Modeling & Control of Hybrid Systems

Chapter 5 – Switched Control

Overview

1. Introduction & motivation for hybrid control
2. Stabilization of switched linear systems
3. Time-controlled switching & pulse width modulation
4. Sliding mode control
5. Stabilization by switching control
1. Introduction & motivation for hybrid control

Several “classical” control methods for continuous-time systems are hybrid:

- variable structure control
- sliding mode control
- relay control
- gain scheduling
- bang-bang time-optimal control
- fuzzy control

→ common characteristic: **switching**
1.1 Motivation for switched controllers

Theorem (Brockett’s necessary condition)

Consider system
\[ \dot{x} = f(x, u) \quad \text{with} \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ f(0,0) = 0 \]

where \( f \) is smooth function

If system is asymptotically stabilizable (around \( x = 0 \)) using continuous feedback law \( u = \alpha(x) \),
then image of every open neighborhood of \( (x, u) = (0,0) \) under \( f \) contains open neighborhood of \( x = 0 \)
1.1 Motivation for switched controllers (continued)

- For non-holonomic integrator: \( \dot{x} = u \)
  \( \dot{y} = v \)
  \( \dot{z} = xv - yu \)

- Is asymptotically stabilizable (see later)

- Satisfies Brockett’s necessary condition?
  
  - if \( f_1 = f_2 = 0 \) then \( f_3 = 0 \)
  
  - hence, \((0, 0, \varepsilon)\) cannot belong to image of \( f \) for any \( \varepsilon \neq 0 \)
    
    \( \rightarrow \) image of open neighborhood of \((x, y, z; u, v) = (0, 0, 0; 0, 0)\)
    
    under \( f \) does \textit{not} contain open neighborhood of \((x, y, z) = (0, 0, 0)\)

  - so non-holonomic integrator cannot be stabilized by \textit{continuous} feedback

  \( \rightarrow \) hybrid control schemes necessary to stabilize it!
1.2 Switching control/logic

- Controller 1
- Controller 2
- Controller $m-1$
- Controller $m$

Supervisor

Plant

$u_1, u_2, \ldots, u_{m-1}, u_m$
1.2 Switching control/logic (continued)

\[ \Sigma_{\text{sup}} \]

\[ \Sigma_{\text{ctrl}}(\sigma) \]

\[ \sigma \rightarrow \text{shared controller state variables} \]

\[ u \rightarrow \text{plant} \]

\[ y \rightarrow \text{supervisor} \]
1.2 Switching control/logic (continued)

Main problem: **Chattering** (i.e., very fast switching)

1. **Hysteresis switching logic**
   - let \( h > 0 \), let \( \pi_\sigma \) be a performance criterion (to be minimized)
   - if supervisor changes value of \( \sigma \) to \( q \), then \( \sigma \) is held *fixed* at \( q \) until \( \pi_p + h < \pi_q \) for some \( p \)
     \[ \rightarrow \sigma \text{ is set equal to } p \]
   - \( \Rightarrow \) threshold parameter \( h > 0 \) prevents infinitely fast switching
   - similar idea: **boundary layer** around switching surface in sliding mode control

2. **Dwell-time switching logic**
   once symbol \( \sigma \) is chosen by supervisor it remains constant for at least \( \tau > 0 \) time units (\( \tau \): “dwell time”)
2. Stabilization of switched linear systems via suitable switching (Problem C)

\[ \dot{x} = A_i x, \quad i \in I := \{1, 2, \ldots, N\} \]

Find switching rule \( \sigma \) as function of time/state such that closed-loop system is asymptotically stable

### 2.1 Quadratic stabilization via single Lyapunov function

Select \( \sigma(x) : \mathbb{R}^n \rightarrow I := \{1, 2, \ldots, N\} \) such that closed-loop system has single quadratic Lyapunov function \( x^T P x \)

**One solution:** if some convex combination of \( A_i \) is stable:

\[ A := \sum \alpha_i A_i \quad (\alpha_i \geq 0, \sum \alpha_i = 1) \text{ is stable} \]

Select \( Q > 0 \) and let \( P > 0 \) be solution of \( A^T P + PA = -Q \)
Quadratic stabilization (continued)

• From $x^T(A^TP + PA)x = -x^TQx < 0$ it follows that

$$\sum_i \alpha_i [x^T(A_i^TP + PA_i)x] < 0$$

• For each $x$ there is at least one mode with $x^T(A_i^TP + PA_i)x < 0$ or stronger

$$\bigcup_{i \in I} \{x \mid x^T(A_i^TP + PA_i)x \leq -\frac{1}{N}x^TQx\} = \mathbb{R}^n$$

• Switching rule:

$$i(x) := \arg\min x^T(A_i^TP + PA_i)x$$

• Leads possibly to sliding modes. Alternative?
Alternative switching rule for quadratic stabilization

- Modified switching rule (based on hysteresis switching logic):
  - stay in mode $i$ as long as $x^T (A_i^T P + PA_i) x \leq -\frac{1}{2N} x^T Q x$
  - when bound reached, switch to a new mode $j$ that satisfies $x^T (A_j^T P + PA_j) x \leq -\frac{1}{N} x^T Q x$

- There is a lower bound on the duration in each mode!

- No conservatism for 2 modes (necessary & sufficient for this case):

  **Theorem**: If there exists a quadratically stabilizing state-dependent switching law for the switched linear system with $N = 2$, then matrices $A_1$ and $A_2$ have a stable convex combination
2.2 Stabilization via *multiple* Lyapunov functions (Problem C)

**Main idea:** Find function $V_i(x) = x^T P_i x$ that decreases for $\dot{x} = A_i x$ in some region

Define $\mathcal{X}_i := \{x | x^T [A_i^T P_i + P_i A_i] x < 0\}$

If $\bigcup_i \mathcal{X}_i = \mathbb{R}^n$, try to switch to satisfy multiple Lyapunov criterion to guarantee asymptotic stability.

**Find** $P_1$ and $P_2$ such that they satisfy the coupled conditions:

- $x^T (P_1 A_1 + A_1^T P_1) x < 0$ when $x^T (P_1 - P_2) x \geq 0$, $x \neq 0$

and

- $x^T (P_2 A_2 + A_2^T P_2) x < 0$ when $x^T (P_2 - P_1) x \geq 0$, $x \neq 0$

Then $\sigma(t) = \arg \max \{V_i(x(t)) | i = 1, 2\}$ is stabilizing ($V_\sigma$ will be continuous)
2.3 S-procedure

**S-procedure** If there exist $\beta_1, \beta_2 \geq 0$ such that

$$-P_1A_1 - A_1^TP_1 + \beta_1(P_2 - P_1) > 0$$
$$-P_2A_2 - A_2^TP_2 + \beta_2(P_1 - P_2) > 0$$

then $\sigma(t) = \arg \max_i \{V_i(x(t)) \mid i = 1, 2\}$

→ only finds switching sequence (discrete inputs)!

What if also continuous inputs are present?
2.4 Stabilization of switched linear systems with continuous inputs

Switched linear system with inputs:

\[ \dot{x} = A_i x + B_i u, \quad i \in I = \{1, \ldots, N\} \]

Now both \( \sigma: [0, \infty) \rightarrow I \) and feedback controllers \( u = K_i x \) are to be determined.

**Case 1**: Determine \( K_i \) such that closed loop is stable under arbitrary switching (assuming we know mode)!

**Case 2**: Determine both \( \sigma: [0, \infty) \rightarrow I \) and \( K_i \)

**Case 3** (for PWL systems): Given \( \sigma \) as function of state, determine \( K_i \)
Case 1: Stabilization of switched linear system under arbitrary switching

\[ \dot{x} = A_i x + B_i u, \ i \in I = \{1, \ldots, N\} \]

Sufficient condition: find common *quadratic* Lyapunov function \( V(x) = x^T P x \) for some positive definite matrix \( P \) and \( K_1, \ldots, K_N \)

\[ (A_i + B_i K_i)^T P + P (A_i + B_i K_i) < 0 \text{ for all } i = 1, \ldots, N \text{ and } P > 0 \]

→ LMIs (also for Cases 2 and 3)

→ state-based switching in this section, ... next ...
3. Time-controlled switching & pulse width modulation

If dynamical system switches between several subsystems → stability properties of total system may be quite different from those of subsystems

\[ T = 0, x = x_0 \]

\[
\begin{align*}
\text{mode 1} & \quad \dot{x} = A_1 x \\
& \quad \dot{T} = 1 \\
& \quad T \leq \frac{1}{2} \varepsilon \\
\implies & \quad T := 0
\end{align*}
\]

\[
\begin{align*}
\text{mode 2} & \quad \dot{x} = A_2 x \\
& \quad \dot{T} = 1 \\
& \quad T \geq \frac{1}{2} \varepsilon \\
\implies & \quad T := 0
\end{align*}
\]
3.1 Time-controlled switching

- $x(t_0 + \frac{1}{2}\varepsilon) = \exp(\frac{1}{2}\varepsilon A_1)x_0 = x_0 + \frac{\varepsilon}{2}A_1x_0 + \frac{\varepsilon^2}{8}A_1^2x_0 + \cdots$

- $x(t_0 + \varepsilon) = \exp(\frac{1}{2}\varepsilon A_2)\exp(\frac{1}{2}\varepsilon A_1)x_0$

  $= (I + \frac{\varepsilon}{2}A_2 + \frac{\varepsilon^2}{8}A_2^2 + \cdots)(I + \frac{\varepsilon}{2}A_1 + \frac{\varepsilon^2}{8}A_1^2 + \cdots)x_0$

  $= (I + \varepsilon[\frac{1}{2}A_1 + \frac{1}{2}A_2] + \frac{\varepsilon^2}{8}[A_1^2 + A_2^2 + 2A_2A_1] + \cdots)x_0$.

- Compare with

  \[ \exp[\varepsilon(\frac{1}{2}A_1 + \frac{1}{2}A_2)] = I + \varepsilon[\frac{1}{2}A_1 + \frac{1}{2}A_2] + \frac{\varepsilon^2}{8}[A_1^2 + A_2^2 + A_1A_2 + A_2A_1] + \cdots \]

  $\rightarrow$ same for $\varepsilon \approx 0$

- So for $\varepsilon \rightarrow 0$ solution of switched system tends to solution of

  $\dot{x} = (\frac{1}{2}A_1 + \frac{1}{2}A_2)x$  (“averaged” system)

- Possible that $A_1, A_2$ stable, whereas $\frac{1}{2}A_1 + \frac{1}{2}A_2$ unstable, or vice versa
Consider

\[ A_1 = \begin{bmatrix} -0.5 & 1 \\ 100 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -100 \\ -0.5 & -1 \end{bmatrix} \]

- \( A_1, A_2 \) unstable, but matrix \( \frac{1}{2}(A_1 + A_2) \) is stable
  → switched system should be stable if frequency of switching is sufficiently high

- Minimal switching frequency found by computing eigenvalues of the mapping \( \exp\left(\frac{1}{2}\varepsilon A_2\right)\exp\left(\frac{1}{2}\varepsilon A_1\right) \) (Why?)
Example (continued)

→ maximal value of $\varepsilon$: 0.04 (=50 Hz)
Example (continued)

for $\varepsilon = 0.02$
3.2 Pulse width modulation

- Assume mode 1 followed during $h \epsilon$, and mode 2 during $(1 - h)\epsilon$
  → behavior of system is well approximated by system
  \[
  \dot{x} = (hA_1 + (1 - h)A_2)x
  \]

- Parameter $h$ might be considered as control input

- If $h$ varies, should be on time scale that is much slower than the time scale of switching

- If mode 1 is “power on” and mode 2 is “power off”, then $h$ is known as duty ratio

- Power electronics: fast switching theoretically provides possibility to regulate power without loss of energy
  → used in power converters (e.g., Boost converter)
3.2 Pulse width modulation (continued)

- System: $\dot{x} = f(x, u), \quad u \in \{0, 1\}$
- Duty cycle: $\Delta$ (fixed)
- $u$ is switched exactly one time from 1 to 0 in each cycle
- Duty ratio $\alpha$: fraction of duty cycle for which $u = 1$

$$u(\tau) = 1 \quad \text{for } t \leq \tau < t + \alpha \Delta$$
$$u(\tau) = 0 \quad \text{for } t + \alpha \Delta \leq \tau < t + \Delta$$

- Hence, $x(t + \Delta) = x(t) + \int_{t}^{t+\alpha \Delta} f(x(\tau), 1) d\tau + \int_{t+\alpha \Delta}^{t+\Delta} f(x(\tau), 0) d\tau$

- Ideal averaged model ($\Delta \to 0$):

$$\dot{x}(t) = \lim_{\Delta \to 0} \frac{x(t + \Delta) - x(t)}{\Delta} = \alpha f(x(t), 1) + (1 - \alpha) f(x(t), 0)$$
4. Sliding mode control

• Consider \( \dot{x}(t) = f(x(t), u(t)) \) with \( u \) scalar

• Suppose switching feedback control scheme:

\[
    u(t) = \begin{cases} 
    \phi_+(x(t)) & \text{if } h(x(t)) > 0 \\
    \phi_-(x(t)) & \text{if } h(x(t)) < 0 
    \end{cases}
\]

• Surface \( \{x \mid h(x) = 0\} \) is called **switching surface**

• Let \( f_+(x) = f(x, \phi_+(x)) \) and \( f_-(x) = f(x, \phi_-(x)) \), then

\[
    \dot{x} = \frac{1}{2}(1 + v)f_+(x) + \frac{1}{2}(1 - v)f_-(x), \quad v = \text{sgn}(h(x))
\]

• Use solutions in Filippov’s sense if “chattering”
4. Sliding mode control (continued)

• Assume “desired behavior” whenever constraint \( s(x) = 0 \) is satisfied

• Set \( \{ x \mid s(x) = 0 \} \) is called \textit{sliding surface}

• Find control law \( u \) such that

\[
\frac{1}{2} \frac{d}{dt} s^2 \leq -\alpha |s|
\]

where \( \alpha > 0 \)

\( \rightarrow \) squared “distance” to sliding surface decreases
along all system trajectories
Properties of sliding mode control

- Quick succession of switches may occur
  → increased wear, high-frequency vibrations
  ⇒ embed sliding surface in thin boundary layer
    - smoothen discontinuity by replacing $\text{sgn}$ by steep sigmoid function
    - Note: modifications may deteriorate performance of closed-loop system

- Main advantages of sliding mode control:
  – conceptually simple
  – robustness w.r.t. uncertainty in system data

- Possible disadvantage:
  – excitation of unmodeled high-frequency modes
5. Stabilization by switching control

- For multi-model *linear* systems
  → use techniques for quadratic stabilization using single or multiple Lyapunov function

- Stabilization of non-holonomic systems using hybrid feedback control (e.g., non-holonomic integrator)
  → rather ad hoc
    not structured
    complicated analysis and proofs
Stabilization of non-holonomic integrator

- **System:** \( \dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = xv - yu \)

- **Sliding mode control:**
  \[
  u = -x + y \text{sgn}(z) \\
  v = -y - x \text{sgn}(z)
  \]

- **Switching surface:** \( z = 0 \)

- **Lyapunov function for \((x, y)\) subspace:** \( V(x, y) = \frac{1}{2}(x^2 + y^2) \)
  \[
  \dot{V} = x^2 + yz \text{sgn}(z) - y^2 - xy \text{sgn}(z) = -(x^2 + y^2) = -2V \\
  \Rightarrow x, y \rightarrow 0
  \]

- \( \dot{z} = xv - yu = -(x^2 + y^2) \text{sgn}(z) = -2V \text{sgn}(z) \)
  So \(|z|\) will decrease and reach 0 provided that
  \[
  2 \int_0^\infty V(\tau)d\tau > |z(0)|
  \]
  \( \rightarrow z \) will reach 0 in finite time
Stabilization of non-holonomic integrator (continued)

• Since $V(t) = V(0)e^{-2t} = \frac{1}{2}(x^2(0) + y^2(0))e^{-2t}$
  condition for system to be asymptotically stable is
  $$\frac{1}{2}(x^2(0) + y^2(0)) \geq |z(0)|$$
  \rightarrow \text{defines parabolic region} \quad \mathcal{P} = \{(x, y, z) \mid 0.5(x^2 + y^2) \leq |z|\}

• If initial conditions do not belong to $\mathcal{P}$ then sliding mode control asymptotically stabilizes system

• If initial state is inside $\mathcal{P}$:
  – first use control law (e.g., nonzero constant control) to steer system outside $\mathcal{P}$
  – then use sliding mode control
  \rightarrow \text{hybrid control scheme}
6. Summary

- Problem C: stabilization → construct switching signal $\sigma$
  - single Lyapunov function → find convex combination that is stable
  - multiple Lyapunov functions → “max”-switching law, S-procedure
  - with continuous inputs → also find state feedback ($K_i$) → LMIs

- Pulse width modulation
- Sliding mode control
- Stabilization of non-holonomic integrator