Recasting the MPC problem for MLD systems as a mixed-integer quadratic programming problem

To illustrate how the MLD-MPC problem can be recast into a mixed-integer quadratic programming problem (see equations (6.22)–(6.23) on p. 112 of the lecture notes), we consider the following simple MLD system with no output, without any real-valued auxiliary variable, and with a scalar input, state, and binary auxiliary variable:

$$x(k+1) = \alpha x(k) + \beta u(k) + \gamma \delta(k) \tag{1}$$

$$ax(k) + bu(k) + c\delta(k) \leqslant d \quad . \tag{2}$$

In addition, let the MPC objective function be

$$J(k) = \|\tilde{x}(k) - \tilde{x}_{\mathrm{eq}}\|_P^2 + \|\tilde{u}(k) - \tilde{u}_{\mathrm{eq}}\|_Q^2 + \|\tilde{\delta}(k) - \tilde{\delta}_{\mathrm{eq}}\|_R^2$$

with $N_{\mathrm{p}} = 3$ and

$$\tilde{x}(k) = \begin{bmatrix} \hat{x}(k+1|k) \\ \hat{x}(k+2|k) \\ \hat{x}(k+3|k) \end{bmatrix}, \quad \tilde{\delta}(k) = \begin{bmatrix} \hat{\delta}(k|k) \\ \hat{\delta}(k+1|k) \\ \hat{\delta}(k+2|k) \end{bmatrix}, \quad \tilde{u}(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ u(k+2) \end{bmatrix},$$

with $\tilde{x}_{\mathrm{eq}}$, $\tilde{u}_{\mathrm{eq}}$, and $\tilde{\delta}_{\mathrm{eq}}$ being the equilibrium values of $\tilde{x}(k)$, $\tilde{u}(k)$, and $\tilde{\delta}(k)$ respectively, and P, Q, and R being positive definite matrices. In addition, let us assume that $N_{\mathrm{c}} = N_{\mathrm{p}}$.
Now the aim is to write the given MLD-MPC optimization problem in an explicit form. Note that at time step k the state $x(k)$ is given and $u(k)$, $u(k+1)$, and $u(k+2)$ are the independent decision variables.

The first step is to express $\hat{x}(k+1|k)$, $\hat{x}(k+2|k)$, $\hat{x}(k+3|k)$, and $\hat{\delta}(k|k)$, $\hat{\delta}(k+1|k)$, $\hat{\delta}(k+2|k)$ as a function of $x(k)$ and $u(k)$, $u(k+1)$, and $u(k+2)$.
We first use (2) to determine $\hat{\delta}(k|k)$:

$$ax(k) + bu(k) + c\hat{\delta}(k|k) \leqslant d \quad . \tag{3}$$

Next, we use (1) to determine $\hat{x}(k+1|k)$:

$$\hat{x}(k+1|k) = \alpha x(k) + \beta u(k) + \gamma \hat{\delta}(k|k) \quad . \tag{4}$$

Subsequently, we again use (2), this time to determine $\hat{\delta}(k+1|k)$:

$$a\hat{x}(k+1|k) + bu(k+1) + c\hat{\delta}(k+1|k) \leqslant d \quad .$$

In this inequality $\hat{x}(k+1|k)$ can be eliminated using (4):

$$a\alpha x(k) + a\beta u(k) + bu(k+1) + a\gamma\hat{\delta}(k|k) + c\hat{\delta}(k+1|k) \leqslant d \quad . \tag{5}$$

Next, we use (1) to determine $\hat{x}(k+2|k)$:

$$\hat{x}(k+2|k) = \alpha\hat{x}(k+1|k) + \beta u(k+1) + \gamma\hat{\delta}(k+1|k) \quad .$$

In this equation $\hat{x}(k+1|k)$ can be eliminated using (4):

$$\hat{x}(k+2|k) = \alpha^2 x(k) + \alpha\beta u(k) + \beta u(k+1) + \alpha\gamma\hat{\delta}(k|k) + \gamma\hat{\delta}(k+1|k) \quad . \tag{6}$$

Next, we use (2) to determine $\hat{\delta}(k+2|k)$:

$$a\hat{x}(k+2|k) + bu(k+2) + c\hat{\delta}(k+2|k) \leqslant d \quad .$$

In this inequality $\hat{x}(k+2|k)$ can be eliminated using (6):

$$a\alpha^2 x(k) + a\alpha\beta u(k) + a\beta u(k+1) + bu(k+2) + a\alpha\gamma\hat{\delta}(k|k) + a\gamma\hat{\delta}(k+1|k) + c\hat{\delta}(k+2|k) \leqslant d \quad . \tag{7}$$

Next, we use (1) to determine $\hat{x}(k+3|k)$:

$$\hat{x}(k+3|k) = \alpha\hat{x}(k+2|k) + \beta u(k+2) + \gamma\hat{\delta}(k+2|k) \quad .$$

In this equation $\hat{x}(k+2|k)$ can be eliminated using (6):

$$\begin{aligned} \hat{x}(k+3|k) = {} & \alpha^3 x(k) + \alpha^2\beta u(k) + \alpha\beta u(k+1) + \beta u(k+2) \\ & + \alpha^2\gamma\hat{\delta}(k|k) + \alpha\gamma\hat{\delta}(k+1|k) + \gamma\hat{\delta}(k+2|k) \quad . \end{aligned} \tag{8}$$

<u>**Note**</u>: In general we will thus get equations of the form

$$\begin{aligned} & a\alpha^\ell x(k) + a\alpha^{\ell-1}\beta u(k) + a\alpha^{\ell-2}\beta u(k+1) + \ldots + a\beta u(k+\ell-1) + bu(k+\ell) \\ & \quad + a\alpha^{\ell-1}\gamma\hat{\delta}(k|k) + a\alpha^{\ell-2}\gamma\hat{\delta}(k+1|k) + \ldots + a\gamma\hat{\delta}(k+\ell-1) + c\hat{\delta}(k+\ell|k) \leqslant d \end{aligned}$$

for $\ell = 0, \ldots, N_\mathrm{p} - 1$ and

$$\begin{aligned} \hat{x}(k+\ell+1|k) = {} & \alpha^{\ell+1} x(k) + \alpha^\ell\beta u(k) + \alpha^{\ell-1}\beta u(k+1) + \ldots + \alpha\beta u(k+\ell-1) + \beta u(k+\ell) \\ & + \alpha^\ell\gamma\hat{\delta}(k|k) + \alpha^{\ell-1}\gamma\hat{\delta}(k+1|k) + \ldots + \alpha\gamma\hat{\delta}(k+\ell-1|k) + \gamma\hat{\delta}(k+\ell|k) \end{aligned}$$

for $\ell = 0, \ldots, N_\mathrm{p} - 1$. $\Diamond$

Let us now go one with rewriting the MLD-MPC problem in an explicit form. If we define

$$\tilde{V}(k) = \begin{bmatrix} \hat{\delta}(k|k) \\ \hat{\delta}(k+1|k) \\ \hat{\delta}(k+2|k) \\ u(k) \\ u(k+1) \\ u(k+2) \end{bmatrix}$$

then the equations (4), (6), (8) and the inequalities (3), (5), (7) can be written in a more compact matrix-vector notation as follows:

$$\tilde{x}(k) = M_1\tilde{V}(k) + M_2 x(k) \tag{9}$$

$$F_1\tilde{V}(k) \leqslant F_2 + F_3 x(k) \quad . \tag{10}$$

Now we use (9) to eliminate $\tilde{x}$ from the expression of $J(k)$. We have

$$\begin{aligned}
J(k) &= (\tilde{x}-\tilde{x}_{\text{eq}})^{\text{T}} P(\tilde{x}-\tilde{x}_{\text{eq}}) + (\tilde{V}-\tilde{V}_{\text{eq}})^{\text{T}} \underbrace{\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}}_{=:\, S} (\tilde{V}-\tilde{V}_{\text{eq}}) \\
&= (M_1\tilde{V}(k) + M_2 x(k) - \tilde{x}_{\text{eq}})^{\text{T}} P(M_1\tilde{V}(k) + M_2 x(k) - \tilde{x}_{\text{eq}}) + (\tilde{V}(k)-\tilde{V}_{\text{eq}})^{\text{T}} S(\tilde{V}(k)-\tilde{V}_{\text{eq}}) \\
&= \tilde{V}^{\text{T}}(k) M_1^{\text{T}} P M_1 \tilde{V}(k) + x^{\text{T}}(k) M_2^{\text{T}} P M_2 x(k) + \tilde{x}_{\text{eq}}^{\text{T}} P \tilde{x}_{\text{eq}} \\
&\quad + 2x^{\text{T}}(k) M_2^{\text{T}} P M_1 \tilde{V}(k) - 2\tilde{x}_{\text{eq}}^{\text{T}} P M_1 \tilde{V}(k) - 2x^{\text{T}}(k) M_2^{\text{T}} P \tilde{x}_{\text{eq}} \\
&\quad + \tilde{V}^{\text{T}}(k) S \tilde{V}(k) - 2\tilde{V}_{\text{eq}}^{\text{T}} S \tilde{V}(k) + \tilde{V}_{\text{eq}}^{\text{T}} S \tilde{V}_{\text{eq}} \\
&= \tilde{V}^{\text{T}}(k) S_1 \tilde{V}(k) + 2(S_2 + x^{\text{T}}(k) S_3)\tilde{V}(k) + \underbrace{x^{\text{T}}(k) S_4 x(k) + 2S_5 x(k) + s_6}_{=:\, s_7(k)} \ .
\end{aligned}$$

Since the term $s_7(k)$ does not depend on $\tilde{V}(k)$, it does not influence the optimal value of $\tilde{V}(k)$, and therefore it can be omitted. So we end up with a problem of the form

$$\begin{aligned}
&\min_{\tilde{V}(k)} \ \tilde{V}^{\text{T}}(k) S_1 \tilde{V}(k) + 2(S_2 + x^{\text{T}}(k) S_3)\tilde{V}(k) \\
&\text{subject to } F_1 \tilde{V}(k) \leqslant F_2 + F_3 x(k) \ ,
\end{aligned}$$

which is a mixed-integer quadratic programming (MIQP) problem.
For 1-norm or ∞-norm we get a mixed-integer linear programming (MILP) problem (see the practical assignment).