Optimal Weighting Functions for Gain-Scheduled Control

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Outline

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  - gain-scheduled controller
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- Optimal weighting functions
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- Conclusions and future research

Introduction

- Gain-scheduling – attractive, conceptually simple method.
- Local model $\longrightarrow$ local controller, interpolation.
- Mainstream literature – stable design:

$$\dot{\mathbf{x}} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i(\mathbf{x}) w_j(\mathbf{x}) (\mathbf{A}_i - \mathbf{B}_i \mathbf{K}_j)\mathbf{x}$$

- Design of weighting functions not satisfactorily addressed.
- Significant gain in performance possible.

Model-based gain-scheduling control

Model: $\textbf{If } \mathbf{z}(t) \textbf{ is } \mathcal{A}_i \textbf{ then } y(t) = G_i u(t), \quad i = 1, 2 \ldots, n$

Controller: $\textbf{If } \mathbf{z}(t) \textbf{ is } \mathcal{B}_i \textbf{ then } u(t) = C_i y(t), \quad i = 1, 2 \ldots, n$

parameter vectors: $\mathbf{g}_i \in \mathbb{R}^{n_g}$, $\mathbf{c}_i \in \mathbb{R}^{n_c}$
complete model and controller: $\mathbf{G} = [\mathbf{g}_1 \ldots \mathbf{g}_n]$, $\mathbf{C} = [\mathbf{c}_1 \ldots \mathbf{c}_n]$

Interpolation:

$$\mathbf{g}(\mathbf{z}) = \sum_i v_i(\mathbf{z})\mathbf{g}_i = \mathbf{G}\mathbf{v}(\mathbf{z}) \quad \text{with} \quad \mathbf{1}^T\mathbf{v}(\mathbf{z}) = 1$$

and

$$\mathbf{c}(\mathbf{z}) = \sum_i w_i(\mathbf{z})\mathbf{c}_i = \mathbf{C}\mathbf{w}(\mathbf{z}) \quad \text{with} \quad \mathbf{1}^T\mathbf{w}(\mathbf{z}) = 1$$

Control design

Local controller parameters:

$$\mathbf{C} = D(\mathbf{G}; H)$$

◇ D design method (pole placement, optimization, etc.)
◇ H performance specification (closed-loop reference model)

Controller weighting functions:

model weighting functions are usually shared by the controller, i.e.:
$\mathbf{w}(\mathbf{z}) = \mathbf{v}(\mathbf{z})$, $\forall \mathbf{z}$

$\Rightarrow$ sub-optimal closed-loop behavior

Example 1: first-order system

$$G_1(s) = \frac{K_1}{\tau_1 s + 1}, \qquad G_2(s) = \frac{K_2}{\tau_2 s + 1}$$

proportional gain $\rightarrow$ characteristic equation:

$$\tau s + 1 + cK = 0$$

the corresponding controller gain:

$$c = -\frac{1 + \tau p}{K}$$

Example 1: first-order system

interpolation of model parameters:

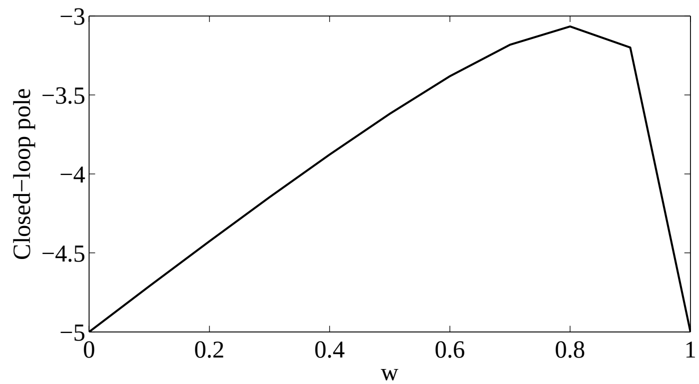
$$K(z) = K_1 + v(z)(K_2 - K_1)$$
$$\tau(z) = \tau_1 + v(z)(\tau_2 - \tau_1)$$

computed controller gain:

$$c_{\mathrm{nl}} = -\frac{1 + \tau(z)p}{K(z)}$$

$\neq$ interpolated gain $c(z) = c_1 + v(z)(c_2 - c_1)$

Closed loop pole variation with w(z)



with $c(z) = c_1 + w(z)(c_2 - c_1)$

Are there better ways of scheduling?

- Schedule model parameters, design controller on-line:

$$\mathbf{c}_{\text{nl}}(\mathbf{z}) = D(\mathbf{g}(\mathbf{z}); H), \quad \forall \mathbf{z}$$

infeasible for computationally demanding design methods, or for methods requiring interaction of the designer

- Design optimal weighting functions (off-line).

Optimal weighting functions

Solve:

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{1}^T \end{bmatrix} \mathbf{w}(\mathbf{z}) = \begin{bmatrix} \mathbf{c}_{\text{nl}}(\mathbf{z}) \\ 1 \end{bmatrix}, \qquad \forall \mathbf{z}$$

$n_c + 1$ equations with n_w unknowns

where:
n_c . . . number of parameters in the controller
n_w . . . number of interpolated controllers

Solutions for optimal weighting functions

- Unique solution if $n_c = n_w - 1$:

$$\mathbf{w}(\mathbf{z}) = \begin{bmatrix} \mathbf{C} \\ \mathbf{1}^T \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{c}_{\text{nl}}(\mathbf{z}) \\ 1 \end{bmatrix}, \qquad \forall \mathbf{z}$$

- Multiple solutions if $n_c < n_w - 1$: choose some WF

- Approximate solution if $n_c > n_w - 1$:

$$\mathbf{w}(\mathbf{z}) = \arg\min \|H - H(\mathbf{z})\|, \quad \forall \mathbf{z},$$

with $H(\mathbf{z}) = \dfrac{G(v(z))C(w(z))}{1 + G(v(z))C(w(z))}$

Construct separate schedulers

$1:$ **If** $\mathbf{z}(t)$ **is** $\mathcal{B}_i^1$ **then** $c_1 = c_{i,1}, \quad i = 1, 2 \ldots, n$

$2:$ **If** $\mathbf{z}(t)$ **is** $\mathcal{B}_i^2$ **then** $c_2 = c_{i,2}, \quad i = 1, 2 \ldots, n$

⋮

⋮

$n_c:$ **If** $\mathbf{z}(t)$ **is** $\mathcal{B}_i^{n_c}$ **then** $c_{n_c} = c_{i,n_c}, \quad i = 1, 2 \ldots, n$

Example 1 (continued)

controller gain: $c = -\dfrac{1+\tau p}{K}$

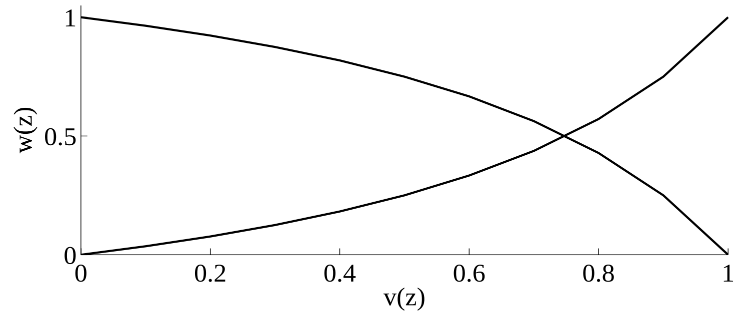
model parameters:

$$K(z) = K_1 + v(z)(K_2 - K_1), \quad \tau(z) = \tau_1 + v(z)(\tau_2 - \tau_1)$$

$$-\frac{1 + [\tau_1 + v(z)(\tau_2 - \tau_1)]p}{K_1 + v(z)(K_2 - K_1)} = c_1 + w(z)(c_2 - c_1)$$

$$\boxed{w(z) = -\frac{c_1}{c_2 - c_1} - \frac{1 + [\tau_1 + v(z)(\tau_2 - \tau_1)]p}{(c_2 - c_1)[K_1 + v(z)(K_2 - K_1)]}}$$

Example 1: optimal weighting functions



Example 2: PI controller

$$\tau\dot{y} + y = -KK_p\frac{1}{T}\int y - KK_p y$$

characteristic equation: $\tau\ddot{y} + (1 + KK_p)\dot{y} + \frac{KK_p}{T}y = 0$

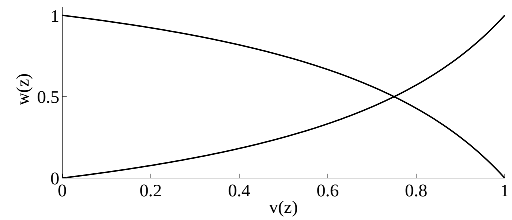
desired poles: $(s - p_1)(s - p_2) = 0$

computed controller parameters:

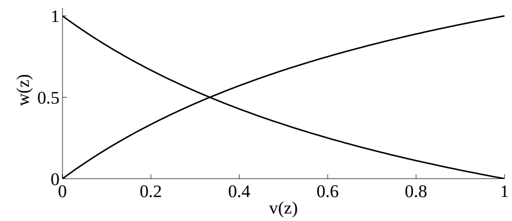
$$K_p(z) = \frac{-(p_1 + p_2)\tau(z) - 1}{K(z)}, \qquad T(z) = \frac{-(p_1 + p_2)\tau(z) - 1}{p_1 p_2 \tau(z)}$$

PI controller: optimal weighting functions

K_p :



T :



Remarks

- Analytic expression for WF possible in special cases (e.g., a simple pole-placement formula).
- Otherwise, discretize the domain of $\mathbf{z}$, compute weighting functions point-wise.
- Nonlinear model can be used instead of local models to compute $\mathbf{c}_{\mathrm{nl}}(\mathbf{z})$.
- If $\mathbf{z}$ is a function of the plant state, the GS controller is functionally equivalent to a feedback-linearizing controller.
- For slowly varying $\mathbf{z}$, closed-loop stability guaranteed by design. Otherwise use LMI techniques to assess stability.

Stability of LPV Systems

$$\dot{\mathbf{x}} = \sum_{i=1}^{n} w_i(z) A_i x$$

Sufficient condition for asymptotic stability: $\exists P = P^T$ s.t.

$$P > 0$$
$$A_i^T P + P A_i < 0 \quad \forall i = 1, 2, \ldots, n$$

Proof: Lyapunov function $V(x) = x^T P x$

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = \sum_{i=1}^{n} w_i(z)\; x^T (A_i^T P + P A_i) x$$

Closed Loop Stability

Model: **If** $\mathbf{x}$ is F_i **then** $\dot{\mathbf{x}} = \mathbf{A}_i \mathbf{x} + \mathbf{B}_i \mathbf{u}$

$$\dot{x} = \sum_{i=1}^{n} v_i(z) A_i x + \sum_{i=1}^{n} v_i(z) B_i u$$

Controller: **If** $\mathbf{x}$ is F_i **then** $\mathbf{u} = -\mathbf{K}_i \mathbf{x}$

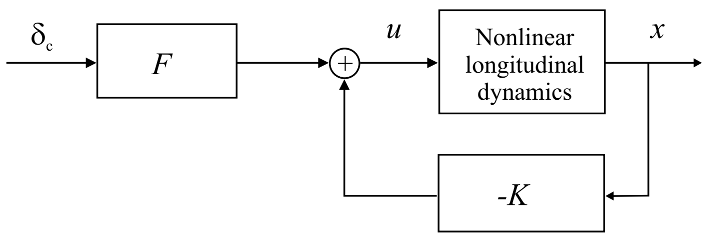
$$u = \sum_{j=1}^{n} w_j(z) K_i x$$

Closed loop: $\dot{\mathbf{x}} = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i(\mathbf{x}) w_j(\mathbf{x}) (\mathbf{A}_i - \mathbf{B}_i \mathbf{K}_j) \mathbf{x}$

Realistic application example

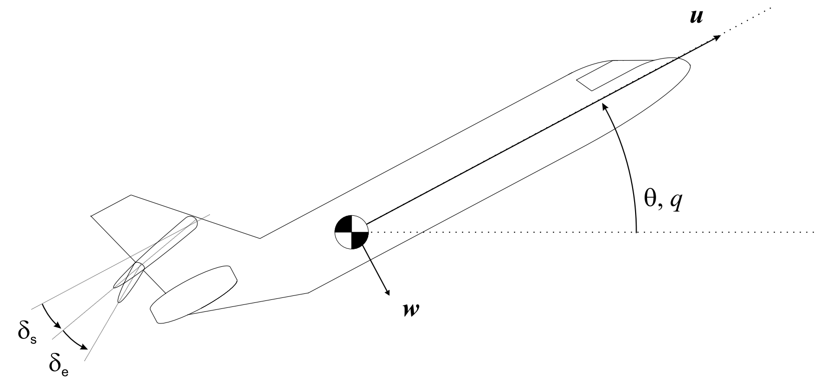
IAI Galaxy business jet

Longitudinal (pitch rate) control

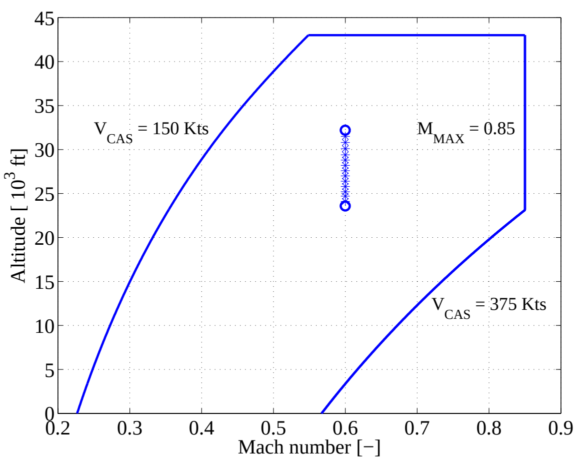


$$\mathbf{x} = \begin{pmatrix} \text{forward speed} \\ \text{downward speed} \\ \text{pitch rate} \\ \text{pitch attitude} \end{pmatrix}, \qquad \delta_c = \text{column deflection}$$

State variables



Scheduling variables z



Performance specifications

- reference model:

$$H = \frac{1}{(s^2 + 2\zeta_{\rm sp}\omega_{\rm sp} + \omega_{\rm sp}^2)(s^2 + 2\zeta_{\rm ph}\omega_{\rm ph} + \omega_{\rm ph}^2)}$$

- column deflection – pitch rate gain
- feedforward filter zeros – response shaping

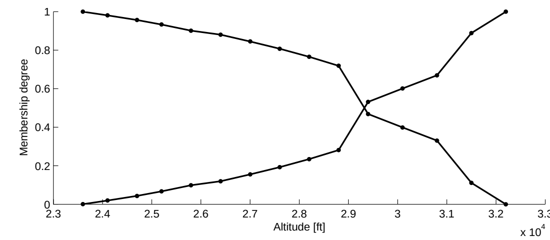
pole placement design

Scheduled controller parameters

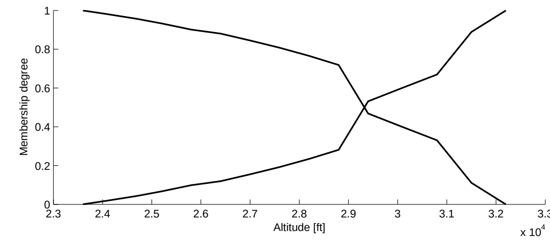
parameter	symbol	$\mathbf{c}_1$	$\mathbf{c}_2$
feedforward gain	K_{ff}	0.967	0.801
filter poles	p_1	-0.013	-0.018
	p_2	-0.834	-0.484
feedback gains	k_1	-0.055	-0.166
	k_2	0.169	-0.201
	k_3	-38.03	-67.97
	k_4	-3.478	-9.281

Optimal weighting functions

K_{ff} :

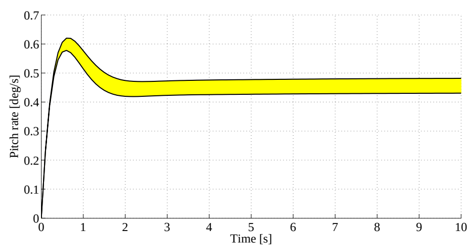


p_1 :

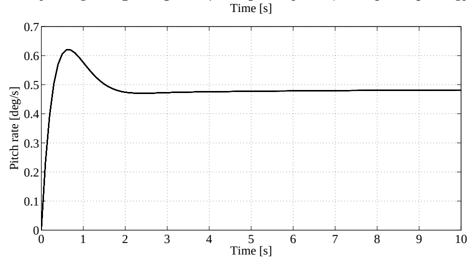


Performance : step response

linear WFs :



optimal WFs :



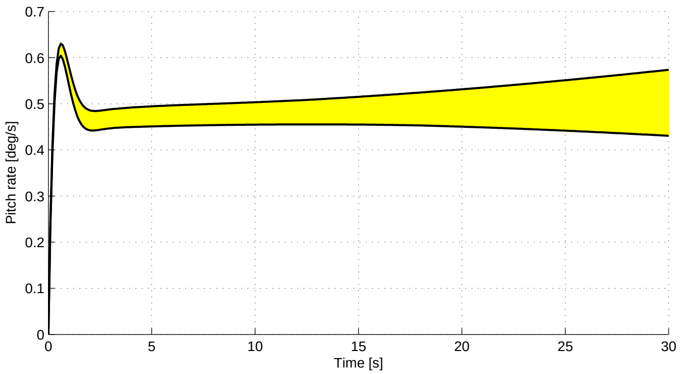
Lyapunov stability analysis

$$(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j)^T\mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j) < 0 \quad \text{for} \quad i = 1, 2, \quad j = 1, 2$$

No $\mathbf{P}>0$ can be found

Nothing can be said on stability with arbitrary weighting functions.

Closer inspection

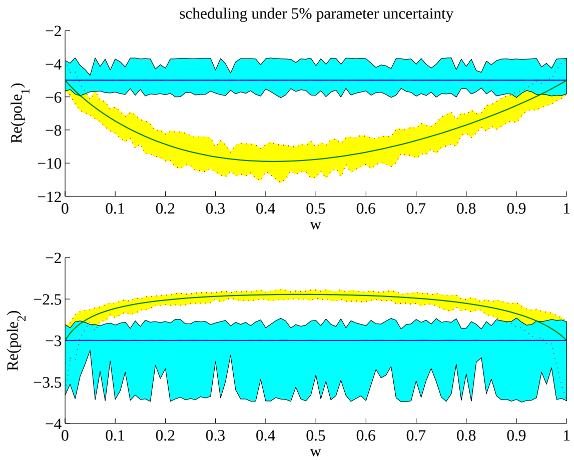


Although pilots not concerned with the phugoid mode, we do not like to have an unstable system on board . . .

Conclusions and future research

- Optimal weighting functions for GS control:
  - computed off-line (real-time performance)
  - invariant closed-loop behavior
  - stability guaranteed for slowly varying $\mathbf{z}(t)$
- Open issues:
  - robustness ($\approx$ feedback linearization)
  - uncertainty in weighting functions
  - numerical aspects

Closed-loop poles under uncertainty



Stable design

$$(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j)^T\mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j) < 0 \quad \text{for} \quad i = 1, 2, \quad j = 1, 2$$
$$\mathbf{P} > 0$$

LMI region: vertical strip $[-3, 0] \cap$ angle $[-\pi/7, \pi/7]$

closed-loop poles:

operating point 1	operating point 2	desired
$-1.54 + 0.65j$	-1.65	$-2.26 + 1.70j$
$-1.54 - 0.65j$	-1.16	$-2.26 - 1.70j$
-0.17	-0.21	$-0.02 + 0.01j$
-0.04	-0.05	$-0.02 - 0.01j$