Worked solutions for the Exam of October 2014 "Optimization in Systems and Control" (SC4091)

QUESTION 1: Optimization methods I

Please note that for some questions more than one answer might be correct. However, below only one answer is listed. Furthermore, the footnotes are for further clarification only and are not considered to be a required part of the answers.
The various εs appearing in the stopping criteria below are all assumed to be small positive numbers.

Answers

P1. (a) *Multi-start* barrier function approach + line search method with Levenberg-Marquardt direction (M10)

(b) Although the objective function of this problem is convex, the constraint is not convex[1], since in the standard form we get $3+x_1-2x_2+x_3-2x_4-\|x\|_1 \leq 0$, of which the first affine part does not affect convexity while the second part ($-\|x\|_1$) is not convex. The gradient and Hessian of the objective function and the Jacobian of the constraints can be computed analytically. This implies that — from the list of available methods — the best suited optimization algorithm is a *multi-start* barrier function approach in combination with a line search method with the Levenberg-Marquardt direction (M10).

(c) The most appropriate stopping criterion is[2]: there exists a μ_k such that

$$\begin{aligned}
&\| \nabla f(x_k) + \nabla g(x_k)\, \mu_k \|_2 \leq \varepsilon_1 \\
&| \mu_k^{\mathrm{T}} g(x_k) | \leq \varepsilon_2 \\
&\mu_k \geq -\varepsilon_3 \\
&g(x_k) \leq \varepsilon_4
\end{aligned}$$

where x_k is the current iteration point, f is the objective of the minimization problem, and g is the inequality constraint function (written in the form $g(x) \leq 0$).

P2. (a) Simplex algorithm for linear programming (M1)

[1] Alternatively, by expanding the 1-norm the constraint can be rewritten as a union of $2^4 = 16$ affine/convex constraints, and then the problem can be recast into solving 16 times a convex optimization problem. So we could also use the ellipsoid algorithm (M11) 16 times. More specifically, a constraint of the form $\|x\|_1 \geq L$ where L is an affine function, can be rewritten as $|x_1|+|x_2|+|x_3|+|x_4| \geq L$ or equivalently $x_1+x_2+x_3+x_4 \geq L$ **or** $x_1+x_2+x_3-x_4 \geq L$ **or** $x_1+x_2-x_3+x_4 \geq L$ **or** … **or** $-x_1-x_2-x_3-x_4 \geq L$, i.e., the union of $2^4 = 16$ affine constraints.

[2] There are no equality constraints; so the equality constraint function h does not appear here.

(b) Since log is a monotonically increasing function, we can also minimize its argument instead. So the new objective function becomes $1+|x_1|+3|x_2|+|x_3|+2|x_4|+3|x_5|$. The first term (i.e., the constant 1) does not influence the optimum and can be omitted. Note that the scaling factors of the other terms are positive. By introducing a variable $\alpha \in \mathbb{R}^5$ the problem can then be recast into the constrained problem $\min\limits_{\alpha,x}\ \alpha_1+3\alpha_2+\alpha_3+2\alpha_4+3\alpha_5$ subject to $\alpha_i \geq |x_i|$ or equivalently $\alpha_i \geq x_i$ and $\alpha_i \geq -x_i$ for $i=1,2,\ldots,5$, which are affine constraints.
The constraint $(5+3x_1+x_2-x_3+x_4-x_5)^3 \leq 27$ can be rewritten as $5+3x_1+x_2-x_3+x_4-x_5 \leq 3$, which is an affine constraint.
The constraint $\max(|3+2x_1+3x_2|,|4x_3+3x_4|,|2x_5+8|) \leq 15$ can be rewritten as

$$\begin{aligned}
&|3+2x_1+3x_2| \leq 15\\
&|4x_3+3x_4| \leq 15\\
&|2x_5+8| \leq 15
\end{aligned}$$

or equivalently

$$\begin{aligned}
&-15 \leq 3+2x_1+3x_2 \leq 15\\
&-15 \leq 4x_3+3x_4 \leq 15\\
&-15 \leq 2x_5+8 \leq 15 \quad ,
\end{aligned}$$

which are affine constraints.
Hence, we have a linear programming problem and therefore the most suited optimization algorithm is the simplex algorithm (M1).

(c) Since the simplex algorithm finds the optimal solution in a finite number of steps, no stopping criterion is required[3].

P3. (a) Ellipsoid algorithm (M11)

(b) The objective function can be written as $x_1^2+(x_1+x_2)^2+(2x_3+x_4)^2+6x_4^2-3x_1-4x_2+8x_3+1$. Quadratic functions and affine functions are convex; moreover, a sum with positive weights of convex functions is also convex. Hence, the objective function is convex[4].
The function exp is convex, and a sum of convex functions is also convex. Hence, the first constraint is convex.
The second and the third constraint are affine and thus convex.
Hence, we have a convex optimization problem. The gradients of the objective function of the constraint functions can be computed analytically. Therefore, the most appropriate

[3]However, in practice a maximum number of iterations is usually specified.

[4]Alternatively, the Hessian of the objective function can be computed: $H=\begin{bmatrix}4&2&0&0\\2&2&0&0\\0&0&8&4\\0&0&4&14\end{bmatrix}$ and it can be verified that H is positive definite.

algorithm is the ellipsoid algorithm (M11).

(c) A suitable stopping criterion is[5]

$$|\tilde{f}(x_k) - \tilde{f}(x^*)| \le \varepsilon_f \quad \text{and} \quad g(x_k) \le \varepsilon_g \ ,$$

where x_k is the current iteration point, x^* is the (yet unknown) optimum of this optimization problem, $\tilde{f}$ denotes the objective function (after rewriting the problem as a convex minimization problem), and $\tilde{g}$ denotes the constraint function (with the constraint recast in the form $\tilde{g}(x) \le 0$ with $\tilde{g}$ convex).

P4. (a) *Multi-start* barrier function approach + line search method with Levenberg-Marquardt direction (M10)

(b) The objective function is non-convex due to the terms x_1x_3 and x_4x_5 (which cannot be absorbed into a square here).
The first and the third constraints can be rewritten as affine constraints:

$$\begin{aligned} &x_1 + 2x_2 + 8x_3 - 9x_4 + 8x_5 \le 2 \\ &-3 \le x_i \le 3 \quad \text{for } i = 1, \ldots, 5 \end{aligned}$$

The second constraint can be written as the union of two affine constraints (so actually the constraint is non-convex):

$$\begin{aligned} &-x_1 + 3x_2 - x_3 + 6x_4 + x_5 \ge 2 \\ \text{or} \quad &-x_1 + 3x_2 - x_3 + 6x_4 + x_5 \le -2 \end{aligned}$$

As we have a nonlinear, non-convex objective function, a multi-start optimization method is required. The gradient and Hessian of the objective function can be computed analytically. So in this case and given the list of available algorithms, which contains an algorithm (M10) that is Hessian-based, a *multi-start* barrier function approach in combination with a line search method with the Levenberg-Marquardt direction (M10) is the most suited optimization algorithm.

(c) The most appropriate stopping criterion is: there exists a μ_k such that

$$\begin{aligned} &\| \nabla f(x_k) + \nabla g(x_k)\, \mu_k \|_2 \le \varepsilon_1 \\ &| \mu_k^{\mathrm{T}} g(x_k) | \le \varepsilon_2 \\ &\mu_k \ge -\varepsilon_3 \\ &g(x_k) \le \varepsilon_4 \end{aligned}$$

where x_k is the current iteration point, f is the objective function of the minimization problem, and g is the inequality constraint function (written in the form $g(x) \le 0$).

[5] In addition, $\|x_k - x^*\|_2 \le \varepsilon_x$ could also be added.

P5. (a) *Multi-start* Lagrange method + Davidon-Fletcher-Powell quasi-Newton algorithm (M7)

(b) The constraint is non-convex (due to the sin term). So we have a nonlinear, non-convex optimization problem with an equality constraint. It is not possible to use the constraint to eliminate one of the variables. The gradient and Hessian of the objective function can be computed analytically. So in this case a *multi-start* Lagrange method in combination with the Davidon-Fletcher-Powell quasi-Newton algorithm (M7) is the most appropriate optimization approach.

(c) The most appropriate stopping criterion is:

$$\begin{aligned} &\| \nabla f(x_k) + \nabla h(x_k)\, \lambda_k \|_2 \leq \varepsilon_1 \\ &|h(x_k)| \leq \varepsilon_2 \end{aligned}$$

where x_k is the current iteration point, f is the objective function of the minimization problem, and h is the equality constraint function (written in the form $h(x) = 0$).

P6. (a) Simplex algorithm for linear programming (M1)

(b) By introducing an auxiliary variable t, the maximization of the objective function can be rewritten as $\max_{t,x} t$ subject to $t \leq 4x_1 - 3x_2 + 8x_3 - 5$ and $t \leq -2x_1 + 7x_2 - x_3 + 1$.
It can be shown that the constraint $7|x_1| + 2|x_2| + 5|x_3| \leq 10$ is equivalent to the following system of affine constraints[6]:

$$\begin{aligned} x_1 + 2x_2 + 5x_3 &\leq 10 \\ 7x_1 + 2x_2 - 5x_3 &\leq 10 \\ 7x_1 - 2x_2 + 5x_3 &\leq 10 \\ 7x_1 - 2x_2 - 5x_3 &\leq 10 \\ -7x_1 + 2x_2 + 5x_3 &\leq 10 \\ -7x_1 + 2x_2 - 5x_3 &\leq 10 \\ -7x_1 - 2x_2 + 5x_3 &\leq 10 \\ -7x_1 - 2x_2 - 5x_3 &\leq 10 \ . \end{aligned}$$

The constraint $3^{2x_1 - 8x_2 + 5x_3} \geq 27$ is equivalent to the affine constraint $2x_1 - 8x_2 + 5x_3 \geq 3$. So we have a linear programming problem and the best suited algorithm is the simplex algorithm (M1).

(c) Since the simplex algorithm finds the optimal solution in a finite number of steps, no stopping criterion is required.

[6] This is allowed since the coefficients of $|x_i|$ are all positive. Compare this with P1, where the coefficients of $|x_i|$ were negative after rewriting the constraint in the form $g(x) \leq 0$.

P7. (a) *Multi-start* Levenberg-Marquardt algorithm (M4)

(b) The objective function is convex, but the constraint is non-convex. By considering the constraint as a quadratic equation in the variable x_1, we obtain:

$$x_1 = \frac{-2x_3 \pm \sqrt{4x_3^2 + 4(9x_3^2 + 5x_2^2 + 2x_2x_3 + 4x_3x_4 + 12x_4^2 + 9)}}{2}$$

(note that the term under the square-root sign is always nonnegative). Using this expression to eliminate x_1 results in two unconstrained optimization problems; the solution with the lowest objective function value will then yield the optimal solution of the original optimization problem. As the unconstrained optimization problems have a non-convex, nonlinear objective function, we need a multi-start approach. The gradient and Hessian of the objective function and the Jacobian of the constraints can be computed analytically. So the most appropriate algorithm is the *multi-start* Levenberg-Marquardt algorithm (M4).

(c) The most appropriate stopping criterion is

$$\|\nabla \tilde{f}(\tilde{x}_k)\|_2 \leq \varepsilon \ ,$$

where $\tilde{x}_k$ and $\tilde{f}$ are respectively the current iteration point and the objective function of the optimization problems obtained after elimination of the variable x_1.

P8. (a) Ellipsoid algorithm (M11)

(b) The arctan function is a monotonically increasing function. So we can also minimize its argument. The expression $4x_1^2 + 2x_1x_2 + 2x_1x_3 + 4x_2^2 + 2x_2x_3 + 4x_3^2 - x_1 + x_2 - x_3$ can be written as $(x_1 + x_2 + x_3)^2 + 3x_1^2 + 3x_2^2 + 3x_3^2 - x_1 + x_2 - x_3$, which is a sum of convex, quadratic functions and of an affine function. As the sum of convex functions is also convex, the given quadratic expression is thus convex.
The first constraint is also convex since the left-hand side of the inequality is the sum of convex, quadratic functions and since the sum of convex functions is also convex.
The second constraint can be rewritten as $|x_1 + x_2 + x_3| - 4 \leq 0$, which is convex since the absolute value function is convex and since a convex function with an affine argument is also convex.
So we have a convex optimization problem. The gradients of the objective function of the constraint functions can be computed analytically. Hence, the most appropriate algorithm is the ellipsoid algorithm (M11).

(c) A suitable stopping criterion is[7]

$$|\tilde{f}(x_k) - \tilde{f}(x^*)| \leq \varepsilon_f \quad \text{and} \quad g(x_k) \leq \varepsilon_g \ ,$$

where x_k is the current iteration point, x^* is the (yet unknown) optimum of this optimization problem, $\tilde{f}$ denotes the objective function (after rewriting the problem as a convex

[7] In addition, $\|x_k - x^*\|_2 \leq \varepsilon_x$ could also be added.

minimization problem), and $\tilde{g}$ denotes the constraint function (with the constraint written in the form $\tilde{g}(x) \leq 0$ with $\tilde{g}$ convex).

P9. (a) *Multi-start* genetic algorithm (M12)

(b) The roots of a 14-th degree polynomial can in general not be expressed in closed form and they should thus be computed numerically. Moreover, the objective function is not convex (due to the non-convex relation between the parameters z and the roots ρ of the polynomial). So we have a nonlinear non-convex optimization problem. Therefore, a multi-start optimization method is required. The gradient and Hessian of the objective function cannot be computed analytically. The numerical computation of the roots will in general be time-consuming and as such it is better not to use numerical computation of the gradient and Hessian of the objective function, but rather we should select a gradient-free optimization method. So in this case a *multi-start* genetic algorithm (M12) is the most suited optimization algorithm[8].

(c) A suitable stopping criterion is an upper bound on the number of generations.

P10. (a) Ellipsoid algorithm (M11)

(b) The maximization problem can be recast as a minimization problem: $\min_x (3x_1^2 + 2x_2^2 + 8x_3^2 - x_1 - 3x_2 + 4x_3 - 8)^5$. As the function $x \mapsto x^5$ is a monotonically increasing function, we can minimize its argument instead. This results in the minimization of the convex, quadratic objective function $3x_1^2 + 2x_2^2 + 8x_3^2 - x_1 - 3x_2 + 4x_3 - 8$.
The first constraint can be rewritten as (see also P6 above):

$$\begin{aligned}
x_1 + x_2 + x_3 &\leq 3 \\
x_1 + x_2 - x_3 &\leq 3 \\
x_1 - x_2 + x_3 &\leq 3 \\
x_1 - x_2 - x_3 &\leq 3 \\
-x_1 + x_2 + x_3 &\leq 3 \\
-x_1 + x_2 - x_3 &\leq 3 \\
-x_1 - x_2 + x_3 &\leq 3 \\
-x_1 - x_2 - x_3 &\leq 3 \quad ,
\end{aligned}$$

which is a system of affine constraints.
The second constraint can be rewritten as $\cosh(2x_1 + 3x_2 - 7x_3) \leq 5$ or equivalently $-\operatorname{acosh}(5) \leq 2x_1 + 3x_2 - 7x_3 \leq \operatorname{acosh}(5)$, where acosh denotes the inverse of cosh.
So we have a convex, quadratic objective function with affine constraints, i.e., a convex quadratic programming problem. As the modified simplex algorithm is not in the list, the best choice is the ellipsoid algorithm (M11).

[8] Also note that the objective function does not involve minimization of an error; so the Gauss-Newton least-squares algorithm cannot be used here.

(c) A suitable stopping criterion is[9]

$$|\tilde{f}(x_k) - \tilde{f}(x^*)| \leq \varepsilon_f \quad \text{and} \quad g(x_k) \leq \varepsilon_g \ ,$$

where x_k is the current iteration point, x^* is the (yet unknown) optimum of this optimization problem, $\tilde{f}$ denotes the objective function (after rewriting the problem as a convex minimization problem), and $\tilde{g}$ denotes the constraint function (with the constraint written in the form $\tilde{g}(x) \leq 0$ with $\tilde{g}$ convex).

[9] In addition, $\|x_k - x^*\|_2 \leq \varepsilon_x$ could also be added.

QUESTION 2: Optimization methods II

Answer for Task 1

- The constant 8 does not change the position of the optimum and can thus be omitted. In order to obtain a minimization problem we rewrite the maximization as $\min_{x \in \mathbb{R}^4} -7x_1 - 2x_2 + 2x_3 + 9x_4$.
- By introducing nonnegative slack variables the constraints (except for $x_1 \geq 0$, which will be merged into $x \geq 0$) can be transformed into equality constraints:

$$\begin{aligned}
&x_1 - 2x_2 + 3x_3 + 8x_4 + s_1 = 6 \\
&-x_1 + 3x_2 - 6x_3 + x_4 - s_2 = 2 \\
&x_1 - x_3 + s_3 = 5 \\
&x_1 + s_4 = 9 \\
&x_2 + s_5 = 1
\end{aligned}$$

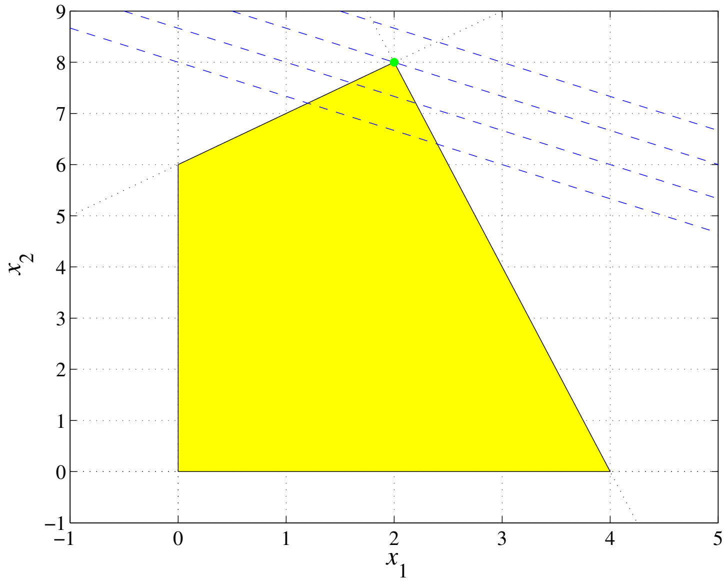
  with $s \geq 0$.
- Note that x_2, x_3, and x_4 are real-valued and can thus also become negative. To obtain only nonnegative variables, we split x_2, x_3, and x_4 into their positive and negative parts $x_2 = x_2^+ - x_2^-$, $x_3 = x_3^+ - x_3^-$, $x_4 = x_4^+ - x_4^-$ with $x_2^+, x_2^-, x_3^+, x_3^-, x_4^+, x_4^- \geq 0$, and rewrite the above equations, resulting in

$$\begin{aligned}
\min_{\tilde{x} \in \mathbb{R}^{12}} \; & -7x_1 - 2x_2^+ + 2x_2^- + 2x_3^+ - 2x_3^- + 9x_4^+ - 9x_4^- \\
\text{s.t.} \; & x_1 - 2x_2^+ + 2x_2^- + 3x_3^+ - 3x_3^- + 8x_4^+ - 8x_4^- + s_1 = 6 \\
& -x_1 + 3x_2^+ - 3x_2^- - 6x_3^+ + 6x_3^- + x_4^+ - x_4^- - s_2 = 2 \\
& x_1 - x_3^+ + x_3^- + s_3 = 5 \\
& x_1 + s_4 = 9 \\
& x_2^+ - x_2^- + s_5 = 1 \\
& \tilde{x} \geq 0
\end{aligned}$$

  where $\tilde{x} = \begin{bmatrix} x_1 & x_2^+ & x_2^- & x_3^+ & x_3^- & x_4^+ & x_4^- & s_1 & s_2 & s_3 & s_4 & s_5 \end{bmatrix}^{\mathrm{T}}$.

Answer for Task 2

The problem can be solved in a graphical way. The feasible set has four vertex points, namely $(0,0)$, $(4,0)$, $(0,6)$, $(2,8)$ corresponding to the (feasible) intersections of the boundaries of the constraints (see also the figure below). By considering the line $2x_1 + 3x_2 = c$ and determining the largest value of c for which this line still intersects with the feasible region (see the dashed lines in the figure below), one finds that the maximum is reached in the vertex point $x^* = (2,8)$; the corresponding value of the objective function is 28.



Answer for Task 3

Here we essentially expect a summary of Section 2.2 of the lecture notes. Be sure to mention the following elements:

- transformation into standard form,
- basic solutions & vertices of the feasible region,
- the constraint matrix A is split into two parts B and N with B square and invertible,
- columns of B and N will be swapped,
- rules for selecting the columns and an intuitive interpretation of these rules,
- stop criterion,
- termination in a finite number of steps.

QUESTION 3: Controller design

Answer for Task 1

We have[10]

$$y = P(d + K(r - y)) = Pd + PKr - PKy$$
$$u = K(r - P(d + u)) = -PKd + Kr - KPu$$

and thus

$$\begin{bmatrix} y \\ u \end{bmatrix} = \underbrace{\frac{1}{1+PK}\begin{bmatrix} P & PK \\ -PK & K \end{bmatrix}}_{G} \begin{bmatrix} d \\ r \end{bmatrix} \quad .$$

Answer for Task 2

From the answer for Task 1 it follows that $H_{ry} = G_{12} = \dfrac{PK}{1+PK}$. Replacing K by $\dfrac{Q}{1-PQ}$ yields

$$H_{ry} = \frac{PQ}{1 - PQ + PQ} = PQ \quad .$$

So for $P = 2$ we get $H_{ry} = 2Q$.

(a) The design specification $4 \leq \|H_{ry}\|_\infty \leq 8$ can be recast as $2 \leq \|Q_{ry}\|_\infty \leq 4$. To show that this design specification is *not* closed-loop convex in Q, we construct a counter-example, i.e., we provide two values Q_1 and Q_2 for which the condition is satisfied and a value $\lambda \in [0,1]$ such that the condition is *not* satisfied for $\lambda Q_1 + (1-\lambda)Q_2$.
To this aim, take $Q_1 = 3$ and $Q_2 = -Q_1 = -3$. Clearly, both Q_1 and Q_2 satisfy the design specification. However, for $\lambda = 0.5$ their convex combination $\frac{1}{2}Q_1 + \frac{1}{2}Q_2 = 0$ does not satisfy the design specification. Hence, the given design specification is not closed-loop convex.

(b) The design specification $4 \leq \text{Re}\{H_{ry}\} \leq 8$ can be recast as $2 \leq \text{Re}\{Q\} \leq 4$. To show that this design specification is closed-loop convex in Q, we consider Q_1 and Q_2 such that $2 \leq \text{Re}\{Q_1\} \leq 4$ and $2 \leq \text{Re}\{Q_2\} \leq 4$, and we show that $2 \leq \text{Re}\{\lambda Q_1 + (1-\lambda)Q_2\} \leq 4$ for all $\lambda \in [0,1]$.
We have

$$\begin{aligned}\text{Re}\{\lambda Q_1 + (1-\lambda)Q_2\} &= \text{Re}\{\lambda Q_1\} + \text{Re}\{(1-\lambda)Q_2\} \\ &= \lambda \text{Re}\{Q_1\} + (1-\lambda)\text{Re}\{Q_2\} \quad .\end{aligned}$$

So $\text{Re}\{\lambda Q_1 + (1-\lambda)Q_2\} = \lambda \text{Re}\{Q_1\} + (1-\lambda)\text{Re}\{Q_2\} \leq \lambda 4 + (1-\lambda)4 \leq 4$. In a similar way we can show that $\text{Re}\{\lambda Q_1 + (1-\lambda)Q_2\} \geq 2$. Hence, $\lambda Q_1 + (1-\lambda)Q_2$ satisfies the design specification for all $\lambda \in [0,1]$ and therefore the given design specification is closed-loop convex.

[10]The arguments k and q are omitted next for the sake of compactness of notation.