Stability of Switched and Jump-Flow Systems

Maurice Heemels
Control Systems Technology group
Department of Mechanical Engineering
Technische Universiteit Eindhoven
m.heemels@tue.nl

DISC Course on Modeling and Control of Hybrid Systems 2017

Outline of lecture

- Switched systems
- Recall: stability of smooth systems
- 3 Different problems
- Jump-flow systems
- Summary

Switched systems

$$\dot{x} = f_\sigma(x)$$

$\{f_1(x), f_2(x), \ldots, f_N(x)\}$ family of smooth vector fields from $\mathbb{R}^n$ to $\mathbb{R}^n$

Switching signal $\sigma : [0,\infty) \to \{1,2,\ldots,N\}$ piecewise constant function of time

- Function of time t: $\sigma(t)$
- Function of state $x(t)$: $\sigma(x)$
- Combinations: $\sigma(t,x)$

No generalized solutions concepts including chattering, infinitely fast switching, sliding motions $\to$ include as additional modes!

Although we get back to this!

Switched linear systems

Switched linear system $\quad \dot{x} = A_\sigma x$

Piecewise linear system
Switching is only state-dependent

$$\dot{x} = A_i x \text{ when } x \in \mathscr{X}_i,$$

where $\mathscr{X}_i \subseteq \mathbb{R}^n$ are polyhedra (given by a finite number of inequalities ($a_k^\top x \geq b_k$, $k = 1,\ldots,K$)

- Well-posedness: cells form partitioning of $\mathbb{R}^n$ (necessary condition only)

$$\cup_{i=1}^n \mathscr{X}_i = \mathbb{R}^n \text{ and interior}(\mathscr{X}_i) \cap \text{interior}(\mathscr{X}_j) = \emptyset$$

- **Piecewise affine (PWA) systems**

$$\dot{x} = A_i x + a_i, \text{ when } E_i x \geq e_i,\ i \in I := \{1,\ldots,N\}.$$

Problem formulation

• Global asymptotic stability (GAS) of a system with state x:

Something like .... $\lim_{t\longrightarrow\infty} x(t) = 0$ for all initial states x_0.

GUAS: global uniform asymptotic stability: uniform in σ

Problem A : Find conditions for which the switched system is GAS for any switching signal (GUAS) ("robust stability").

Problem B : Show that the switched system is GAS for a given switching strategy or a class of switching strategies.

Problem C : Construct a switching signal that makes the switched system GAS (i.e. a stabilization problem).

$\longrightarrow$ Two classes for problem B: (i) state-based (ii) time-restricted

$\longrightarrow$ Problem C will be treated in the lecture on Hybrid Control.

Formal definitions for problem A

$$\dot{x} = f_\sigma(x)$$

$\{f_1(x), f_2(x), \ldots, f_N(x)\}$ family of smooth vector fields from $\mathbb{R}^n$ to $\mathbb{R}^n$

• 0 should be an equilibrium of the switched system under arbitrary switching, implying that $f_i(0) = 0$ for all $i = 1, \ldots, N$.

• Lyapunov stability of the origin

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x_0,\|x_0\|\leq\delta}\ \forall_{\sigma:\mathbb{R}_{\geq 0}\to\{1,\ldots,N\}}\ \forall_{t\geq 0}\ \|x_{x_0,\sigma}(t)\| \leq \varepsilon$$

• Global attractivity $\forall_{x_0\in\mathbb{R}^n}\ \forall_{\sigma:\mathbb{R}_{\geq 0}\to\{1,\ldots,N\}}\ x_{x_0,\sigma}(t) \to 0$

• Global uniform attractivity

$$\forall_{R>0}\ \forall_{\varepsilon>0}\ \exists_{T>0}\ \forall_{x_0,\|x_0\|\leq R}\ \forall_{\sigma:\mathbb{R}_{\geq 0}\to\{1,\ldots,N\}}\ \forall_{t>T}\ \|x_{x_0,\sigma}(t)\| \leq \varepsilon$$

• Global Uniform Asymptotic Stability: Lyapunov stability and global uniform attractivity

• Global Uniform Exponential Stability (GUES)

$$\exists_{c>0}\ \exists_{\lambda>0}\ \forall_{x_0\in\mathbb{R}^n}\ \forall_{\sigma:\mathbb{R}_{\geq 0}\to\{1,\ldots,N\}}\ \|x_{x_0,\sigma}(t)\| \leq ce^{-\lambda t}\|x_0\|$$

Back to basics: Lyapunov theory for stability of continuous systems

$f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous!

Theorem 1 Let $x = 0$ be an equilibrium of $\dot{x} = f(x)$ (i.e. $f(0) = 0$) and let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

- $V(x) \to \infty$ as $\|x\| \to \infty$ (V is *radially unbounded*);
- $V(0) = 0$ and $V(x) > 0$, if $x \neq 0$ (i.e. V is positive definite); and
- $\dot{V}(x) = L_f V(x) := \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x) < 0$ for all $x \neq 0$.

Then $x = 0$ is GAS.

Converse theorem: If $x = 0$ is a GAS equilibrium of $\dot{x} = f(x)$ (i.e. $f(0) = 0$), then there exists a Lyapunov function V

Back to basics: Lyapunov functions for GES

Theorem

Let $x = 0$ be an equilibrium of $\dot{x} = f(x)$ (i.e. $f(0) = 0$) and let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with

- there exist $p \in \mathbb{N}$, $c_1 > 0$, $c_2 > 0$ s.t. $c_1\|x\|^p \leq V(x) \leq c_2\|x\|^p$ for all $x \in \mathbb{R}^n$
- there exists $c_3 > 0$ such that $\dot{V}(x) = L_f V(x) := \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x) \leq -c_3\|x\|^p$ for all $x \in \mathbb{R}^n$

Then $x = 0$ is GES.

For linear systems $\dot{x} = Ax$ quadratic Lyapunov functions $V(x) = x^\top Px$ and $p = 2$ are useful

Back to basics: Lyapunov functions for GES

Discrete-time case. $x^+(t) = x(t+1)$ for all $t \in \mathbb{N}$.

Theorem

Let $x = 0$ be an equilibrium of $x^+ = f(x)$ (i.e. $f(0) = 0$) and let $V : \mathbb{R}^n \to \mathbb{R}$ be a function with

- there exist $p \in \mathbb{N}$, $c_1 > 0$, $c_2 > 0$ s.t. $c_1\|x\|^p \leq V(x) \leq c_2\|x\|^p$ for all $x \in \mathbb{R}^n$
- there exists $c_3 > 0$ such that $\Delta V(x) := V(f(x)) - V(x) \leq -c_3\|x\|^p$ for all $x \in \mathbb{R}^n$

Then $x = 0$ is GES.

For linear systems $x^+ = Ax$ quadratic Lyapunov functions $V(x) = x^\top Px$ and $p = 2$ are useful

Stability of linear systems

Consider the linear system $\dot{x} = Ax$ and consider a quadratic LF $V(x) = x^\top Px$ with P symmetric ($P = P^\top$) positive definite, i.e.

- $x^\top Px > 0$ for all $x \neq 0$ and P symmetric
- all eigenvalues of P are positive
- all leading principal minors $det P_{JJ} > 0$ for all $J = \{1, \ldots, j\}$ for $j = 1, \ldots, n$.
- $P = H^\top H$ for an invertible matrix H

- Q is called negative definite, if $-Q$ is positive definite.

$$\dot{V}(x) = \dot{x}^\top Px + x^\top P\dot{x} = x^\top [A^\top P + PA]x < 0 \text{ for all } x \neq 0$$

Hence, $A^\top P + PA$ should be a negative definite matrix: $A^\top P + PA < 0$

Stability of linear systems - continued

Theorem 2 Equivalent:

- $\dot{x} = Ax$ is GAS;
- A is a Hurwitz matrix (all eigenvalues in open left half plane)
- there is a quadratic Lyapunov function $V(x) = x^\top Px$ for some positive definite matrix P such that the Lyapunov inequality $A^\top P + PA < 0$ holds.

Moreover, for every Hurwitz A and for any $Q > 0$ there is a $P > 0$ such that the following Lyapunov equality holds

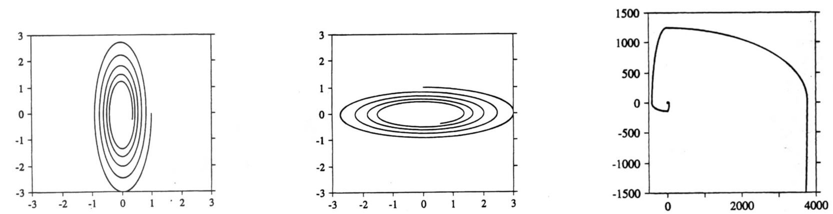
$$A^\top P + PA = -Q$$

$\longrightarrow$ Similar "Lyapunov" results for discrete-time (linear) systems

True hybrid problem: Combining stable dynamics → stable?

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1x_2 < 0 \\ A_2 x, & \text{if } x_1x_2 > 0 \end{cases}$$

$$A_1 = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix};\ A_2 = \begin{pmatrix} -1 & 100 \\ -10 & -1 \end{pmatrix}.\ \text{Eigenvalues } = -1 \pm 31.6j$$



Problem A: When is switched system UGAS for any switching signal?

Also for constant switching signals $\sigma(t) = i$ for all t

$$\Downarrow$$

$\dot{x} = f_i(x)$ should be GAS

$$\Downarrow$$

There is a radially unbounded Lyapunov function for each i!

Common Lyapunov function approach

• Try to find one shared Lyapunov function that decreases along any of the submodels:

A C^1-function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *common Lyapunov function* for $\dot{x} = f_\sigma(x)$ with $\sigma \in \{1, \ldots, N\}$ if

$$\dot{V}(x) = L_{f_i}V(x) = \frac{\partial V}{\partial x} f_i(x) < 0, \text{ when } x \neq 0 \text{ and for all } i = 1, \ldots, N.$$

Theorem 3 If all the smooth submodels share a positive definite radially unbounded common Lyapunov function, then the switched system is GUAS.

Question: What about sliding modes?

A converse theorem

Necessary and sufficient condition:

Theorem 4 If the switched system is GUAS, then all f_i share a positive definite radially unbounded common Lyapunov function.

Hence, no conservatism in result!

Switched linear systems: a common quadratic LF approach

Stability of switched linear systems of the form

$$\dot{x} = A_\sigma x, \quad \sigma \in \{1, \ldots, N\}$$

Common LF of quadratic type $V(x) = x^\top Px$ for positive definite P?

$$\dot{V}(x) = L_{f_i}V(x) := \frac{\partial V}{\partial x} f_i(x) = x^\top [PA_i + A_i^\top P]x < 0 \text{ for all } x \neq 0 \text{ and } i$$

Hence, we obtain **linear matrix inequalities** (LMIs)

$$A_i^\top P + PA_i < 0 \text{ for all } i = 1, \ldots, N \text{ and } P > 0$$

• LMIs can be efficiently solved (SEDUMI/YALMIP)!

Converse quadratic LF theorem?

Asymptotic stability of switched linear system $\dot{x} = A_\sigma x \Rightarrow$ existence common quadratic Lyapunov function???

The answer is negative

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix};\ A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix},$$

which is GUAS, but no common quadratic LF by infeasibility condition

However, there is a common LF that is homogeneous of degree 2:

$$V(x) = \max_{i=1,2,\dots,k} (l_i^\top x)^2$$

Conditions implying existence common quadratic LF

Theorem 5 If the matrices $\{A_1, \dots, A_N\}$ commute pairwise

$$\text{for all } i, j, \text{ it holds that } A_i A_j = A_j A_i.$$

and are all Hurwitz, then there exists a common quadratic Lyapunov function $P = P_N$, that can be found from solving the following set of Lyapunov equalities successively:

$$\begin{aligned} A_1^\top P_1 + P_1 A_1 &= -I \\ A_2^\top P_2 + P_2 A_2 &= -P_1 \\ A_3^\top P_3 + P_3 A_3 &= -P_2 \\ \vdots \quad &\ \ \vdots \\ A_N^\top P_N + P_N A_N &= -P_{N-1}. \end{aligned}$$

More involved conditions exist (cf. references in lecture notes!)

Jump-flow systems

$$\begin{aligned} \dot{x} &= f(x), \\ x^+ &= g(x), \end{aligned}$$

DISCUSSION: What about GAS/GES under arbitrary jumping/flowing?

Summary

- Stability of submodels $\not\Rightarrow$ stability! “Hybrid problem”
- Recalled Lyapunov theory for continuous systems (discrete/continuous-time)
- Problem A: GUAS for arbitrary switchings:
  - common LF approach
  - converse theorem
  - switched linear: common quadratic LF
  - discussion on jump-flow systems under arbitrary flowing/jumping
- Linear matrix inequalities (nice design tool!)

Problem B: Is the switched system GAS for given switching strategies?

$\longrightarrow$ Two classes of switching strategies:

1. induced by state space partitioning (piecewise linear systems)
2. time-restricted switching signals (dwell-time conditions)

Piecewise smooth systems

Piecewise smooth systems: $\dot{x} = \begin{cases} f_1(x), & \text{when } x \in \mathscr{X}_1 \\ f_2(x), & \text{when } x \in \mathscr{X}_2 \\ \vdots \\ f_N(x), & \text{when } x \in \mathscr{X}_N \end{cases}$

Sufficient: existence of common Lyapunov function

This might be conservative ... let's see if we can find relaxations ...

Relaxation 1: Decrease of Lyapunov function only in active region

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 \le 0 \\ A_2 x, & \text{if } x_1 x_2 > 0, \end{cases} \text{ with } A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}; A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

• No common quadratic LF.
• However, for $V(x) = x_1^2 + x_2^2$ it holds that $\dot{V} < 0$ along the nonzero solutions of the switched system, which implies GAS.

Relaxation w.r.t. common LF approach: Indeed, we only need

- $\dot{V}(x) = L_{A_1 x} V(x) = \frac{\partial V(x)}{\partial x} A_1 x < 0$ if $x_1 x_2 \le 0$
- $\dot{V}(x) = L_{A_2 x} V(x) = \frac{\partial V(x)}{\partial x} A_2 x < 0$ if $x_1 x_2 > 0$.

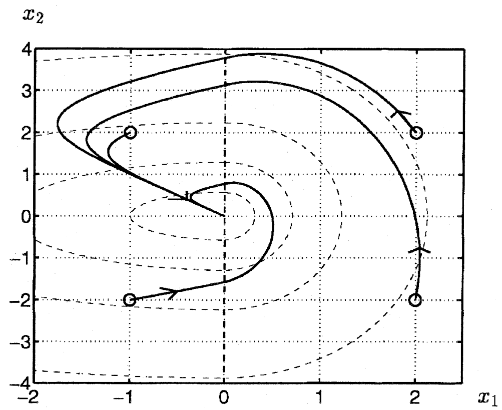
Hence, general set-up:

Find V pos. def. s.t. $L_{f_i} V(x)$ is only negative, where $\dot{x} = f_i(x)$ can be active.

Relaxation 2: Multiple Lyapunov functions

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 \le 0 \\ A_2 x, & \text{if } x_1 > 0, \end{cases} \text{ where } A_1 = \begin{pmatrix} -5 & -4 \\ -1 & -2 \end{pmatrix}; A_2 = \begin{pmatrix} -2 & -4 \\ 20 & -2 \end{pmatrix}.$$

• No common LF



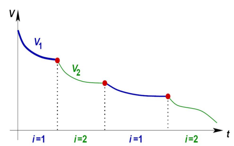
Relaxation 2: multiple Lyapunov function

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 \le 0 \\ A_2 x, & \text{if } x_1 > 0, \end{cases} \text{ where } A_1 = \begin{pmatrix} -5 & -4 \\ -1 & -2 \end{pmatrix}; A_2 = \begin{pmatrix} -2 & -4 \\ 20 & -2 \end{pmatrix}.$$

No common LF $\rightarrow$ However, one can use two quadratic LF $V_i(x) = x^\top P_i x$ with

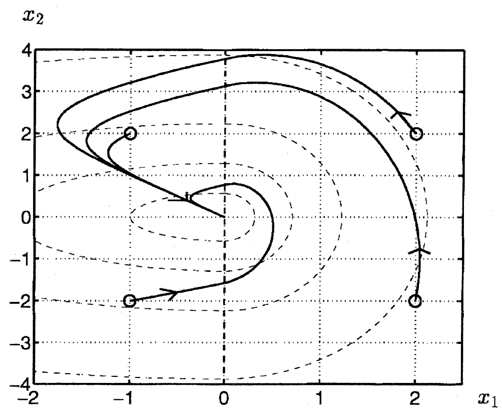
$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}; P_2 = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}.$$

V_i is LF for $\dot{x} = A_i x$ (i.e. $A_i^\top P_i + P_i A_i < 0$ and $P_i > 0$, $i = 1, 2$)



$$V(x) = \begin{cases} V_1(x) = x_1^2 + 3x_2^2, & \text{when } i = 1 \text{ is active subsystem, i.e. } x_1 \le 0 \\ V_2(x) = 10x_1^2 + 3x_2^2, & \text{when } i = 2 \text{ is active subsystem, i.e. } x_1 > 0 \end{cases}$$

which is a continuous piecewise quadratic Lyapunov function.



Relaxation 3: Lyapunov function positive definite

$$V(x) = \begin{cases} V_1(x) = x_1^2 + 3x_2^2, & \text{when } i = 1 \text{ is active subsystem, i.e. } x_1 \le 0 \\ V_2(x) = 10x_1^2 + 3x_2^2, & \text{when } i = 2 \text{ is active subsystem, i.e. } x_1 > 0 \end{cases}$$

$V(x)$ must be positive definite, but since $V(x) = V_i(x)$ when $x \in \mathscr{X}_i$ the constituting functions $V_i(x)$ should only be positive when $x \in \mathscr{X}_i$.

Hence,

$$x \in \mathscr{X}_i, x \neq 0 \Rightarrow V_i(x) > 0$$

and possibly $V_i(x) < 0$ when $x \notin \mathscr{X}_i$.

The three relaxations for piecewise linear systems

$$\dot{x} = A_i x \text{ if } x \in \mathscr{X}_i$$

There are several relaxations possible w.r.t. common quadratic LF:

- One can require that the derivative $L_{A_i x} V(x)$ of $V(x) = x^\top P_i x$ is only negative in the region where the subsystem is active.
- One can use multiple Lyapunov functions, say $V_i(x) = x^\top P_i x$, for each submodel and "connect them" in a suitable way.
- One can require that the Lyapunov function $V_i(x) = x^\top P_i x$ is only positive definite in its active region.

Two "regional conditions:"

- $x \in \mathscr{X}_i$ and $x \neq 0$ should imply $x^\top [A_i^\top P_i + P_i A_i] x < 0$
- $x \in \mathscr{X}_i$ and $x \neq 0$ should imply $x^\top P_i x > 0$

How to cope with this? S-procedure!

S-procedure

Aim: $V(x) = x^\top Px$, $P > 0$ s.t. $x^\top[A_i^\top P + PA_i]x < 0$ for $0 \neq x \in \mathscr{X}_i$.

Find: $S_i(x)$ based on $\mathscr{X}_i$ in the sense that $S_i(x) \geq 0$ when $x \in \mathscr{X}_i$

Next: search for $\beta \geq 0$ satisfying

$$x^\top A_i^\top Px + x^\top PA_i x + \beta S_i(x) < 0 \text{ for all } x$$

Result: Since $S_i(x)$ might be negative outside $\mathscr{X}_i$, less conservative than $A_i^\top P + PA_i < 0$.

Computationally interesting: $S_i(x) = x^\top S_i x$, then LMI:

find $\beta_i \geq 0$ and $P > 0$ such that

$$A_i^\top P + PA_i + \beta_i S_i < 0.$$

Example and extension

Consider the region $\mathscr{X}_i = \{x \in \mathbb{R}^2 \mid E_i x \geq 0\}$ where $E_i = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$

Take U_i a matrix with nonnegative elements, e.g.

$$U_i = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}$$

then $\tilde{S}_i(x) = x^\top E_i^\top U_i E_i x \geq 0$ when $x \in \mathscr{X}_i$

Reason: we are multiplying a row vector $x^\top E_i$ and a column vector $U_i E_i x$ with non-negative elements ...

Hence, to guarantee $x^\top[A_i^\top P_i + P_i A_i]x < 0$ when $x \in \mathscr{X}_i$, it suffices to find a matrix U_i with nonnegative elements such that

$$A_i^\top P_i + P_i A_i + E_i^\top U_i E_i < 0.$$

Second relaxation: multiple Lyapunov functions

Aim: $V(x) = V_i(x) = x^\top P_i x$ when $x \in \mathscr{X}_i$ where $V_i(x)$ is such that for each dynamics $\dot{x} = A_i x$

- $\dot{V}_i < 0$ when $x \in \mathscr{X}_i$
- V is continuous over the boundary[a].

$$\dot{x} = A_i x, \text{ when } E_i x \geq 0, \; i \in I.$$

Assumption:

- cells partition state space, no sliding modes[b]
- Let the switching planes

$$\mathscr{X}_i \cap \mathscr{X}_j \subseteq \ker H_{ij} = \{x \mid H_{ij}x = 0\} = \text{im} Z_{ij} = \{Z_{ij} v \mid v \in \mathbb{R}^{m_{ij}}\}$$

for certain matrices H_{ij} and Z_{ij}.

[a] Not needed in discrete-time case!

[b] For extensions including sliding modes, see [Heemels, Weiland, "Input-to-state stability and interconnections of discontinuous dynamical systems," Automatica 2008]

Second relaxation - continued

Observation for first relaxation: Take $S_i = E_i^\top U_i E_i$ with U_i nonnegative entries

$$A_i^\top P_i + P_i A_i + E_i^\top U_i E_i < 0$$

To guarantee continuity of the *piecewise quadratic Lyapunov function*

$$V(x) = x^\top P_i x, \text{ when } x \in \mathscr{X}_i, i.e.$$

$$x^\top P_i x = x^\top P_j x \text{ for all } x \in \mathscr{X}_i \cap \mathscr{X}_j \subseteq \text{im} Z_{ij}.$$

Hence as any $x \in \mathscr{X}_i \cap \mathscr{X}_j$ can be written as $x = Z_{ij} v$ for some v it suffices to have

$$v^\top Z_{ij}^\top P_i Z_{ij} v = v^\top Z_{ij}^\top P_j Z_{ij} v$$

or

$$Z_{ij}^\top [P_i - P_j] Z_{ij} = 0.$$

$$\mathscr{S} := \{(i,j) \in \{1,\ldots,N\} \times \{1,\ldots,N\} \mid i \neq j \text{ and } \mathscr{X}_i \cap \mathscr{X}_j \neq \{0\}\}.$$

Third relaxation and assembling the stuff!

Observation $x^\top P_i x > 0$ only when $0 \neq x \in \mathscr{X}_i$

Theorem
If one can find symmetric matrices W_i and U_i with nonnegative entries and such that P_i satisfy

$$A_i^\top P_i + P_i A_i + E_i^\top U_i E_i < 0,\ i = 1,\ldots,N \qquad \text{(decreasing in region where active)}$$

and

$$P_i - E_i^\top W_i E_i > 0,\ i = 1,\ldots,N \qquad \text{(positive in region where active)}$$

and

$$Z_{ij}^\top [P_i - P_j] Z_{ij} = 0,\ (i,j) \in \mathscr{S}. \qquad \text{(continuity)}$$

then every ordinary trajectory (without sliding modes) tends to zero exponentially.

- Linear matrix inequalities!!!
- Note that one does not have to take W_i and U_i full matrices. One could reduce the solution space by only allowing diagonal matrices.

Alternative for warranting continuity for Relaxation 2

Assumption:

- there exist matrices F_i such that $F_i x = F_j x$ for all $x \in \mathscr{X}_i \cap \mathscr{X}_j$.

To guarantee continuity of the *piecewise quadratic Lyapunov function*

$$V(x) = x^\top P_i x,\ \text{when } x \in \mathscr{X}_i, i.e.$$

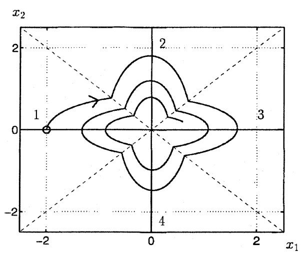
$$x^\top P_i x = x^\top P_j x \text{ for all } x \in \mathscr{X}_i \cap \mathscr{X}_j.$$

one can impose:

- $P_i = P_j$ (gives quadratic Lyapunov function)
- $P_i = P_j + F_{ij}^\top T_{ij} + T_{ij} F_{ij}$ where $F_{ij} x = 0$ when $x \in \mathscr{X}_i \cap \mathscr{X}_j$ (e.g. $F_{ij} = F_i - F_j$)
- $P_i = F_i^\top T F_i$, $i \in I$ for some symmetric matrix T.

Example of the "Flower system"

$$A_1 = A_3 = \begin{pmatrix} -0.1 & 1 \\ -5 & -0.1 \end{pmatrix};\ A_2 = A_4 = \begin{pmatrix} -0.1 & 5 \\ -1 & -0.1 \end{pmatrix}.$$

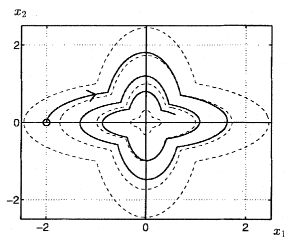


$$E_1 = -E_3 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix};\ E_2 = -E_4 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$H_{12} = H_{21} = H_{34} = H_{43} = [1\ 1]$$
$$H_{23} = H_{32} = H_{41} = H_{14} = [-1\ 1]$$
$$Z_{12} = Z_{21} = Z_{34} = Z_{43} = [-1\ 1]^\top$$
$$Z_{23} = Z_{32} = Z_{41} = Z_{14} = [1\ 1]^\top$$

Flower system - continued

$$P_1 = P_3 = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix};\ P_2 = P_4 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$



For alternative continuity guarantee, one could use $F_i := [E_i^\top I]^\top$

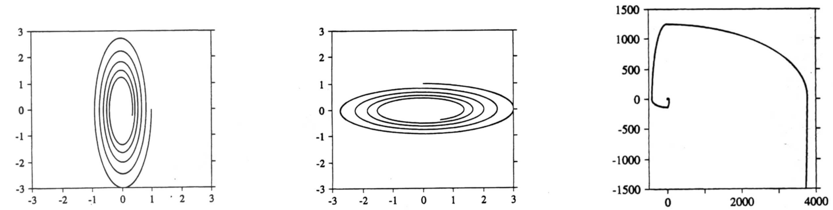
Problem B: Switched systems with dwell time conditions

Switched systems with dwell time conditions

"Dwell time:" duration / how long a system stays ("dwells") in a certain mode

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 < 0 \\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases}$$

$$A_1 = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix}; A_2 = \begin{pmatrix} -1 & 100 \\ -10 & -1 \end{pmatrix}. \text{ Eigenvalues } = -1 \pm 31.6j$$



"Dwell time too short"

Stability under slow switching for switched linear systems

Consider a switched linear system $\dot{x} = A_i x$, $i = 1, \ldots, N$, where each matrix A_i being Hurwitz.

For each $i = 1, \ldots, N$ it holds that $\|e^{A_i t} x_0\| \leq c_i e^{-\lambda_i t}\|x_0\|$, where $\lambda_i > 0$

Hence, there also exist $c > 0$ and $\lambda > 0$ such that

$$\|e^{A_i t} x_0\| \leq c e^{-\lambda t}\|x_0\|$$

This implies that if we stay long enough in each mode such that $ce^{-\lambda t} < 1$, we have global asymptotic stability.

For switching signals with minimal dwell time of at least τ_d time units with $\tau_d > \frac{1}{\lambda}\ln c$, the system is (uniformly) GAS

$\longrightarrow$ More elaborate conditions exist, also for nonlinear switched system using Lyapunov functions

E.g. average dwell time assumptions

Switched systems with average dwell time assumptions

$$\dot{x} = f_i(x) \quad i = 1, \ldots, N$$

For a switching signal $\sigma : [0, \infty) \rightarrow \{1, \ldots, N\}$ we denote the number of switchings (discontinuities) in σ in the interval (t, T) by $N_\sigma(t, T)$.

We say that σ has the average dwell time τ_a if there exists a positive number N_0 such that

$$N_\sigma(t, T) \leq N_0 + \frac{T - t}{\tau_a} \quad \text{for all } T \geq t \geq 0$$

- $N_0 = 0$: no switching
- $N_0 = 1$: σ cannot switch twice on interval of length smaller than τ_a: minimal dwell time of τ_a!

Switched systems with average dwell time assumptions

$$\dot{x} = f_i(x) \qquad i = 1, \ldots, N$$

$$N_\sigma(t,T) \leq N_0 + \frac{T-t}{\tau_a} \qquad \text{for all } T \geq t \geq 0$$

Theorem Suppose there exist C^1 functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, constants $0 < a < b$, $p \in \mathbb{N}$, $\mu \geq 1$ and $\lambda_0 > 0$ such that for all $i = 1, \ldots, N$

- $a\|x\|^p \leq V_i(x) \leq b\|x\|^p$ for all x
- $\frac{\partial V_i}{\partial x} f_i(x) \leq -\lambda_0 V_i(x)$ for all x
- $V_i(x) \leq \mu V_j(x)$ for all i, j and all x

Then the switched system is (uniformly) GAS for all σ with average dwell time

$$\tau_a > \frac{\ln \mu}{\lambda_0}$$

and N_0 arbitrary.

Question: What happens if $\mu = 1$?

Jump-flow systems

$$\begin{aligned} Flow: \quad \dot{x} &= F(x), \text{ when } x \in C \\ Jumps: \quad x^+ &= G(x), \text{ when } x \in D \end{aligned}$$

Jump-flow systems

$$\begin{aligned} Flow: \quad \dot{x} &= F(x), \text{ when } x \in C \\ Jumps: \quad x^+ &= G(x), \text{ when } x \in D \end{aligned}$$

Survey: [Goebel, Sanfelice, Teel, CSM, 2009]

Various application domains:

- Networked control systems [Heemels, Teel, vdWouw, Nesic, TAC, 2010], [Carnevale, Teel, Nesic, TAC, 2007], [Nesic, Teel, TAC, 2004]
- Reset control systems [Aangenent et al, IJRNC, 2010], [Nesic, Zaccarian, Teel, Automatica, 2008]
- Event-triggered control systems [Donkers & Heemels, TAC 2012, CDC 2010], [Borgers & Heemels, TAC 2014], [Heemels, Donkers, Teel, TAC 2013], [Heemels, Teel, Dullerud, TAC 2016], [Dolk, Borgers, Heemels, TAC 2017]
- Hybrid automata

Stability of Jump-flow systems

$$\begin{aligned} Flow: \quad \dot{x} &= F(x), \text{ when } x \in C \\ Jumps: \quad x^+ &= G(x), \text{ when } x \in D \end{aligned}$$

Definition Given a compact set $\mathscr{A} \in \mathbb{R}^n$. The function $V : \text{dom } V \to \mathbb{R}$ is a Lyapunov function candidate if

(i.) V is continuous and nonnegative on $(C \cup D) \setminus \mathscr{A} \subset \text{dom } V$,

(ii.) V is continuously differentiable on an open set $\mathscr{O}$ satisfying $C \setminus \mathscr{A} \subset \mathscr{O} \subset \text{dom } V$,

(iii.) $\lim_{x \to \mathscr{A}, x \in \text{dom } V \cap (C \cup D)} V(x) = 0$.

Stability of Jump-flow systems

$$\begin{aligned} \textit{Flow}: \quad \dot{x} &= F(x), \text{ when } x \in C \\ \textit{Jumps}: \quad x^+ &= G(x), \text{ when } x \in D \end{aligned}$$

Definition Given a compact set $\mathscr{A} \in \mathbb{R}^n$. The function $V : \text{dom } V \to \mathbb{R}$ is a Lyapunov function candidate if

(i.) V is continuous and nonnegative on $(C \cup D)\backslash\mathscr{A} \subset \text{dom } V$,

(ii.) V is continuously differentiable on an open set $\mathscr{O}$ satisfying $C\backslash\mathscr{A} \subset \mathscr{O} \subset \text{dom } V$,

(iii.) $\lim_{x \to \mathscr{A}, x \in \text{dom } V \cap (C \cup D)} V(x) = 0$.

Theorem Given a compact set $\mathscr{A} \subset \mathbb{R}^n$ satisfying $G(D \cap \mathscr{A}) \subset \mathscr{A}$. If there exists a Lyapunov function candidate V s.t.

- the sublevel sets of V on dom $V \cap (C \cup D)$ are compact
- $\langle \nabla V(x), F(x) \rangle < 0$ for all $x \in C\backslash\mathscr{A}$
- $V(G(x)) - V(x) < 0$ for all $x \in D\backslash\mathscr{A}$,

then $\mathscr{A}$ is GAS (if all solutions are forward complete).

GES of jump-flow systems:

$$\begin{aligned} \textit{Flow}: \quad \dot{x} &= F(x), \text{ when } x \in C \qquad \mathscr{A} = \{0\} \\ \textit{Jumps}: \quad x^+ &= G(x), \text{ when } x \in D \end{aligned}$$

Theorem: Suppose there exist a function $V : \mathbb{R}^n \to \mathbb{R}$, which is continuously differentiable on $\mathbb{R}^n$ s.t.

(i.) there exist $p \in \mathbb{N}$, $c_1 > 0$, $c_2 > 0$ s.t. $c_1\|x\|^p \le V(x) \le c_2\|x\|^p$ for all $x \in \mathbb{R}^n$

(ii.) there exists $c_3 > 0$ with $\dot{V}(x) = L_F V(x) := \frac{\partial V}{\partial x} F(x) \le -c_3\|x\|^p$ for all $x \in C$

(iii.) it holds that $V(G(x)) \le V(x)$ for all $x \in D$

Then 0 is GES (if all solutions are defined for all $t \in [0, \infty)$) in the sense that $\|x(t)\| \le ce^{-\lambda t}\|x_0\|$ for some $c, \lambda > 0$.

Question: But what if (iii.) does not hold?

Stability of jump-flow systems: Time-dependent jumping

$$\begin{aligned} \textit{Flow}: \quad \dot{x}(t) &= F(x(t)), \text{ when } t \neq t_i, \; i = 0,1,2,\dots \\ \textit{Jumps}: \quad x(t^+) &= G(x(t)), \text{ when } t = t_i, \; i = 0,1,2,\dots \end{aligned}$$

- Jump times $0 \le t_0 < t_1 < t_2 < t_3 < \dots$
- For $0 \le t < T$ we denote by $N(t,T)$ the number of jump times in $(t,T]$.

Suppose there exist a function $V : \mathbb{R}^n \to \mathbb{R}$, which is continuously differentiable on $\mathbb{R}^n$ and satisfies:

(i.) there exist $p \in \mathbb{N}$, $c_1 > 0$, $c_2 > 0$ s.t. $c_1\|x\|^p \le V(x) \le c_2\|x\|^p$ for all $x \in \mathbb{R}^n$

(ii.) there exists $c \in \mathbb{R}$ such that $\dot{V}(x) \le -cV(x)$ for all $x \in \mathbb{R}^n$

(iii.) there exists $d \in \mathbb{R}$ such that $V(G(x)) \le e^{-d}V(x)$ for all $x \in \mathbb{R}^n$

Question: now what?

Stability of jump-flow systems: Time-dependent jumping

$$\begin{aligned} \textit{Flow}: \quad \dot{x}(t) &= F(x(t)), \text{ when } t \neq t_i, \; i = 0,1,2,\dots \\ \textit{Jumps}: \quad x(t^+) &= G(x(t)), \text{ when } t = t_i, \; i = 0,1,2,\dots \end{aligned}$$

- Jump times $0 \le t_0 < t_1 < t_2 < t_3 < \dots$
- For $0 \le t < T$ we denote by $N(t,T)$ the number of jump times in $(t,T]$.

Terminology:

- We say that the jump times $\{t_i\}_{i\in\mathbb{N}}$ satisfy the minimal average inter-jump time condition for $\tau^* > 0$ and $N_0 > 0$ ($\{t_i\}_{i\in\mathbb{N}} \in \mathscr{S}_{\text{avg}}[\tau^*, N_0]$), if for all $T > t \ge 0$

$$N(t,T) \le N_0 + \frac{T-t}{\tau^*}$$

"on average, at most one jump per interval of length τ^*"

- We say that the jump times $\{t_i\}_{i\in\mathbb{N}}$ satisfy the maximal average inter-jump time condition for $\tau^* > 0$ and $N_0 > 0$ ($\{t_i\}_{i\in\mathbb{N}} \in \mathscr{S}_{\text{r-avg}}[\tau^*, N_0]$), if for all $T > t \ge 0$

$$N(t,T) \ge -N_0 + \frac{T-t}{\tau^*}$$

"on average, at least one jump per interval of length τ^*"

Stability of jump-flow systems: Time-dependent jumping

$$\begin{aligned} \textit{Flow}: \quad \dot{x}(t) &= F(x(t)), \text{ when } t \neq t_i,\ i = 0,1,2,\ldots \\ \textit{Jumps}: \quad x(t^+) &= G(x(t)), \text{ when } t = t_i,\ i = 0,1,2,\ldots \end{aligned}$$

Suppose there exist a function $V : \mathbb{R}^n \to \mathbb{R}$, which is continuously differentiable on $\mathbb{R}^n$ and satisfies:

(i.) there exist $p \in \mathbb{N}$, $c_1 > 0$, $c_2 > 0$ s.t. $c_1\|x\|^p \leq V(x) \leq c_2\|x\|^p$ for all $x \in \mathbb{R}^n$

(ii.) there exists $c \in \mathbb{R}$ such that $\dot{V}(x) \leq -cV(x)$ for all $x \in \mathbb{R}^n$

(iii.) there exists $d \in \mathbb{R}$ such that $V(G(x)) \leq e^{-d}V(x)$ for all $x \in \mathbb{R}^n$

Theorem The jump-flow system is GES under the following conditions

- When $d < 0$ and $c > 0$ and $\{t_i\}_{i\in\mathbb{N}} \in \mathscr{S}_{\text{avg}}[\tau^*, N_0]$ for all $\tau^* > \frac{|d|}{c}$ and any $N_0 > 0$
- When $d > 0$ and $c < 0$ and $\{t_i\}_{i\in\mathbb{N}} \in \mathscr{S}_{\text{r-avg}}[\tau^*, N_0]$ for all $\tau^* < \frac{d}{|c|}$ and any $N_0 > 0$

Summary of stability of HS

- Stability of submodels $\not\Rightarrow$ stability! "Hybrid problem"
- Problem A
  - UGAS under arbitrary switching
  - Common Lyapunov function
- Problem B: state-dependent switching
  - PWL systems
  - Continuous piecewise quadratic (PWQ) Lyapunov functions (3 relaxations)
- Problem B: time-dependent switching
  - All subsystems GAS
  - Minimal and average dwell time restrictions
- Jump-flow systems: Problem A and B (state-based and time-based)