Exam — October 2013
"Optimization in Systems and Control" (SC4091)

QUESTION 1: Optimization methods I (60 points)

Consider the following optimization problems:

P1. $$\max_{x\in\mathbb{R}^3} \; -\exp(x_1^2+2x_1x_2+x_2^2+x_3^2-x_1-3x_2+4x_3)$$
$$\text{s.t. } \|x\|_1 \leq 1$$
$$\log(4+x_1+x_2-x_3) \leq 1$$

P2. $$\min_{x\in\mathbb{R}^4} (2x_1+4x_2+3x_3+2x_4+7)^2+\cosh\big((x_1+x_2-4x_3-x_4)^6\big)$$
$$\text{s.t. } \|x\|_2 \geq 4$$

P3. $$\min_{x\in\mathbb{R}^5} \big(1+|x_1|+3|x_2|+|x_3|+2|x_4|+3|x_5|\big)^3$$
$$\text{s.t. } (5+3x_1+x_2-x_3+x_4-x_5)^4 \leq 81$$
$$\min(3+2x_1+3x_2, 4x_3+3x_4, 2x_5+8) \geq 2$$

P4. $$\min_{x\in\mathbb{R}^3} \frac{1}{(x_1+3x_2+4x_3)^6}+0.0001\cosh|x_1+4x_2+5x_3-100|$$
$$\text{s.t. } x_1+3x_2+4x_3 \geqslant 1$$
$$\log_{10}\big((x_1-2)^2+(x_2-3)^2+(x_3-4)^2\big) \leqslant 2$$

P5. $$\min_{x\in\mathbb{R}^3} 4x_1^2+2x_2^2+2x_1x_2+6x_3^2-2x_1x_3$$
$$\text{s.t. } \exp(2x_1^2+3x_2^2+5x_3^2)+(4x_1+5x_2+6x_3)^2+x_1-x_3 = 100$$

P6. $$\max_{x\in\mathbb{R}^3} \min(|4x_1-3x_2+8x_3-5|, |-2x_1+7x_2-x_3+1|)$$
$$\text{s.t. } 2^{7x_1-2x_2+5x_3} \leqslant 64$$
$$\|x\|_\infty \leqslant 3$$

P7. $$\min_{x\in\mathbb{R}^3} \arctan\big(4x_1^2+2x_1x_2+2x_1x_3+4x_2^2+2x_2x_3+4x_3^2-x_1+x_2-x_3\big)$$
$$\text{s.t. } (3x_1-x_2+6x_3)^4 \leqslant 625$$
$$\sqrt{|x_1+x_2+x_3|} \leqslant 3$$

P8. $\max_{v \in \mathbb{R}^{10}} \|G\|_\infty$

where the (stable) transfer function G depends on v and is defined by

$$G(z,v) = \frac{z^2 + \dfrac{v_1 v_2 - v_3 v_5}{1 + v_1^2 + v_2^2 + v_6^2} z - \dfrac{v_4 v_9 - v_7 v_{10}}{1 + v_7^2 + v_8^2}}{\left(z - \dfrac{1}{2 + v_2^2 + 2v_5^2 + 3v_9^2}\right)\left(z - \dfrac{1}{3 + 2v_4^2 + 8v_{10}^2}\right)\left(z + \dfrac{1}{4 + v_7^2}\right)}$$

with z the z-transform variable.

P9. $\min_{x \in \mathbb{R}^5} x_1^4 + x_2^2 + 3x_1x_3 + 2x_4x_5$

$$\begin{aligned} \text{s.t. } & |x_1 + 2x_2 + 8x_3 - 9x_4 + 8x_5| \leq 7 \\ & -x_1 + 3x_2 - x_3 + 6x_4 + x_5 \geq 2 \\ & \|x\|_\infty \leq 3 \end{aligned}$$

P10. $\min_{x \in \mathbb{R}^4} \sqrt[3]{(x_1 + 6x_2 + 8x_3 - 9x_4 - 10)^2}$

s.t. $x_1 + x_1x_3^2 + 3x_2^2 + 2x_2^2x_3^2 + 4x_3^2x_4^2 + 8x_4^4 = 5$

Tasks:

1. For each of the problems P1 up to P10:

   (a) Determine the best suited optimization method for each of the given problems. Select one of the following methods:

   M1 : Simplex algorithm for linear programming
   M2 : Gradient projection method with quadratic line minimization
   M3 : Steepest descent method
   M4 : Davidon-Fletcher-Powell quasi-Newton algorithm
   M5 : Line search method with Levenberg-Marquardt direction and cubic line minimization
   M6 : Lagrange method + line search method with Levenberg-Marquardt direction and cubic line minimization
   M7 : Lagrange method + Nelder-Mead method
   M8 : Barrier function method + Powell directions and variable step size line minimization
   M9 : Penalty function method + line search method with steepest descent direction and variable-step line minimization
   M10 : Simulated annealing algorithm
   M11 : Interior point method
   M12 : Genetic algorithm

   - ***Explain*** why the selected algorithm is the *best suited* for the given problem.
   - Note that you may have to perform some extra steps first in order to *simplify* the optimization problem. If you do this, then you have to ***indicate*** it clearly.
   - If it is necessary to use the selected algorithm in a *multi-start local minimization* procedure, you have to ***indicate*** this clearly.
   - Give only one solution! If more than one solution is given, the worst one will be taken.

   (b) Determine the most appropriate stopping criterion for the selected optimization method and ***explain*** why it is the most appropriate!

QUESTION 2: Optimization methods II (25 points)

<u>Tasks:</u>

1. Explain how the steepest descent method for unconstrained optimization works. Give the iteration formula and illustrate your explanation by drawing one or more pictures.
   <u>Note</u>: It is not required to explain in detail how the line minimization part of the method works, i.e., it is sufficient to briefly indicate how the line minimization part of the approach works ***in general***, but you do not have to select and explain a particular line minimization method.

2. Now consider the following optimization problem:

$$\min_{(x,y)\in\mathbb{R}^2} 2x^2 + y^2 - xy - 8x + 10 \quad . \tag{1}$$

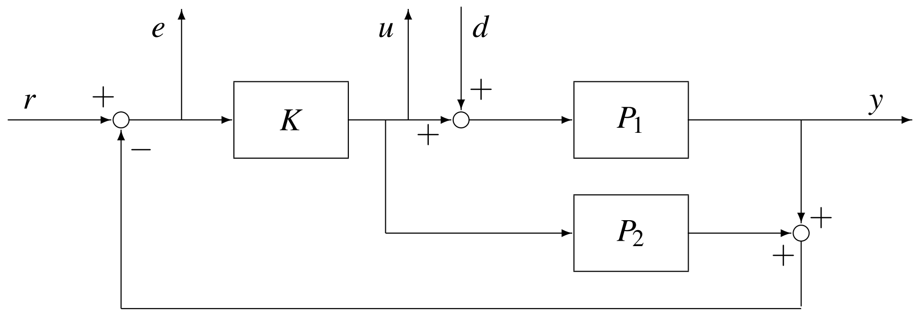
   a) Is the objective function of this optimization problem convex or not? Why?

   b) Compute all the points (x^*, y^*) in which a local optimum of the given optimization problem is reached. Characterize these points (are they maximima or minima?). Also compute the function values corresponding to these points.

   c) Now solve the problem (1) using the steepest descent method and perform two iteration steps:
   So compute the first two search directions for the steepest descent method, perform the subsequent line minimizations, and give the values of the objective function after each line minimization.
   The starting point is equal to $(x_0, y_0) = (0, 0)$ and the line minimization method is assumed to find the *exact* line minimum.

QUESTION 3: Controller design (15 points)

Consider the following closed-loop system consisting of two stable systems with transfer functions P_1 and P_2, and a controller with transfer function K:



There are two external input signals: the process noise d and the reference signal r. There are three external output signals: u (the output of the controller), y (the output of the first system), and e (the error signal).

Tasks:

a) Determine the transfer function matrix M such that

$$\begin{bmatrix} u(k) \\ y(k) \\ e(k) \end{bmatrix} = M(q) \begin{bmatrix} r(k) \\ d(k) \end{bmatrix}$$

where q denotes the forward shift operator.

b) Consider a class $\mathscr{K}$ of controllers defined by

$$K(q) = \frac{Q(q)}{1 - P_1(q)Q(q) - P_2(q)Q(q)}$$

with Q a stable transfer function.
Show that any controller in $\mathscr{K}$ will internally stabilize the given closed-loop system.

End of the exam