol{x}_{\text{Small}}; \boldsymbol{x}_{\text{Big}}] = [0.15; 0.85]$

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}(\boldsymbol{\delta})\boldsymbol{x}(t) + \boldsymbol{B}u(t) \text{ with} \\
\boldsymbol{A}(\boldsymbol{\delta}(t)) &= \begin{bmatrix} p_2\delta_1(t) & 0 \\ p_2\delta_1(t) & -p_3\delta_2(t) \end{bmatrix}.
\end{aligned} \tag{62}$$

Parameters $[p_1; p_2; p_3] = [0.17; 0.15; 0.13]$ characterizing the given plant can be directly measured.

### B. Control Results

Fig. 5(a) shows that for the tracking of a reference signal $\boldsymbol{r}(t)$, the simple PDC controller exhibits a permanent steady-state error. This is because of a significant model-plant mismatch. This problem can be remedied by using the extended fuzzy controller with an integrator in the forward channel. Results shown in Figs. 5(b)–7 are obtained with this extended fuzzy controller. According to Section IV-A, the entire design problem of the EFS was stated in terms of LMIs. Figs. 5(b) and 6 demonstrate the effect of the parameter $\varphi$ for the controller with regionally constrained eigenvalues of the underlying locally valid subsystems.

Fig. 6 depicts a typical problem that occurs when a well-tuned controller from simulations is applied to the real process. Undesired oscillations appeared in the lower working-range of the output-variable $y = x_2$. This local deterioration of performance is obviously due to the mismatch between the white-box simulation model and the physical process in some operating regions. The exact nature of this mismatch has not been investigated. Instead, the fuzzy controller has been locally tuned; the performance requirements of the corresponding local controllers have been decreased. The result is shown in Fig. 7.

Note that all the presented fuzzy controllers are based on state variables estimated by a fuzzy observer as described in Section IV-B, since not all states are measurable and thus cannot be used for local state feedback or fuzzy gain-scheduling. Hence, the simplifying assumptions for the fuzzy observer design enabling its analysis and synthesis have been validated both through simulation and real-world experiment. The estimated states used for the control are depicted in each of the simulation diagrams, denoted by $x_1^e(t)$ and $x_2^e(t)$ respectively.

## VI. Conclusion

The proposed method has been applied in simulations and real-time experiments. The results show that the designed controller achieves good performance. The calculations of all the parameters of this model-based fuzzy controller and observer have been automated by means of an LMI-solver such that the stability and the desired performance (the speed of response, no overshoots, no steady-state error) of the closed-loop system were achieved. The proposed nonlinear controller design has been parameterized by a small number of tuning parameters having a physical meaning (lower and upper bound on the speed of response, suppressing of overshoots). This feature, together with the locally-oriented structure of the overall controller, enables a good insight into the controller's working—it makes the tuning effective and simple if model-plant mismatches appear.

The proposed fuzzy controllers (FS and EFS) are in principle very simple. This makes their implementation quite straightforward regarding the required hardware, sampling period, etc. The computation of the controller's parameters, however, is a rather time-consuming optimization process that cannot be done in real-time and therefore it is rather not appropriate for adaptive control (except for slower systems).

A possible limitation of this approach is the use of the Lyapunov method, which is conservative. In practice, it can happen that the desired performance cannot be achieved, or even the whole controller design problem can be infeasible. The reason for this is that this method guarantees all properties of the control system defined by the corresponding LMIs for all rates of changes of the scheduling variables within the WWR. This is also in the case that these variables only vary very slowly. Another restriction is the fact that only quadratic Lyapunov functions have been considered. Some design strategies how to overcome the infeasibility problem are given in [17], [18], and [10].

## APPENDIX

Starting from the formalism of Lyapunov and exponential stability used in the theorems in this paper, we can state additional conditions to be satisfied for the considered systems to be *input–output stable*. Using a theorem from [19, Ch. 6], it is quite straightforward to prove that for the considered class of quasi linear parameter varying systems, BIBO stability is guaranteed as well.

*Theorem VII.1:* Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}), \boldsymbol{x}(0) = x_0 \tag{63}$$

$$\boldsymbol{y} = \boldsymbol{h}(t, \boldsymbol{x}, \boldsymbol{u}). \tag{64}$$

Let $D = \{\boldsymbol{x} \in \mathbb{R}^n \mid ||\boldsymbol{x}|| < r\}$, $D_u = \{\boldsymbol{u} \in \mathbb{R}^m \mid ||\boldsymbol{u}|| < r_u\}$, $\boldsymbol{f} : [0,\infty) \times D \times D_u \to \mathbb{R}^n$ be piecewise continuous in $t$ and locally Lipschitz in $(\boldsymbol{x}, \boldsymbol{u})$ and $\boldsymbol{h} : [0,\infty) \times D \times D_u \to \mathbb{R}^n$ be piecewise continuous in $t$ and continuous in $(\boldsymbol{x}, \boldsymbol{u})$. Suppose that the following hold true.

- $\boldsymbol{x}_0 = 0$ is an exponentially stable equilibrium point of (63), and there is a Lyapunov function $V(t, \boldsymbol{x})$ that satisfies

$$c_1||\boldsymbol{x}||^2 \le V(t, \boldsymbol{x}) \le c_2||\boldsymbol{x}||^2 \tag{65}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}}\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{0}) \le c_3||\boldsymbol{x}||^2 \tag{66}$$

$$\left\|\frac{\partial V}{\partial \boldsymbol{x}}\right\| \le c_4||\boldsymbol{x}|| \tag{67}$$

  $\forall\,(t, \boldsymbol{x}) \in [0,\infty] \times D$ for some positive constants $c_1, c_2, c_4$ and negative $c_3$.
- $\boldsymbol{f}$ and $\boldsymbol{h}$ satisfy the inequalities

$$||\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}) - \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{0})|| \le L||\boldsymbol{u}|| \tag{68}$$

$$||\boldsymbol{h}(t, \boldsymbol{x}, \boldsymbol{u})|| \le n_1||\boldsymbol{x}|| + n_2||\boldsymbol{u}|| \tag{69}$$

  $\forall\ (t, \boldsymbol{x}, \boldsymbol{u}) \in [0,\infty] \times D \times D_u$ for some nonnegative constants $L, n_1, n_2$.

Then, for each $\boldsymbol{x}_0$ with $||\boldsymbol{x}_0|| \le r\sqrt{c_1/c_2}$, the system (63)–(64) is small-signal finite-gain $\mathcal{L}_p$ stable for each $p \in [1,\infty]$. In particular, for each $\boldsymbol{u}$, the output $\boldsymbol{y}$ satisfies

$$||\boldsymbol{y}||_{\mathcal{L}_p} \le \gamma||\boldsymbol{u}||_{\mathcal{L}_p} + \beta \tag{70}$$

where $\gamma$ and $\beta$ are positive constants. Furthermore, if the origin is globally exponentially stable and all the assumptions hold globally, then the system (63)–(64) is finite-gain $\mathcal{L}_p$ stable for each $p \in [1,\infty]$ and inequality (70) holds for each $\boldsymbol{x}_0 \in \mathbb{R}^n$ and $\boldsymbol{u} \in \mathbb{R}^m$ (BIBO stability).

*Proof:* See [19, Ch. 6] regarding stability of state-space models.

Theorem VII.1 shows that if the origin $\boldsymbol{x}_0 = \boldsymbol{0}$ of the unforced system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{0}) \tag{71}$$

is an exponentially stable equilibrium, then, under the above assumptions on $\boldsymbol{f}$ and $\boldsymbol{h}$, the system (63)–(64) will be input–output stable. Corollary VII.1 points out that the class of qLPV systems considered in our paper always satisfies Theorem VII.1.

*Corollary VII.1:* The following class of systems is considered:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t, \boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}(t, \boldsymbol{x})\boldsymbol{u} \tag{72}$$

$$\boldsymbol{y} = \boldsymbol{C}(t, \boldsymbol{x})\boldsymbol{x} + \boldsymbol{D}(t, \boldsymbol{x})\boldsymbol{u}. \tag{73}$$

In accordance with Theorem VII.1, the system (72)–(73) is piecewise continuous in $t$ and continuous in $(\boldsymbol{x}, \boldsymbol{u})$. Recall that the considered control systems described by (72)–(73) have a global exponentially stable equilibrium $\boldsymbol{x}_0 = 0$. Furthermore, a (quadratic) Lyapunov function satisfying (65)–(67) has been found as a part of the presented design procedure. Therefore, if also conditions (68)–(69) hold, (72)–(73) is said input–output stable. Hence, compare (63)–(64) with (72)–(73) and substitute into (68)–(69). As a result, one gets

$$||\boldsymbol{B}(t, \boldsymbol{x})\boldsymbol{u}|| \le L||\boldsymbol{u}|| \tag{74}$$

$$||\boldsymbol{C}(t, \boldsymbol{x})\boldsymbol{x} + \boldsymbol{D}(t, \boldsymbol{x})\boldsymbol{u}|| \le n_1||\boldsymbol{x}|| + n_2||\boldsymbol{u}||. \tag{75}$$

Clearly, (74)–(75) are satisfied for any bounded $(\boldsymbol{x}, \boldsymbol{u})$ if $\boldsymbol{B}(\cdot), \boldsymbol{C}(\cdot)$ and $\boldsymbol{D}(\cdot)$ are bounded. This is always true as $\boldsymbol{A}(\cdot)$, $\boldsymbol{B}(\cdot)$, $\boldsymbol{C}(\cdot)$ and $\boldsymbol{D}(\cdot)$ are bounded in a polytope, $\boldsymbol{x}$ is bounded according to the definition of the wide-working range and $\boldsymbol{u}$ is a bounded reference signal.

## ACKNOWLEDGMENT

The authors would like to thank the anonymous referees for their criticism which helped to improve this paper. The first author would like to thank the German Academic Exchange Service, Bonn, Germany for the financial support in 1998 during his stay with the Control Laboratory at Delft University of Technology, The Netherlands, where the real-time experiments were performed.

## REFERENCES

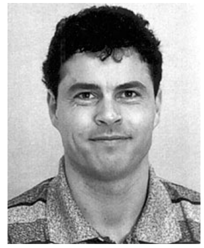
[1] R. Palm, D. Driankov, and H. Hellendoorn, *Model Based Fuzzy Control*. Berlin, Germany: Springer-Verlag, 1997.
[2] D. Driankov, R. Palm, and U. Rehfuess, "A Takagi-Sugeno fuzzy gain-scheduler," in *Proc. IEEE Int. Conf. Fuzzy Systems*, 1996, pp. 1053–1059.
[3] M. Fischer, O. Nelles, and R. Isermann, "Predictive control based on local linear fuzzy models," *Int. J. Syst. Sci.*, vol. 29, no. 7, pp. 679–697, 1998.
[4] R. Babuška, *Fuzzy Modeling for Control*. Norwell, MA: Kluwer, 1998.
[5] T. A. Johansen, K. J. Hunt, P. J. Gawthrop, and H. Fritz, "Off-equilibrium linearization and design of gain scheduled control with application to vehicle speed control," *Control Eng. Prac.*, vol. 6, pp. 167–180, 1998.
[6] K. J. Hunt and T. A. Johansen, "Design and analysis of gain-scheduled local controller networks," *Int. J. Control*, vol. 66, no. 5, pp. 619–651, 1997.
[7] T. Takagi and M. Sugeno, "Fuzzy identification of system and its applications to modeling and control," *IEEE Trans. Syst., Man, Cybern.*, vol. 15, pp. 116–132, Jan. 1985.

[8] P. M. Thomson, "Classical/$H_2$ solution for a robust control design benchmark problem," *J. Guid., Control, Dyna.*, vol. 18, no. 1, pp. 160–169, 1995.
[9] K. Tanaka, T. Ikeda, and H. O. Wang, "Fuzzy regulators and fuzzy observers," *IEEE Trans. Fuzzy Syst.*, vol. 6, pp. 250–265, Apr. 1998.
[10] P. Korba, "A gain-scheduling approach to model-based fuzzy control," Ph.D. dissertation, Gerhard Mercator Universität-GH-Duisburg, Duisburg, Germany, 2000.
[11] H. O. Wang, K. Tanaka, and M. Griffin, "An approach to fuzzy control of nonlinear systems: Stability and design issues," *IEEE Trans. Fuzzy Syst.*, vol. 4, pp. 14–23, Feb. 1996.
[12] S. Boyd, L. E. Ghaoui, E. Feron, and V. Belakrishnan, "Linear matrix inequalities in system and control theorie," in *SIAM: Studies In Applied Mathematics*. Philadelphia, PA: SIAM, 1994, vol. 15.
[13] K. Tanaka and M. Sugeno, "Stability analysis and design of fuzzy control systems," *Fuzzy Sets Syst.*, vol. 45, pp. 135–156, 1992.
[14] S. Kawamoto, K. Tada, A. Ishigame, and T. Taniguchi, "An approach to stability analysis of second order fuzzy systems," in *Proc. IEEE Int. Conf. Fuzzy Syst.*, 1992, pp. 1427–1434.
[15] X.-J. Ma, Z.-Q. Sun, and Y.-Y. He, "Analysis and design of fuzzy controller and observer," *IEEE Trans. Fuzzy Syst.*, vol. 6, pp. 41–51, Feb. 1998.
[16] M. Chilali and P. Gahinet, "$H_\infty$-design with pole placement constrains: An LMI approach," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 358–367, Mar. 1996.
[17] P. Korba and P. M. Frank, "An applied optimization-based gain-scheduled fuzzy control," in *Proc. Amer. Control Conf.*, Chicago, IL, June 2000, pp. 3383–3387.
[18] P. Korba, H. Werner, and P. M. Frank, "A LMI-based fuzzy gain-scheduling for the TORA benchmark nonlinear control problem," presented at the Control 2000, J. Maciejovski, Ed., Cambridge, U.K., Sept. 2000.
[19] H. K. Khalil, *Nonlinear Systems*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1996.

**Petr Korba** received the M.Sc. degree in electrical engineering from the Czech Technical University, Prague, Czech Republic, and the Ph.D. degree (with honors) from the University of Duisburg, Duisburg, Germany, in 1995 and 2000, respectively.

He was a Visiting Scientist in control engineering at the University of Technology in Delft, The Netherlands, in 1998, and at the University of Manchester Institute of Science and Technology (UMIST), Manchester, U.K., in 1999. He worked as a Research Associate and Member of Staff at UMIST, Control Systems Centre, until 2001. He is currently with Asea Brown Boveri (ABB) Switzerland, Ltd., Corporate Research. His research interests include model-based fuzzy control, gain-scheduling, and robust control and their applications.

Dr. Korba received the 2000 American Control Conference Best Session Award. He was awarded two scholarships of the German Academic Exchange Service, Bonn, Germany, and a scholarship of three Rotary Clubs in Duisburg, Germany.



**Robert Babuška** received the M.Sc. degree in control engineering from the Czech Technical University, Prague, Czech Republic, and the Ph.D. degree from the Delft University of Technology, The Netherlands, in 1990 and 1997, respectively.

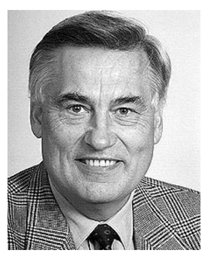
Currently, he is a Professor at the Delft Center for Systems and Control, Faculty of Mechanical Engineering, Delft University of Technology. He has coauthored more than 30 journal papers and book chapters, and has published a research monograph *Fuzzy Modeling for Control* (Norwell, MA: Kluwer, 1998). His research interests include the use of fuzzy set techniques and neural networks in nonlinear system identification and control.

Dr. Babuška is currently serving as an Associate Editor of the IEEE TRANSACTIONS ON FUZZY SYSTEMS and *Engineering Applications of Artificial Intelligence*. He is also an Area Editor of *Fuzzy Sets and Systems*.

**Henk B. Verbruggen** received the M.Sc. degree in electrical engineering from the Delft University of Technology, The Netherlands, in 1963.

Since 1963, he has been a Staff Member of the Control Engineering Laboratory at the Electrical Engineering department of the Delft University of Technology. In 1980, he was appointed a Full Professor. His research interests include model-based predictive control, fuzzy logic and neural networks for modeling, control, fault detection and controller reconfiguration. He is author and coauthor of more than 200 publications.

Mr. Verbuggen served as Chairman of the Coordinating Committee on Computer Control of IFAC, as an Associate Editor of the IFAC journal *Engineering Applications of AI*, and as an Area Editor of *Fuzzy Sets and Systems*. He has been involved in a number of EC-sponsored research projects and working groups.



**Paul M. Frank** received the Dipl.-Ing. degree in electrical engineering in 1959, the Doctor Ing. degree in 1966 and the Habilitation degree, all from the University of Karlsruhe, Karlsruhe, Germany, in 1959, 1966, and 1973, respectively. He has also received three honorary doctoral degrees, from the University of Iasi, Romania (1994), the Universit'e de Haute Alsace, Mulhouse, France (1997), and the Technical University of Cluj-Napoca, Romania (1998).

He was an Assistant and Associate Professor at the University of Karlsruhe from 1959 to 1976. From 1974 to 1975, he spent a year as a scholar and Guest Professor at the University of Washington, Seattle. From 1976 to 1999, he was a Full Professor and Head of the Department of Measurement and Control at the Gerhard-Mercator-University, Duisburg, Germany, where he has been Professor Emeritus since 1999. He holds the position of Honorary President of the German-French Institute of Automation and Robotics IAR. From 1977 to 2000, he was a Permanent Guest Lecturer at the Ecole Nationale Sup'erieure de Physique de Strasbourg ENSPS, France. In 1986, he founded the company Amira GmbH, which produces and sells laboratory setups for practical courses in control engineering. He has published or edited seven books and published more than 460 papers in technical journals and international conferences, and is Co-editor of several technical journals. His main interests are in fault diagnosis, robust control systems, sensitivity theory, and fuzzy techniques.

Dr. Frank is a Member of the International Affairs Committee of IEEE, a Member (former Board Member) of the VDI/VDE-GMA, and was President of EUCA from 1999 to 2001.