Modeling and Control of Hybrid Systems

DISC Assignment 2017

- Make a report in Word or LaTeX containing your solutions. All solutions should be clearly motivated. Only end results of calculations are not sufficient!
- Hand in your report as **pdf** via the DISC website.
- **DEADLINE:** Friday the **3rd of March 2017**.
- Good luck!

Problem 1 ***Well-posedness***

Consider the following theorems (slide 22 of the lecture 'Solution Concepts and Well-posedness of Hybrid Systems'):

Theorem 1 *A hybrid automaton is non-blocking, if the following condition holds:*

(C1) for all $(q, x) \in \texttt{Reach} \cap \texttt{Out}$*, there exists* $e = (q, q') \in E$ *with* $x \in G(e)$*.*

In case the automaton is deterministic, this condition is also necessary.

Theorem 2 *A hybrid automaton is deterministic, if and only if for all* $(q, x) \in \texttt{Reach}$

- *if* $x \in G((q, q'))$ *for some* $(q, q') \in E$*, then* $(q, x) \in \texttt{Out}$*;*
- *if* $(q, q') \in E$ *and* $(q, q'') \in E$ *with* $q' \neq q''$ *, then* $x \notin G((q, q')) \cap G((q, q''))$*; and*
- *if* $(q, q') \in E$ *and* $x \in G((q, q'))$*, then there is at most one* $x' \in X$ *with* $(x, x') \in R((q, q'))$*.*

(a) Show that Condition (C1) is a necessary condition for a deterministic hybrid automaton to be non-blocking.

(b) Condition (C1) is a sufficient, but not a necessary condition for a non-deterministic hybrid automaton to be non-blocking. Show this by providing an example of a non-deterministic and non-blocking hybrid automaton which does not satify (C1).

Consider the hybrid automaton in Figure 1.

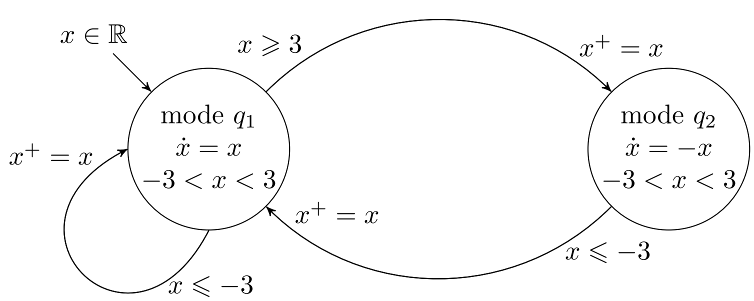

Figure 1: Hybrid automaton of Problem 1.

(c) Compute the sets `Reach` and `Out` as defined in slides 20 and 21 of the lecture 'Solution Concepts and Well-posedness of Hybrid Systems' for the hybrid automaton of Figure 1.

(d) Is the hybrid automaton of Figure 1 deterministic? Is it non-blocking?

Problem 2 ***Stability of discrete-time switched linear systems with arbitrary switching***

Consider a discrete-time switched linear system of the form

$$x_{k+1} = A_{\sigma(k)} x_k \tag{1}$$

where $A_1, \ldots, A_N \in \mathbb{R}^{n \times n}$ are matrices and $\sigma : \{0, 1, 2, \ldots\} \mapsto \{1, \ldots, N\}$ is the switching signal.

(a) Write down the conditions that a *common* Lyapunov function should satisfy for the system (1). Note that the existence of a common Lyapunov function implies global asymptotic stability (GAS) under arbitrary switching. (Hint: in discrete time the Lyapunov function should satisfy $\Delta V(x_k) := V(x_{k+1}) - V(x_k) < 0$ for all $x_k \neq 0$. Why?)

(b) Reformulate the conditions found in the previous part for a common *quadratic* Lyapunov function as linear matrix inequalities. If we have N modes (N linear subsystems), then how many LMIs do we get?

(c) Consider a discrete-time switched linear system with two modes ($N = 2$) and matrices given by

$$A_1 = \begin{bmatrix} 0.6200 & -0.2848 \\ -0.0712 & 0.8336 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.5125 & -0.2855 \\ 1.4277 & 0.5125 \end{bmatrix}.$$

This switched linear system is globally asymptotically stable (GAS) under arbitrary switching. You can verify that there does not exist a common quadratic Lyapunov function (use the MPT3 toolbox and try to find one). Find another way of proving stability for this system using a mode-dependent quadratic Lyapunov function of the form $V(x_k) = x_k^\top P_{\sigma(k)} x_k$, where $\sigma(k) \in \{1, 2\}$ denotes the active mode at time k. Derive linear matrix inequalities that are sufficient for GAS under arbitrary switching using this multiple quadratic Lyapunov function. Solve these for the particular example using the MPT3 toolbox.

(d) If instead of $N = 2$ we have N modes, how many LMIs do we get?

(e) Can we also use a mode-dependent Lyapunov function to prove GAS for a *continuous-time* switched linear system with arbitrary switching? If yes, derive which LMIs should be satisfied. If no, explain why not.

Problem 3 ***Inter-jump time restrictions for jump-flow systems***

Consider the jump-flow system

$$\begin{aligned} \dot{x}(t) &= Ax(t), \text{ when } t \neq t_i, \; i \in \{0, 1, 2, \ldots\} \\ x(t^+) &= Rx(t), \text{ when } t = t_i, \; i \in \{0, 1, 2, \ldots\}, \end{aligned} \tag{2}$$

which jumps at times $0 \leq t_0 < t_1 < t_2 < \ldots$. This system is globally exponentially stable (GES) when the conditions on slide 49 of the lecture 'Stability of Switched and Jump-Flow Systems' are satisfied.

(a) For the system (2), transform the conditions (i.)-(iii.) on slide 49 of the lecture 'Stability of Switched and Jump-Flow Systems' into LMIs. Use a quadratic Lyapunov function $V(x) = x^\top Px$.

For the two cases below, provide numerical values for the maximal or minimal average inter-jump time $\tau^* > 0$ such that the jump-flow system (2) is GES. Try to find the least conservative values for τ^*, and *explain your method.*

(b) $A = A_1 := \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $R = R_1 = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.7 \end{bmatrix}$

(c) $A = A_2 := \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ and $R = R_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Problem 4 ***Stability of a jump-flow system with state-based switching***

Consider the jump-flow system given by

$$\dot{x} = Ax, \quad \text{when } x \in \mathcal{C} \tag{3a}$$

$$x^+ = Rx, \quad \text{when } x \in \mathcal{D}, \tag{3b}$$

where $x \in \mathbb{R}^n$, A and R are constant matrices in $\mathbb{R}^{n \times n}$ and $\mathcal{C}$ and $\mathcal{D}$ are subsets of $\mathbb{R}^n$. In the hybrid automata framework this system is represented by H_{jump} given by $(Q, X, f, \texttt{Init}, \texttt{Inv}, E, G, R)$ with

- $Q = \{q_1\}$
- $X = \mathbb{R}^n$ where $\mathbb{R}$ the denotes the set of all real numbers
- $f(q_1, x) = Ax$
- $\texttt{Init} = Q \times X$
- $\texttt{Inv}(q_1) = \mathcal{C}$
- $E = \{(q_1, q_1)\}$
- $G((q_1, q_1)) = \mathcal{D}$
- $R((q_1, q_1)) = \{(x^-, x^+) \in X \times X \mid x^+ = Rx^-\}$

We assume throughout this problem that $\mathcal{C} \cup \mathcal{D} = \mathbb{R}^n$.

(a) Draw a picture of the hybrid automaton H_{jump}.

(b) Consider the hybrid automaton H_{jump} and suppose that $\mathcal{C} := \{x \in \mathbb{R}^n \mid Ex \geq 0 \text{ or } Ex \leq 0\}$ and $\mathcal{D} := \{x \in \mathbb{R}^n \mid Jx \geq 0 \text{ or } Jx \leq 0\}$ with $n = 2$ (where the inequalities $Jx \geq 0$ and $Ex \geq 0$ hold component wise) and

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}; \; R = \begin{bmatrix} 0.6 & 0 \\ 0.25 & 0 \end{bmatrix}; \; E = \begin{bmatrix} 1 & -2 \\ -0.1 & 1 \end{bmatrix}; \; J = \begin{bmatrix} 1 & -2 \\ 0.1 & -1 \end{bmatrix}.$$

Draw phase portrait to show how the system behaves.

(c) Consider the hybrid automaton H_{jump} and suppose that we consider the general case where $\mathcal{C} := \{x \in \mathbb{R}^n \mid Ex \geq 0 \text{ or } Ex \leq 0\}$ and $\mathcal{D} := \{x \in \mathbb{R}^n \mid Jx \geq 0 \text{ or } Jx \leq 0\}$, where $E \in \mathbb{R}^{n \times n}$ and $J \in \mathbb{R}^{n \times n}$. We assume that the corresponding H_{jump} is non-blocking, deterministic and produces global solutions for $t \to \infty$. Derive linear matrix inequalities (LMIs) that guarantee that $V(x) = x^\top Px$ is a Lyapunov function showing global asymptotic stability (GAS) of the hybrid automaton H_{jump}. Use the S-procedure twice.

(d) Consider now H_{jump} with $\mathcal{C}$ and $\mathcal{D}$ as in (c) and the numerical data as given in part (b). Solve the LMIs as derived in part (c) using the Yalmip/Sedumi LMI-toolbox to show that the system H_{jump} with the numerical data in (b) is GAS.

(e) Consider H_{jump} with $\mathcal{C}$ and $\mathcal{D}$ as in (c) and the numerical data given by

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}; \; R = \begin{bmatrix} 2.4\alpha & 0 \\ \alpha & 0 \end{bmatrix}; \; E = \begin{bmatrix} 1 & -2 \\ -0.1 & 1 \end{bmatrix}; \; J = \begin{bmatrix} 1 & -2 \\ 0.1 & -1 \end{bmatrix}$$

in which $\alpha \in \mathbb{R}$ is some nonnegative real number. Approximate the maximal value of α for which H_{jump} is GAS using the LMIs found in (c). It is sufficient to provide a close estimate of the maximal value. Hence, you can try to solve the LMIs in (c) after you *fixed* the value of α. Choose various values of α to find a good approximation of the true maximal value.