Solving Linear Matrix Inequalities using the Multi Parametric Toolbox

Maurice Heemels
Niek Borgers
Daniel Deenen
Annemiek van der Maas
d.p.borgers@tue.nl
d.a.deenen@tue.nl
a.v.d.maas@tue.nl

Technische Universiteit Eindhoven
University of Technology

Motivating Examples

- Quadratic Lyapunov Function

$$\text{Find } P \text{ s.t.} \begin{cases} A^\top P + PA < 0 \\ P > 0 \end{cases}$$

- Common Quadratic Lyapunov Function

$$\text{Find } P \text{ s.t.} \begin{cases} A_1^\top P + PA_1 < 0 \\ A_2^\top P + PA_2 < 0 \\ P > 0 \end{cases}$$

- PWL system, continuity conditions and S-procedure

$$\text{Find } P_i, U_i \text{ and } W_i \text{ s.t.} \begin{cases} A_i^\top P_i + P_i A_i + E_i^\top U_i E_i < 0 \\ P_i - E_i^\top W_i E_i > 0 \\ Z_{ij}^\top [P_i - P_j] Z_{ij} = 0 \qquad (i,j) \in \mathcal{J} \end{cases}$$

Example: Quadratic Lyapunov Function

Given

$$A = \begin{bmatrix} -1 & 2 & 0 \\ -3 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \quad Q = 0.1I$$

Find P such that

$$\begin{cases} A^\top P + PA + Q \leq 0 \\ P > 0 \end{cases}$$

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optimize(L); %solve the LMIs
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optimize(L); %solve the LMIs
P = value(P) %convert P to normal numbers
```

Always check if *P* really satisfies all constraints!

General Procedure

- Define unknowns: `sdpvar(..,..)` (square matrices are symmetric by default)
- Define constraints `Li` for $i = 1, \ldots, N$
- Combine constraints by adding (`L = L1+L2`) or concatenating (`L=[L1, L2]`)
- Solve for unknowns: `optimize(L)`
- Convert solutions: `value(..)`
- Check solutions: Substitute in constraints and check manually!

Commands and Technicalities

- Equality operator: $==$
- Inequality operators (non-strict only!): $<=$, $>=$
- Condition on matrix (eigenvalues): `L=[M<=0]`
  Only when *M* is symmetric!
- Condition on matrix elements: `L=[M(:)<=0]`
- Solve while minimizing a parameter, matrix elements or eigenvalues: `optimize(L,MinCond)`

Final Notes

For installation files and instructions, see
http://control.ee.ethz.ch/%7empt/3/Main/Installation

For more info, check the YALMIP Wiki
https://yalmip.github.io/tutorial/semidefiniteprogramming/

Or in your command window in Matlab type

- `help sdpvar`
- `help optimize`

On Canvas

- Examples
- This presentation