Minimization of 2-norms and squares versus 1-norms and absolute values

Minimization of x^2 versus $|x|$

We have

$$\arg\min_{x} x^2 = \arg\min_{x} |x|$$

Indeed, since $x^2 = |x|^2$ and since $(\cdot)^2$ is increasing for a nonnegative argument (as occurs here), we can minimize the argument (i.e., $|\cdot|$) instead.

Note that this also extends to e.g. $\arg\min_{(x,y)} (x+y)^2 = \arg\min_{(x,y)} |x+y|$

Minimization of $\|x\|_2^2$ versus $\|x\|_1$

Note that the equivalence given above does not extend to norms.

Consider, e.g., the 2-norm minimization problem

$$\text{P1:} \min_{(x,y)\in\mathbb{R}} x^2+y^2 \quad \text{s.t. } 2x+y=1$$

and the 1-norm minimization problem

$$\text{P2:} \min_{(x,y)\in\mathbb{R}} |x|+|y| \quad \text{s.t. } 2x+y=1$$

For P1 we get $y = 1-2x$ and thus

$$\begin{aligned}\min_{x\in\mathbb{R}} x^2+(1-2x)^2 &= \min_{x\in\mathbb{R}} x^2+1-4x+4x^2 \\ &= \min_{x\in\mathbb{R}} 5x^2-4x+1\end{aligned}$$

The zero-gradient condition yields $10x-4=0$ or $x^* = \dfrac{2}{5} = 0.4$.

For P2 we also have $y = 1-2x$ and thus

$$\min_{x\in\mathbb{R}} |x|+|1-2x| := f(x)$$

which can either be solved graphically or by considering 4 subproblems:

$$\begin{aligned}
&\text{P2a: } \min_{x\in\mathbb{R}} x+1-2x = \min_{x\in\mathbb{R}} -2x+1 && \text{if } x \geqslant 0 \text{ and } x \leqslant 1/2 \\
&\text{P2b: } \min_{x\in\mathbb{R}} x-1+2x = \min_{x\in\mathbb{R}} 3x-1 && \text{if } x \geqslant 0 \text{ and } x \geqslant 1/2 \\
&\text{P2c: } \min_{x\in\mathbb{R}} -x+1-2x = \min_{x\in\mathbb{R}} -3x+1 && \text{if } x \leqslant 0 \text{ and } x \leqslant 1/2 \\
&\text{P2d: } \min_{x\in\mathbb{R}} -x-1-2x = \min_{x\in\mathbb{R}} -3x-1 && \text{if } x \leqslant 0 \text{ and } x \geqslant 1/2
\end{aligned}$$

P2a yields $x = 1/2$ as optimal solution with $f(1/2) = 1/2$.
P2b yields $x = 1/2$ as optimal solution with $f(1/2) = 1/2$.
P2c yields $x = 0$ as optimal solution with $f(0) = 1$.
P2d is infeasible.
So the x-component of the optimal solution for P2 is $x^* = 1/2 = 0.5$.

So P1 and P2 have *different* minimizers.