Hybrid control

Maurice Heemels
Control Systems Technology group
Department of Mechanical Engineering
Eindhoven University of Technology
m.heemels@tue.nl

DISC Course on Modeling and Control of Hybrid Systems 2017

Outline of lecture 3

- Problem C: Construct a stabilizing switching sequence, a discrete control problem
  - State-dependent switching
  - Time-dependent switching
- Continuous (and discrete) control problems
- Observer design
- Summary

Problem C ...

Problem A : Find conditions for which the switched system is UGAS for any switching signal.

Problem B : Show that the switched system is GAS for a given switching strategy or a class of switching strategies.

Problem C : Construct a switching signal that makes the switched system GAS (i.e. a stabilization problem).

State-based switching

Stabiliz. switched linear systems via suitable switching (Pr. C)

$$\dot{x} = A_i x, \quad i \in I := \{1, 2, \ldots, N\}$$

Find switching rule σ as function of time / state such that closed loop is asymptotically stable.

Quadratic stabilization via a single Lyapunov function

Select $\sigma(x) : \mathbb{R}^n \to I := \{1, 2, \ldots, N\}$ s.t. closed loop has single quadratic Lyapunov function $x^T P x$.

One solution: convex combination of A_i is stable

$$A := \sum \alpha_i A_i \quad (\alpha_i \geq 0, \ \sum \alpha_i = 1) \text{ is stable}$$

Select $Q > 0$ and let $P > 0$ be solution of $A^T P + PA = -Q$.

Quadratic stabilization - continued

From $x^T(A^TP+PA)x = -x^TQx < 0$ it follows that

$$\sum_i \alpha_i[x^T(A_i^TP+PA_i)x] \;<\; 0.$$

- For each x there is at least one mode with $x^T(A_i^TP+PA_i)x<0$ or stronger

$$\bigcup_{i\in I} \underbrace{\{x \mid x^T(A_i^TP+PA_i)x \le -x^TQx\}}_{\mathscr{X}_i} \;=\; \mathbb{R}^n$$

- Switching rule:

$$i(x) \;:=\; \arg\min x^T(A_i^TP+PA_i)x$$

- Leads possibly to sliding modes. Alternative?

Alternative switching rule for quadratic stabilization

- A modified switching rule (based on hysteresis switching logic):

  ⋆ stay in mode i as long as $x^T(A_i^TP+PA_i)x \le -\rho x^TQx$, with $0<\rho<1$.

  ⋆ when bound reached, switch to a new mode j that satisfies

$$x^T(A_j^TP+PA_j)x \le -x^TQx.$$

- There is a lower bound on the duration in each mode!

Theorem 1 If there exists a quadratically stabilizing state-dependent switching law for the switched linear system with $N=2$, then the matrices A_1 and A_2 have a Hurwitz convex combination.

Stabilization via multiple Lyapunov functions (Problem C)

Main idea: Find $V_i(x) = x^TP_ix$ that decreases for $\dot{x} = A_ix$ in some region.

Define $\mathscr{X}_i := \{x \mid x^T[A_i^TP_i + P_iA_i]x < 0\}$.

If $\mathscr{X}_1 \cup \mathscr{X}_2 = \mathbb{R}^n$, try to switch to satisfy multiple Lyapunov criterion to guarantee asymptotic stability.

Find P_1 and P_2 such that they satisfy the coupled conditions:

$$x^T(P_1A_1 + A_1^TP_1)x < 0 \text{ when } x^T(P_1-P_2)x \ge 0,\ x \ne 0$$

and

$$x^T(P_2A_2 + A_2^TP_2)x < 0 \text{ when } x^T(P_2-P_1)x \ge 0,\ x \ne 0.$$

Then $\sigma(t) = \arg\max\{V_i(x(t)) \mid i=1,2\}$ stabilizing (V_σ = continuous)

... S-procedure ...

S-procedure There exist $\beta_1, \beta_2 \ge 0$ such that

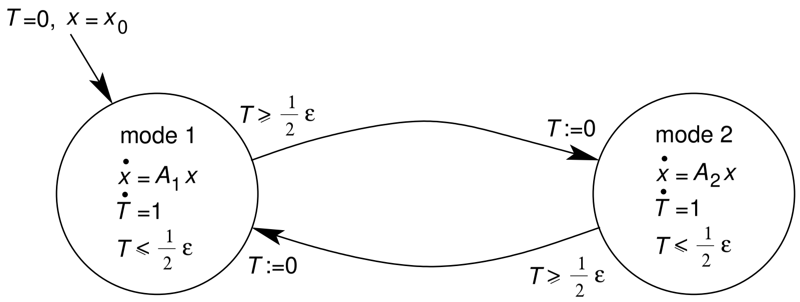
$$-P_1A_1 - A_1^TP_1 + \beta_1(P_2-P_1) > 0$$
$$-P_2A_2 - A_2^TP_2 + \beta_2(P_1-P_2) > 0$$

$\sigma(t) = \arg\min\{V_i(x(t)) \mid i=1,2\}$ when you can find $\beta_1, \beta_2 \le 0$

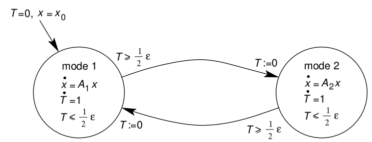
$\longrightarrow$ state-based switching previously ... now ...

Time-controlled switching / pulse width modulation

If dynamical system switches between several subsystems
$\rightarrow$ stability properties of total system may be quite different from those of subsystems



Time-controlled switching



- $x(t_0+\frac{1}{2}\varepsilon) = \exp(\frac{1}{2}\varepsilon A_1)x_0 \;=\; x_0+\frac{\varepsilon}{2}A_1x_0+\frac{\varepsilon^2}{8}A_1^2x_0+\cdots$

$$x(t_0+\varepsilon) = (I+\tfrac{\varepsilon}{2}A_2+\tfrac{\varepsilon^2}{8}A_2^2+\cdots)(I+\tfrac{\varepsilon}{2}A_1+\tfrac{\varepsilon^2}{8}A_1^2+\cdots)x_0$$
$$= (I+\varepsilon[\tfrac{1}{2}A_1+\tfrac{1}{2}A_2]+\tfrac{\varepsilon^2}{8}[A_1^2+A_2^2+2A_2A_1]+\cdots)x_0.$$

- Compare with

$$\exp[\varepsilon(\tfrac{1}{2}A_1+\tfrac{1}{2}A_2)] \;=\; I+\varepsilon[\tfrac{1}{2}A_1+\tfrac{1}{2}A_2]+\tfrac{\varepsilon^2}{8}[A_1^2+A_2^2+A_1A_2+A_2A_1]+\cdots$$

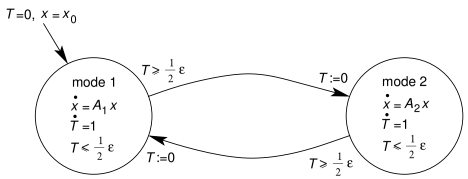
$\longrightarrow$ same for $\varepsilon\approx 0$

- So for $\varepsilon\rightarrow 0$ solution of switched system tends to solution of

$$\dot{x} \;=\; (\tfrac{1}{2}A_1+\tfrac{1}{2}A_2)x \qquad \text{(“averaged” system)}$$

- Possible that A_1 and A_2 are Hurwitz, whereas matrix $\frac{1}{2}A_1+\frac{1}{2}A_2$ is not Hurwitz, or vice versa.

Example

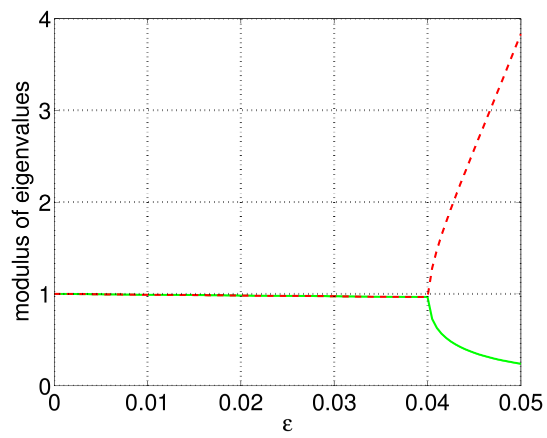


- Consider

$$A_1 \;=\; \begin{bmatrix} -0.5 & 1 \\ 100 & -1 \end{bmatrix}, \quad A_2 \;=\; \begin{bmatrix} -1 & -100 \\ -0.5 & -1 \end{bmatrix}$$

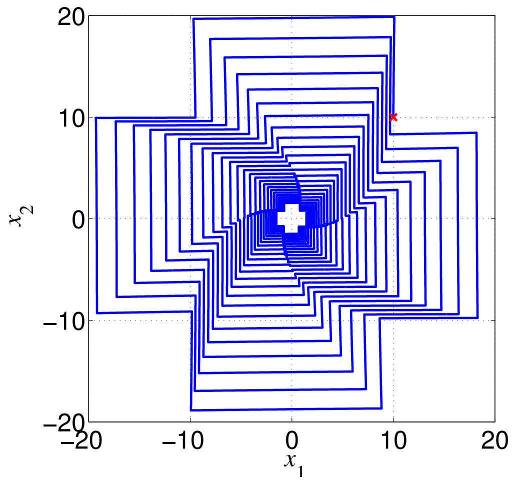
- A_1, A_2 not Hurwitz, but matrix $\frac{1}{2}(A_1+A_2)$ is Hurwitz
  $\rightarrow$ switched system should be stable if frequency of switching is sufficiently high
- Minimal switching frequency found by computing eigenvalues of the mapping $\exp(\frac{1}{2}\varepsilon A_2)\exp(\frac{1}{2}\varepsilon A_1)$ (Why?)

Example (cont.)



$\rightarrow$ maximal value of ε: 0.04 ($50Hz$)

Example (cont.)



Conclusions

- Stability of switched systems
- Problem A: UGAS under arbitrary switching: common Lyapunov functions
- Problem B: Stability under particular switching strategies
  - State-dependent switching (PWL): continuous PWQ Lyap. functions
  - Time-dependent switching: minimal or average dwell time
  - Systems with jumps: jump-flow or impulsive systems
- Problem C: Design of stabilizing switching signals:
  - State-dependent and time-dependent switching design
- In case of switched linear systems LMIs a helpful tool!!

Conclusions

- Stability of switched systems
- Problem A: UGAS under arbitrary switching: common Lyapunov functions
- Problem B: Stability under particular switching strategies
  - State-dependent switching (PWL): continuous PWQ Lyap. functions
  - Time-dependent switching: minimal or average dwell time
  - Systems with jumps: jump-flow or impulsive systems
- Problem C: Design of stabilizing switching signals:
  - State-dependent and time-dependent switching design
- In case of switched linear systems LMIs a helpful tool!!

- Next: include continuous control inputs!

Only find switching sequence (discrete inputs)! What if also continuous inputs are present?

Stabilization of switched linear systems with continuous inputs

Switched linear system with inputs:

$$\dot{x} = A_i x + B_i u,\ i \in I = \{1,\ldots,N\}$$

Now $\sigma : [0,\infty) \to I$ and feedback controllers $u = K_i x$ are to determined.

Case 1: Determine K_i such that closed loop UGAS under arbitrary switching (assuming **known** mode)!

Case 2: Determine both $\sigma : [0,\infty) \to I$ and K_i

Case 3: σ given as function of state (PWL). Determine K_i

Case 1: Stabiliz. of switched linear system under arb. switching

$$\dot{x} = A_i x + B_i u,\ i \in I = \{1,\dots,N\}$$

Sufficient condition: find a common *quadratic* Lyapunov function $V(x) = x^T P x$ for some positive definite matrix P and $K_1,\dots,K_N$.

$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) < 0 \text{ for all } i = 1,\dots,N \text{ and } P > 0$$

$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) < 0 \text{ for all } i = 1,\dots,N \text{ and } P > 0$$

Pre- and postmultiplying by P^{-1}:

$$P^{-1}(A_i + B_i K_i)^T + (A_i + B_i K_i)P^{-1} < 0 \text{ for all } i = 1,\dots,N \text{ and } P^{-1} > 0$$

Linear Matrix Inequalities

$$ZA_i^T + A_i Z + Y_i^T B_i^T + B_i Y_i < 0 \text{ for all } i = 1,\dots,N \text{ and } Z > 0,$$

$P^{-1} =: Z$ and $K_i P^{-1} =: Y_i$. Hence, $P = Z^{-1}$ and $K_i = Y_i Z^{-1}$.

Hence, if LMIs feasible, then $u = K_i x$ leads to UGAS “cloop” under arbitrary switching **knowing** the mode as we use $u = K_i x$ when subsystem i is active!

Case 2: Design of switched feedback and switching sequence

$$\dot{x} = A_i x + B_i u,\ i \in I = \{1,2\}$$

Determine $\sigma : [0,\infty) \to I$ and $u = K_i x$, $i = 1,\dots,N$

Use previous conditions for finding switching sequence

i) Find K_1, K_2 and $\alpha \in [0,1]$ such that $\alpha(A_1 + B_1 K_1) + (1-\alpha)(A_2 + B_2 K_2)$ is stable, i.e.

$$[\alpha(A_1 + B_1 K_1) + (1-\alpha)(A_2 + B_2 K_2)]^T P + P[\alpha(A_1 + B_1 K_1) + (1-\alpha)(A_2 + B_2 K_2)] < 0.$$

For fixed α previous transformation leads to LMIs!

ii) Find $\beta_1 \geq 0$, $\beta_2 \geq 0$, P_1 and P_2 positive definite and gains K_1 and K_2 such that

$$-P_1(A_1 + B_1 K_1) - (A_1 + B_1 K_1)^T P_1 + \beta_1(P_2 - P_1) > 0$$

$$-P_2(A_2 + B_2 K_2) - (A_2 + B_2 K_2)^T P_2 + \beta_2(P_1 - P_2) > 0.$$

Case 3: Design of switched feedback

If switching structure has already been given

$$\dot{x} = A_i x + B_i u, \text{ when } x \in \mathscr{X}_i,$$

$\bigcup_{i=1}^N \mathscr{X}_i = \mathbb{R}^n$ and $\mathscr{X}_i \cap \mathscr{X}_j$ for $i \neq j$ is a (lower-dimensional) boundary.

If $u = K_i x$ when $x \in \mathscr{X}_i$ we obtain closed-loop dynamics

$$\dot{x} = (A_i + B_i K_i)x, \text{ when } x \in \mathscr{X}_i$$

$\longrightarrow V(x) = x^T P x$ $\quad\quad$ $\mathscr{X}_i \subseteq \{x \mid E_i x \geq 0\}$

Find $K_1,\dots,K_N$, $P > 0$ and symmetric U_i with nonnegative entries s.t.

$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) + E_i^T U_i E_i < 0,\ i = 1,\dots,N$$

• Extensions via continuous PWQ Lyapunov functions (BMIs!)
Also discrete-time results!!!

Discrete-time case

$$x_{k+1} = A_i x_k + B_i u_k,\ i \in I = \{1,\dots,N\}$$

Goal: construct switched state feedback $u_k = K_i x_k$, $i \in I = \{1,\dots,N\}$ that stabilizes the closed-loop systems under arbitrary switching (with known mode):

$$x_{k+1} = (A_i + B_i K_i) x_k,\ i \in I = \{1,\dots,N\}$$

Sufficient: find a common *quadratic* Lyapunov function $V(x) = x^T P x$ for positive definite P and $K_1,\dots,K_N$.

$$V(x_{k+1}) - V(x_k) < 0,\ \text{when } x_k \neq 0,\ \text{ i.e.}$$

$$(A_i + B_i K_i)^T P (A_i + B_i K_i) - P < 0,\ i = 1,\dots,N \text{ and } P > 0$$

The free variables K_i and P appear not linearly?

What to do?

Use Schur complements

Schur Complement For all $X \in \mathbb{S}^n, Y \in \mathbb{R}^{m\times n}, Z \in \mathbb{S}^m$, the following statements are equivalent:

a) $Z \succ \Theta,\quad X - Y^T Z^{-1} Y \succ \Theta.$

b) $\begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} \succ \Theta.$

a) $Z \succ \Theta,\quad X - Y^T Z^{-1} Y \succeq \Theta.$

b) $Z \succ \Theta,\quad \begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} \succeq \Theta.$

Proof Assume $Z \succ \Theta$. The *nonsingular matrix*

$$T = \begin{bmatrix} I & \Theta \\ -Z^{-1}Y & I \end{bmatrix}$$

establishes the congruence transformation

$$T^T \begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} T = \begin{bmatrix} X - Y^T Z^{-1} Y & \Theta \\ \Theta & Z \end{bmatrix} \succ \Theta (\succeq \Theta).$$

Applying Schur complements to:

$$P - (A_i + B_i K_i)^T P (A_i + B_i K_i) > 0,\ i = 1,\dots,N \text{ and } P > 0$$

$$\begin{pmatrix} P & (A_i + B_i K_i)^T \\ A_i + B_i K_i & P^{-1} \end{pmatrix} > 0,\ i = 1,\dots,N$$

Pre- and postmultiply now by $\begin{pmatrix} P^{-1} & 0 \\ 0 & I \end{pmatrix}$ yielding

$$\begin{pmatrix} P^{-1} & P^{-1}(A_i + B_i K_i)^T \\ (A_i + B_i K_i) P^{-1} & P^{-1} \end{pmatrix} > 0,\ i = 1,\dots,N$$

Using the linearizing change of variables $P^{-1} =: Z$ and $K_i P^{-1} =: Y_i$ gives LMIs:

$$\begin{pmatrix} Z & Z A_i^T + Y_i^T B_i^T \\ A_i Z + B_i Y_i & Z \end{pmatrix} > 0,\ i = 1,\dots,N$$

In discrete-time one does not need common quadratic Lyapunov function for GAS under arbitrary switching (BZ)

Summary: Stabilization of switched systems with continuous inputs

$$\dot{x} = A_i x + B_i u,\ i \in I = \{1,\dots,N\}$$

Now $\sigma : [0,\infty) \to I$ and feedback controllers $u = K_i x$ are to determined.

$$\dot{x} = (A_i + B_i K_i) x,\ i \in I = \{1,\dots,N\}$$

Case 1: Determine K_i such that closed loop stable under arbitrary switching.

$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) < 0 \text{ for all } i = 1,\dots,N \text{ and } P > 0$$

Case 2: Determine both $\sigma : [0,\infty) \to I$ and K_i

$$[\alpha(A_1 + B_1K_1) + (1-\alpha)(A_2 + B_2K_2)]^T P + P[\alpha(A_1 + B_1K_1) + (1-\alpha)(A_2 + B_2K_2)] < 0.$$

or arg-max based approach.

Case 3: σ given as function of state (PWL). Determine K_i

$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) + E_i^T U_i E_i < 0,\ i = 1,\dots,N$$

Also discrete-time results!!!

Transforming nonlinear MI into LMIs

"Tricks: "

- Pre- and postmultiplying by suitable invertible matrices S^T and S

$$P > 0 \text{ iff } S^T P S > 0$$

- Apply Schur complements
- Change of variables
- Combinations

Summary

- Problem C: Construct a stabilizing switching sequence, a discrete control problem
  - State-dependent switching
    - ∗ Find convex combination that is Hurwitz: single LF
    - ∗ Multiple LF approach, "max"-switching law
  - Time-dependent switching based on Hurwitz convex combination
- Continuous control problem
  - construct K_i for all σ: common P via LMIs!
  - construct K_i and σ: use top 2 approaches (almost LMI for single LF)!
  - construct K_i given σ (PWL): use conditions from stability analysis (BMIs)!

Observer design

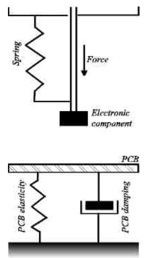
Problem statement

- Consider the system:

$$\dot{x} = \begin{cases} A_1 x + Bu, & \text{if } H^T x \le 0 \\ A_2 x + Bu, & \text{if } H^T x > 0 \end{cases}$$
$$y = Cx,$$

Goal: Design an observer that gives the state estimate $\hat{x}$, using only u,y as inputs

Fig. 1. Fast component mounter (courtesy of Assembleon).



Linear case

$$\begin{aligned}\dot{x} &= Ax + Bu \quad x(0) = x_0\\ y &= Cx\end{aligned}$$

Observer: copy of the system and output injection term

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \quad \hat{x}(0) = \hat{x}_0\\ \hat{y} &= C\hat{x}\end{aligned}$$

Estimated state $\hat{x}$ and observation error $e := x - \hat{x}$

$$\dot{e} = (A - LC)e$$

GAS ($e(t) \to 0$ when $t \to \infty$), when $A - LC$ Hurwitz or, equivalently

$$P > 0 \text{ and } (A - LC)^T P + P(A - LC) < 0 \text{ has a solution}$$

Note that this is equivalent to (A, C) being detectable (sufficient: observable)

Question: Is this a LMI? Why (not)?
Question: How can we influence the decrease rate of e?

Switched linear systems with known mode ...

$$\begin{aligned}\dot{x} &= A_{\sigma(t)}x + B_{\sigma(t)}u\\ y &= C_{\sigma(t)}x \qquad \sigma(t) \in \{1, 2, \ldots N\} \text{ known but arbitrary}\end{aligned}$$

Observer
$$\begin{aligned}\dot{\hat{x}} &= A_{\sigma(t)}\hat{x} + B_{\sigma(t)}u + L_{\sigma(t)}(y - \hat{y})\\ y &= C_{\sigma(t)}\hat{x}\end{aligned}$$

When mode is known...

$$\begin{aligned}\dot{x} &= A_{\sigma(t)}x + B_{\sigma(t)}u\\ y &= C_{\sigma(t)}x \qquad \sigma(t) \in \{1, 2, \ldots N\} \text{ known but arbitrary}\end{aligned}$$

Observer
$$\begin{aligned}\dot{\hat{x}} &= A_{\sigma(t)}\hat{x} + B_{\sigma(t)}u + L_{\sigma(t)}(y - \hat{y})\\ y &= C_{\sigma(t)}\hat{x}\end{aligned}$$

Observation error $e := x - \hat{x}$

$$\dot{e} = (A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)})e$$

Find common Lyap. function $V(e) = e^T P e$ s.t. $\dot{V} < 0$

$$(A_i - L_i C_i)^T P + P(A_i - L_i C_i) < 0 \text{ and } P > 0$$

Often mode is unknown ...

$$\dot{x} = \begin{cases} A_1 x + Bu, & \text{if } H^T x \le 0\\ A_2 x + Bu, & \text{if } H^T x > 0\end{cases}$$
$$y = Cx,$$

- Proposed observer

$$\dot{\hat{x}} = \begin{cases} A_1\hat{x} + Bu + L_1(y - \hat{y}), & \text{if } H^T\hat{x} \le 0\\ A_2\hat{x} + Bu + L_2(y - \hat{y}), & \text{if } H^T\hat{x} > 0\end{cases}$$
$$\hat{y} = C\hat{x}$$

- observation error $e = x - \hat{x}$

$$\dot{e} = \begin{cases} (A_1 - L_1 C)e, & H^T x \le 0, \quad H^T x - H^T e \le 0\\ (A_1 - L_1 C)e - \Delta A x, & H^T x > 0, \quad H^T x - H^T e \le 0\\ (A_2 - L_2 C)e + \Delta A x, & H^T x \le 0, \quad H^T x - H^T e > 0\\ (A_2 - L_2 C)e, & H^T x > 0, \quad H^T x - H^T e > 0,\end{cases}$$

where $\Delta A := A_1 - A_2$

We have N^2 modes in error dynamics because of inclusion of mixed modes

Stabilization of error dynamics

Use a Lyapunov function of the form

$$V(e) = e^T P e, \quad P = P^T > 0$$

and demand $\dot{V} \leq -\mu e^T e$, which yields

- $e^T\{(A_1 - L_1C)^T P + P(A_1 - L_1C) + \mu I\}e \leq 0$,

  when $H^T x \leq 0, H^T(x-e) \leq 0$,

- $\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I & P\Delta A \\ \Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$

  when $H^T x \leq 0, H^T(x-e) \geq 0$,

- $\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_1 - L_1C)^T P + P(A_1 - L_1C) + \mu I & -P\Delta A \\ -\Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$

  when $H^T x \geq 0, H^T(x-e) \leq 0$,

- $e^T\{(A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I\}e \leq 0$

  when $H^T x \geq 0, H^T(x-e) \geq 0$

S-procedure: incorporating regional info

- by requiring

  $$\dot{V} \leq -\mu e^T e \quad \text{everywhere}$$

  global exponential stability of e is achieved

- from $H^T x \leq 0$ and $H^T(x-e) \geq 0$ we have $x^T HH^T(x-e) \leq 0$ or

  $$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$$

- from $H^T x \geq 0$ and $H^T(x-e) \geq 0$ we have $x^T HH^T(x-e) \leq 0$ or

  $$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \geq 0$$

- previous condition can be used to relax requirements on $\dot{V}$ using S-procedure:

  $$x^T S x \geq 0 \Rightarrow x^T T x \geq 0$$

S-procedure

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$$

should imply

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I & P\Delta A \\ \Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$$

Hence, it is sufficient to find $\lambda \geq 0$

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I & P\Delta A \\ \Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq \lambda \begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix}$$

S-procedure

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$$

should imply

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I & P\Delta A \\ \Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0$$

Hence, it is sufficient to find $\lambda \geq 0$

$$\begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I & P\Delta A \\ \Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq \lambda \begin{bmatrix} e \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix}$$

Theorem [Juloski, Heemels, Weiland, IJRNC 2007] If there exist L_1, L_2 and $\lambda \geq 0$, $\mu > 0$ and $P = P^T > 0$ such that

$$\begin{bmatrix} (A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I & P\Delta A + \lambda\frac{1}{2}HH^T \\ \Delta A^T P + \lambda\frac{1}{2}HH^T & -\lambda HH^T \end{bmatrix} \leq 0$$

$$\begin{bmatrix} (A_1 - L_1C)^T P + P(A_1 - L_1C) + \mu I & -P\Delta A + \lambda\frac{1}{2}HH^T \\ -\Delta A^T P + \lambda\frac{1}{2}HH & -\lambda HH^T \end{bmatrix} \leq 0$$

then the error dynamics is exponentially stable.

Question: what happened to the other (non-mixed) modes?

Main result

Theorem If there exist L_1, L_2 and $\lambda \geq 0$, $\mu > 0$ and $P = P^T > 0$ such that

$$\begin{bmatrix} (A_2-L_2C)^T P + P(A_2-L_2C)+\mu I & P\Delta A + \lambda\frac{1}{2}HH^T \\ \Delta A^T P + \lambda \frac{1}{2} HH^T & -\lambda HH^T \end{bmatrix} \leq 0$$

$$\begin{bmatrix} (A_1-L_1C)^T P + P(A_1-L_1C)+\mu I & -P\Delta A + \lambda\frac{1}{2}HH^T \\ -\Delta A^T P + \lambda \frac{1}{2} HH & -\lambda HH^T \end{bmatrix} \leq 0$$

then the error dynamics is exponentially stable.

- Only works for continuous PWL systems

$$H^T x = 0 \quad \Rightarrow \quad A_1 x = A_2 x$$

which implies that $A_2 = A_1 + GH^T$ and thus

$$\dot{x} = A_1 x + G \max(H^T x, 0) + Bu$$

- Absolute stability theory / Popov and circle criteria
- Exploiting continuity and common observer gain $L_1 = L_2$ simpler LMIs
- Similar results for discrete-time systems [Juloski, Heemels, Weiland, IJRNC 2007]
- What can you do when system discontinuous (recover mode, make effect x on e small) [Heemels, Weiland, Juloski, HSCC 2007]
- For systems with friction-like characteristics, see [Doris et al, CST 2008], [De Bruijn et al, Automatica, 2009], [Brogliato, Heemels, TAC09], etc.

Continuous PWA system and common gain

$$\dot{x} = \begin{cases} A_1 x + Bu, & \text{if } H^T x \leq 0 \\ A_2 x + Bu, & \text{if } H^T x > 0 \end{cases}$$
$$y = Cx,$$

- Proposed observer

$$\dot{\hat{x}} = \begin{cases} A_1 \hat{x} + Bu + L(y-\hat{y}), & \text{if } H^T \hat{x} \leq 0 \\ A_2 \hat{x} + Bu + L(y-\hat{y}), & \text{if } H^T \hat{x} > 0 \end{cases}$$
$$\hat{y} = C\hat{x}$$

- Observation error $e = x - \hat{x}$ and $\Delta A := A_1 - A_2$

$$\dot{e} = \begin{cases} (A_1-LC)e, & H^T x \leq 0, \; H^T x - H^T e \leq 0 \\ (A_1-LC)e - \Delta A x, & H^T x > 0, \; H^T x - H^T e \leq 0 \\ (A_2-LC)e + \Delta A x, & H^T x \leq 0, \; H^T x - H^T e > 0 \\ (A_2-LC)e, & H^T x > 0, \; H^T x - H^T e > 0, \end{cases}$$

Theorem [Pavlov et al, book 2005] Suppose there exist $P > 0$ and observer gain L such that $(A_i - LC)^T P + P(A_i - LC) < 0 \quad i = 1,2$, then the error dynamics is exponentially stable

- discrete-time case: [Heemels et al, CDC 2008]

Summary

- Problem C: Construct a stabilizing switching sequence, a discrete control problem
  - State-dependent switching
    - Find convex combination that is Hurwitz: single LF
    - Multiple LF approach, “max”-switching law
  - Time-dependent switching based on Hurwitz convex combination
- Continuous control problem
  - construct K_i for all σ: common P via LMIs!
  - construct K_i and σ: use top 2 approaches (almost LMI for single LF)!
  - construct K_i given σ (PWL): use conditions from stability analysis (BMIs)!
- Observer design
- No complete systematic controller design (except optimization-based, but own problems)
- Open research area ....
- ... also identification, observer design, etc.: see final chapter for further reading!