Exam — November 2018
Optimization in Systems and Control (SC42055)

QUESTION 1: Optimization methods I (60 points)

Consider the following optimization problems:

P1. $$\min_{x\in\mathbb{Z}^4} 5^{4x_1+2x_2-3x_3-8x_4-5}$$
$$\text{s.t. } \|x\|_2 \leq 25$$
$$|7x_1-8x_2+3x_3+6x_4-15| \leq 165$$
$$2x_1-3x_2-x_3+7x_4+9 \geq 0$$

P2. $$\min_{\tau\in\mathbb{R}^{10}} \sum_{k=1}^{7500} \big(z(k,\tau)-z_{\text{ref}}(k)\big)^2$$
where the signal $\{z(k,\tau)\}_{k=1}^{7500}$ is the sampled output (with a sample time step of 0.2 s) of a closed-loop system consisting of a high-order nonlinear plant and a controller with 10 controller parameters (τ_1, τ_2, $\ldots$, τ_{10}) and where $\{z_{\text{ref}}(k)\}_{k=1}^{7500}$ is a given reference signal.

P3. $$\max_{x\in\mathbb{R}^4} -8x_1^2-2x_1x_2-x_2^2-4x_3^2-4x_3x_4-7x_4^2+3x_1-4x_2+8x_3-17$$
$$\text{s.t. } 16 \leq 2^{3x_1}2^{4x_2}2^{x_3}2^{-2x_4} \leq 16^5$$
$$x_1^4+x_2^4+x_3^4+x_4^4 \leq 8100$$
$$\min(x_1+x_2+2x_3+x_4,\, 4x_1-x_2+6x_3-5x_4) \leq 42$$

P4. $$\min_{x\in\mathbb{R}^5} \exp(x_1^2+x_2^2+x_3^2+2x_5)+\max\Big(\sqrt{(2x_1-5x_2+8x_4-2)^2},\, (4x_1-x_2-2x_3-7x_5)^{-4}\Big)$$
$$\text{s.t. } 4x_1-x_2-2x_3-7x_5 \geq 1$$
$$\log_{10}(1-8x_1^2+6x_2^2-x_3-x_4+2x_5) \geq 1000$$
$$8x_1^2+6x_2^2+x_3+x_4-2x_5 \leq 12$$
$$2(x_1^2+x_2^2)^4+4(x_3-x_4+x_5) \leq 10000$$

P5. $$\min_{x\in\mathbb{R}^4} \max\big(2^{x_2+2x_3+x_4^2},\, \cosh(x_1+2x_2-3x_3+8x_4-3)\big)+\left(\frac{1}{2}\right)^{1-x_1^4-2x_2^2-8x_3^8}$$

P6. $\max_{x\in\mathbb{R}^3} -3^{x_1^2-4x_1x_2+x_2^2+8x_3^2-x_1-3x_2+x_3-7}$

s.t. $\|x\|_1 \leq 1$

$\big(\exp(x_1+x_2-x_3)\big)^3 \leq 10^9$

$\max\big(|1+x_1+20x_2+30x_3-7|, (x_1+8x_2-x_3)^2\big) \leq 900$

P7. $\min_{x\in\mathbb{R}^4} (7|x_1|+3|x_2|+|x_3|+2|x_4|+1)^2$

s.t. $(6+3x_1+x_2-x_3-x_4)^4 \leq 16000$

$\max(3+2x_1+3x_2-x_4, 4x_2+3x_4, 2x_1-8) \leq 9$

$\|x\|_\infty \geq 3$

P8. $\max_{x\in\mathbb{R}^4} \big(7-\exp(x_1^2+2x_2^2+x_3^2-2x_3x_4+x_4^2-3x_1)\big)$

s.t. $|x_1|+2|x_2|+3|x_3|+4|x_4| \leq 4$

$\cosh(2x_1+3x_2-4x_3+8x_4) \geq 3$

P9. $\min_{x\in\mathbb{R}^3} 9x_1^2+4x_2^2+2x_1x_2+8x_3^2-2x_1x_3$

s.t. $\exp(3x_1^2+2x_2^2+x_3^2)+(4x_1+5x_2+6x_3)^2+x_1-x_3 = 100$

P10. $\min_{x\in\mathbb{R}^4} 5\log(x_1+2x_2+6x_3+x_4+1)+2\exp\big((x_1+x_2-4x_3-x_4+3)^4-1\big)$

s.t. $\min(x_1+2x_2, 6x_3+x_4) \geq 1$

$x_1^2+x_2^4+x_3^6+x_4^8 \leq 25$

Tasks: For each of the problems P1 up to P10:

(a) Give the *most simple type* of optimization problem that the given problem can be reduced to, *irrespective* of the optimization algorithm used later on (So if you have a nonconvex constrained problem that you will solve using, e.g., a barrier or penalty function method in item (b), you should still mark NCC here). Select one of the following types:

- LP: linear programming problem
- QP: convex quadratic programming problem
- CP: convex optimization problem
- NCU: nonconvex unconstrained optimization problem
- NCC: nonconvex constrained optimization problem
- MILP: mixed-integer linear programming problem

Note that you may have to perform some extra steps first in order to *simplify* the optimization problem. If you do this, then you have to ***indicate*** it clearly in your reply to item (d).

(b) Now determine the *best suited optimization method* for the given problem. Select one of the following methods:

M1 : Simplex algorithm for linear programming
M2 : Modified simplex algorithm for quadratic programming
M3 : Gradient projection method with variable step size line minimization
M4 : Gauss-Newton least squares algorithm
M5 : Levenberg-Marquardt algorithm
M6 : Line search method with the steepest descent direction and quadratic line minimization
M7 : Lagrange method + Davidon-Fletcher-Powell quasi-Newton algorithm
M8 : Lagrange method + Nelder-Mead method
M9 : Penalty function approach + line search method with Powell directions and variable step size line minimization
M10 : Barrier function approach + line search method with Powell directions
M11 : Simulated annealing
M12 : Branch-and-bound method for mixed-integer linear programming

* If it is necessary to use the selected algorithm in a *multi-start optimization* procedure, you have to ***indicate*** this clearly in your reply to item (b) by putting a checkmark in the Multi-start box.

* Give only one solution! If more than one solution is given, the worst one will be taken.

(c) Give the most appropriate stopping criterion for the selected optimization method.
Do *not* reply by referring to your answer to a previous or subsequent problem. Such a statement will considered to be a wrong answer.

(d) ***Explain and motivate*** your reply to items (a) and (b).

QUESTION 2: Optimization methods II (25 points)

<u>Tasks:</u>

1. The first task involves mixed-integer linear optimization problems:

   a) Explain how the branch-and-bound method for mixed-integer linear programming works.

   b) Under which conditions is a mixed-integer linear optimization problem convex? Explain your answer.

   c) For regular linear optimization problems one of the vertices of the set determined by the linear equality and inequality constraints will always be an optimal point. Does that also hold for mixed-integer linear optimization problems. Why (not)?

2. Now consider the following optimization problem:

$$\begin{aligned} \max_{x \in \mathbb{R}^2} \; & x_1 + x_2 \\ \text{s.t.} \; & x_1 + 0.5x_2 \leqslant 1 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

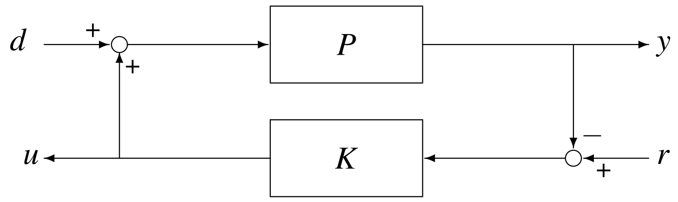
   a) Is this optimization problem convex or not? Why?

   b) Now determine an optimum of the given optimization problem.

   c) Is the optimum you found a global optimum? Why?

QUESTION 3: Controller design (15 points)

Consider the following closed-loop system consisting of a ***stable*** plant P with transfer function $P(q)$ and a controller K with transfer function $K(q)$, where q denotes the forward shift operator:



There are two external input signals: the process noise d and the reference signal r. There are two external output signals: u and y.

Tasks:

1. Determine the transfer function matrix M such that

$$\begin{bmatrix} y(k) \\ u(k) \end{bmatrix} = M(q) \begin{bmatrix} d(k) \\ r(k) \end{bmatrix} .$$

2. Consider a controller of the form

$$\hat{K}(q) = \frac{Q(q)}{1 - P(q)Q(q)}$$

with

$$Q(q) = \theta_1 + \theta_2 q^{-1} + \theta_3 q^{-2}$$

where θ_1, θ_2, and θ_3 are the *real-valued* design parameters of the controller.

Consider the set $\hat{S}$ of all points $(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ for which the closed-loop system with the controller $\hat{K}$ given above is stable (i.e., such that all entries of the transfer function matrix M are stable).

Show that the set $\hat{S}$ is convex.

3. Now let the plant P be a delay, so $P(q) = q^{-1}$, and consider another type of controller defined by

$$\tilde{K}(q) = \frac{1}{\kappa + q^{-1}}$$

where κ is the *real-valued* design parameter of the controller.

Define $\tilde{S}$ as the set of all values $\kappa \in \mathbb{R}$ for which the closed-loop system with the controller $\tilde{K}$ is stable (i.e., such that all entries of the transfer function matrix M are stable).

Show that $\tilde{S}$ is not convex.

End of the exam