Exam — October 2014
"Optimization in Systems and Control" (SC4091)

QUESTION 1: Optimization methods I (60 points)

Consider the following optimization problems:

P1. $$\min_{x\in\mathbb{R}^4} |2x_1+4x_2+3x_3+2x_4+7|+\exp\big((x_1+x_2-4x_3-x_4)^4\big)$$
$$\text{s.t. } \|x\|_1 \geq 3+x_1-2x_2+x_3-2x_4$$

P2. $$\min_{x\in\mathbb{R}^5} \log\big(1+|x_1|+3|x_2|+|x_3|+2|x_4|+3|x_5|\big)$$
$$\text{s.t. } (5+3x_1+x_2-x_3+x_4-x_5)^3 \leq 27$$
$$\max(|3+2x_1+3x_2|, |4x_3+3x_4|, |2x_5+8|) \leq 15$$

P3. $$\min_{x\in\mathbb{R}^4} 2x_1^2+2x_1x_2+x_2^2+4x_3^2+4x_3x_4+7x_4^2-3x_1-4x_2+8x_3+1$$
$$\text{s.t. } \exp(x_1)+\exp(x_2)+\exp(x_3)+\exp(x_4) \leq 2$$
$$x_1+2x_3+x_4 \geq 3$$
$$-7 \leq x_1+x_2-x_3-x_4 \leq 7$$

P4. $$\min_{x\in\mathbb{R}^5} x_1^4+x_2^2+2x_1x_3+2x_4x_5$$
$$\text{s.t. } (x_1+2x_2+8x_3-9x_4+8x_5)^5 \leq 32$$
$$\sqrt{(-x_1+3x_2-x_3+6x_4+x_5)^2} \geq 2$$
$$\|x\|_\infty \leq 3$$

P5. $$\min_{x\in\mathbb{R}^4} x_1^4+7x_2^2-8x_3x_4+2x_3-9$$
$$\text{s.t. } \sin(1+2x_3x_4+x_1^4x_2^2)+3x_2x_3^2+\exp(x_1+x_3-x_4)=1$$

P6. $$\max_{x\in\mathbb{R}^3} \min(4x_1-3x_2+8x_3-5,\ -2x_1+7x_2-x_3+1)$$
$$\text{s.t. } 7|x_1|+2|x_2|+5|x_3| \leq 10$$
$$3^{2x_1-8x_2+5x_3} \geq 27$$

P7. $$\min_{x\in\mathbb{R}^4} \exp\big((x_1+6x_2+8x_3-9x_4-3)^2\big)$$
$$\text{s.t. } x_1^2+2x_1x_3-9x_3^2-5x_2^2-2x_2x_3-4x_3x_4-12x_4^2=9$$

P8. $\min_{x\in\mathbb{R}^3} \arctan\left(4x_1^2+2x_1x_2+2x_1x_3+4x_2^2+2x_2x_3+4x_3^2-x_1+x_2-x_3\right)$

s.t. $3(x_1-x_2)^2+6x_3^2 \leq 36$

$4-|x_1+x_2+x_3| \geq 0$

P9. $\min_{z\in\mathbb{R}^7} \sum_{i=1}^{14} \left(|\rho_i(z)|-i\right)^2$

where $\rho_1(z)$, $\rho_2(z)$, ..., $\rho_{14}(z)$ are the (possibly complex) roots of the following parameterized 14-th degree polynomial in the variable ρ:

$$\rho^{14}+z_1\rho^{10}+z_2\sin z_3\rho^9+\cos(z_1-z_3z_4)\rho^6+\rho^5 + \frac{1+z_3+z_5^2}{2+z_1^2+\sin z_2}\exp z_5\rho^4+\cosh(z_4+z_6)\rho^2+z_7\rho-z_2z_6-z_4$$

where z is the parameter.

P10. $\max_{x\in\mathbb{R}^3} -(3x_1^2+2x_2^2+8x_3^2-x_1-3x_2+4x_3-8)^5$

s.t. $\|x\|_1 \leq 3$

$\cosh^4(2x_1+3x_2-7x_3) \leq 625$

Tasks:

- For each of the problems P1 up to P10:

  (a) Determine the best suited optimization method for each of the given problems. Select one of the following methods:

  M1 : Simplex algorithm for linear programming
  M2 : Gradient projection method with fixed step size line minimization
  M3 : Gauss-Newton least squares algorithm
  M4 : Levenberg-Marquardt algorithm
  M5 : Steepest descent method
  M6 : Line search method with the Broyden-Fletcher-Goldfarb-Shanno direction and cubic line minimization
  M7 : Lagrange method + Davidon-Fletcher-Powell quasi-Newton algorithm
  M8 : Lagrange method + steepest descent method
  M9 : Penalty function approach + line search method with Powell directions and variable step size line minimization
  M10 : Barrier function approach + line search method with Levenberg-Marquardt direction
  M11 : Ellipsoid algorithm
  M12 : Genetic algorithm

  * If it is necessary to use the selected algorithm in a *multi-start local minimization* procedure, you have to ***indicate*** this clearly.
  * Give only one solution! If more than one solution is given, the worst one will be taken.

  (b) ***Explain*** why the selected algorithm is the *best suited* for the given problem.
  Note that you may have to perform some extra steps first in order to *simplify* the optimization problem. If you do this, then you have to ***indicate*** it clearly.

  (c) Give the most appropriate stopping criterion for the selected optimization method.

- **Important**:
  In your written reply, please make sure to answer for each of the problems P1, P2, . . . , P10 first subquestion (a) and immediately thereafter subquestion (b) and (c) for that problem, before moving on to the next problem.
  Also make sure to stick to the given order of the problems (P1, P2, . . . , P10).

QUESTION 2: Optimization methods II (25 points)

The topic of this question is linear programming.

Tasks:

1. Transform the following linear programming problem into the standard form min $c^T x$ subject to $Ax = b$ and $x \geq 0$:

$$\begin{aligned} \max_{x \in \mathbb{R}^4} \ & 7x_1 + 2x_2 - 2x_3 - 9x_4 + 8 \\ \text{s.t.} \ & x_1 - 2x_2 + 3x_3 + 8x_4 \leq 6 \\ & -x_1 + 3x_2 - 6x_3 + x_4 \geq 2 \\ & x_1 - x_3 \leq 5 \\ & 0 \leq x_1 \leq 9 \\ & x_2 \leq 1 \end{aligned}$$

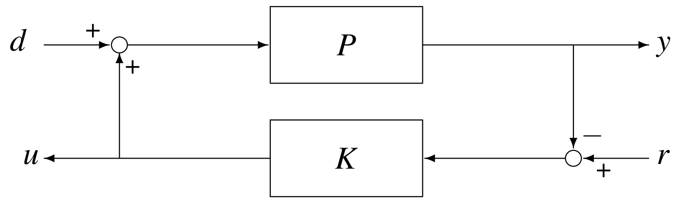
2. Solve the following linear programming problem:

$$\begin{aligned} \max_{x \in \mathbb{R}^2} \ & 2x_1 + 3x_2 \\ \text{s.t.} \ & 4x_1 + x_2 \leq 16 \\ & -x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

3. Explain how the simplex algorithm for linear programming works.

QUESTION 3: Controller design (15 points)

Consider the following closed loop system consisting of a stable plant with transfer function P and a controller with transfer function K:



There are two external input signals: the process noise d and the reference signal r. There are two external output signals: u and y.

<u>Tasks</u>:

1. Determine the transfer function matrix G such that

$$\begin{bmatrix} y(k) \\ u(k) \end{bmatrix} = G(q) \begin{bmatrix} d(k) \\ r(k) \end{bmatrix}$$

where q denotes the forward shift operator.

2. Now let P be defined as $P(q) = 2$, and consider a class of controllers defined by

$$K(q) = \frac{Q(q)}{1 - P(q)Q(q)}$$

where Q is stable. Let H_{ry} be the transfer function in the z domain from the reference signal r to the output y.

(a) First consider the design specification $4 \leq \| H_{ry} \|_\infty \leq 8$. Show that this design specification is *not* closed-loop convex in Q.

(b) Now consider the design specification $4 \leq \mathrm{Re}\{H_{ry}(e^{j\omega})\} \leq 8$ for all $\omega \in \mathbb{R}$. Show that this design specification is closed-loop convex in Q.