Exam — November 2019
Optimization in Systems and Control (SC42055)

QUESTION 1: Optimization methods I (60 points)

Consider the following optimization problems:

P1. $$\min_{x\in\mathbb{R}^4} \sqrt[5]{(x_1+6x_2+8x_3-9x_4-10)^2}$$
$$\text{s.t. } x_1^3+x_3^2+3x_2^2+2x_2^2x_3^2+4x_3^2x_4^2+8x_4^4=5$$

P2. $$\max_{x\in\mathbb{Z}^4} \left(\frac{1}{5}\right)^{4x_1+2x_2-3x_3-8x_4-5}$$
$$\text{s.t. } \|x\|_1 \leq 25$$
$$|7x_1-8x_2+3x_3+6x_4-15| \leq 165$$
$$2x_1-3x_2-x_3+7x_4+10 \geq 1$$

P3. $$\min_{\theta\in\mathbb{R}^{10}} F(\theta)$$
$$\text{s.t. } 0 \leqslant \theta_i \leqslant 1 \quad \text{for all } i$$
where
$$F(\theta)=\int_0^{\theta_1}\int_{\theta_2}^{1}\int_0^{\theta_3} \frac{\cosh(\theta_4 t_1)+\theta_5(1+t_2)^{\theta_6}+\exp(2t_3)(1+\sin(\theta_7 t_1^2))}{|2+t_2\cos(\theta_6 t_2)+\sin(\theta_8 t_1+\theta_9 t_3)^2+\exp(\theta_{10}t_2)|}\, dt_1 dt_2 dt_3$$

P4. $$\min_{x\in\mathbb{R}^4} (7|x_1|+3|x_2|+|x_3|+2|x_4|+1)^2$$
$$\text{s.t. } (6+3x_1+x_2-x_3-x_4)^4 \leq 16000$$
$$\min(3+2x_1+3x_2-x_4, 4x_2+3x_4, 2x_1-8) \geq 9$$
$$\|x\|_1 \leq 3$$

P5. $$\max_{x\in\mathbb{R}^4} -8x_1^2-2x_1x_2-x_2^2-4x_3^2-4x_3x_4-7x_4^2+3x_1-4x_2+8x_3-13$$
$$\text{s.t. } 1 \leq 3^{3x_1}3^{4x_2}3^{x_3}3^{-2x_4} \leq 729$$
$$6x_1^4+x_2^2+x_3^4+2x_4^2 \leq 8100$$
$$\max(x_1+x_2+2x_3+x_4,\ 4x_1-x_2+6x_3-5x_4) \geq 25$$

P6. $\min_{x\in\mathbb{R}^4} |5x_1 - 2x_2 + 6x_3 + 18| + 3\cosh\big(0.001(4x_1 - x_2 + x_3 - 2x_4)^4 - 3\big)$

s.t. $\|x\|_2 \leq 5$

P7. $\max_{x\in\mathbb{R}^3} -5^{x_1^2-4x_1x_2+x_2^2+8x_3^2-x_1-3x_2+x_3-7}$

s.t. $\|x\|_\infty \leq 1$

$\big(\cosh(x_1 + x_2 - x_3)\big)^3 \leq 10^9$

$\max\big(|1 + x_1 + 20x_2 + 30x_3 - 7|, (x_1 + 8x_2 - x_3)^2\big) \leq 900$

P8. $\min_{x\in\mathbb{R}^5} \exp(x_1^2 + x_2^2 + x_3^2 + 2x_5) + \max\big(|2x_1 - 5x_2 + 8x_4 - 2|, (4x_1 - x_2 - 2x_3 - 7x_5)^{-2}\big)$

s.t. $4x_1 - x_2 - 2x_3 - 7x_5 \geq 1$

$\log(1 - 8x_1^2 - 6x_2^2 - x_3 - x_4 + 2x_5) \geq 9$

$8x_1^2 + 6x_2^2 + x_3 + x_4 - 2x_5 \leq 12$

$2(x_1^2 + x_2^2)^4 + 4(x_3 - x_4 + x_5) \leq 10000$

Tasks: For each of the problems P1 up to P8:

(a) Give the *most simple type* of optimization problem that the given problem can be reduced to, *irrespective* of the optimization algorithm used later on (So if you have a nonconvex constrained problem that you will solve using, e.g., a barrier or penalty function method in item (b), you should still mark NCC here). Select one of the following types:

- LP: linear programming problem
- QP: convex quadratic programming problem
- CP: convex optimization problem
- NCU: nonconvex unconstrained optimization problem
- NCC: nonconvex constrained optimization problem
- MILP: mixed-integer linear programming problem

The first checkbox serves to indicate that a problem can be recast as multiple, more simple problems. If you obtain, e.g., 4 linear programming problems, check — next to the LP checkbox — also the first checkbox and fill out 4 on the dots.
Note that you may have to perform some extra steps first in order to *simplify* the optimization problem. If you do this, then you have to ***indicate*** it clearly in your reply to item (e).

(b) For the *simplified* problem you obtained in step (a), indicate whether the objective function and the constraints are linear, quadratic, convex, or nonconvex.

(c) Now determine the *best suited optimization method* for the given problem. Select one of the following methods:

M1 : Simplex algorithm for linear programming
M2 : Modified simplex algorithm for quadratic programming
M3 : Gradient projection method with variable step size line minimization
M4 : Gauss-Newton least squares algorithm
M5 : Davidon-Fletcher-Powell quasi-Newton algorithm
M6 : Levenberg-Marquardt algorithm
M7 : Steepest descent method
M8 : Line search method with the steepest descent direction and quadratic line minimization
M9 : Lagrange method + steepest descent method
M10 : Penalty function approach + line search method with Powell directions and variable step size line minimization
M11 : Barrier function approach + line search method with steepest descent direction
M12 : Branch-and-bound method for mixed-integer linear programming

* If it is necessary to use the selected algorithm in a *multi-start optimization* procedure, you have to ***indicate*** this clearly in your reply to item (b) by putting a checkmark in the Multi-start box.
* Give only one solution! If more than one solution is given, the worst one will be taken.

(d) Give the most appropriate stopping criterion for the selected optimization method.
Do *not* reply by referring to your answer to a previous or subsequent problem. Such a statement will considered to be a wrong answer.

(e) ***Explain and motivate*** your reply to items (a), (b), and (c).

QUESTION 2: Optimization methods II (25 points)

<u>Tasks:</u>

1. Explain how the Levenberg-Marquardt method for unconstrained optimization works. Illustrate your explanation by drawing one or more pictures. Also give the general iteration formula and explain what the advantage is of the Levenberg-Marquardt method compared to the Newton and quasi-Newton methods for unconstrained optimization.

2. Now consider the following optimization problem:

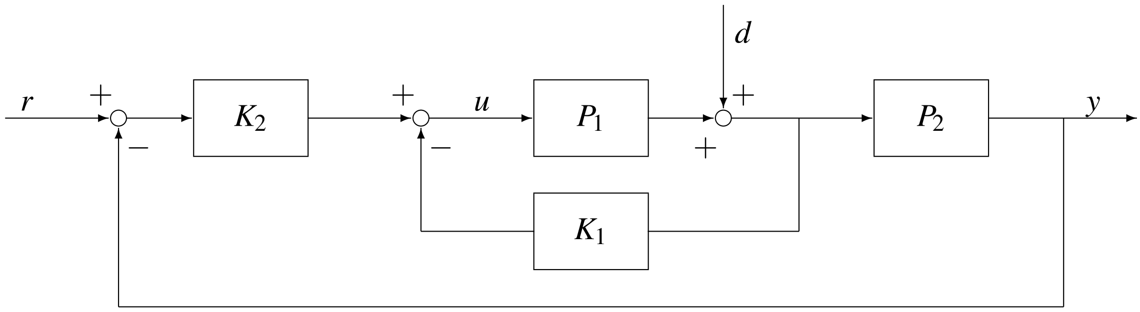
$$\begin{aligned} \max_{(x,y)\in\mathbb{R}^2} \; & x^2 + 2y^2 - xy - 4y + 12 \\ \text{s.t.} \; & x, y \geqslant 0 \\ & x + y \leqslant 2 \; . \end{aligned}$$

   a) Is this optimization problem convex or not? Why?

   b) Now solve the given optimization problem using the gradient projection method and perform two iteration steps:
   Take $(x_0, y_0) = (1, 0)$ as starting point, compute the two subsequent search directions for the gradient projection method, and perform the corresponding line minimizations, where for the line minimization you should determine the *exact* line minimum.

QUESTION 3: Controller design (15 points)

Consider the following closed-loop system consisting of the stable systems P_1 and P_2, and two controllers K_1 and K_2:



The signal r is the reference signal, d is a noise signal, y is the output of the system, and u is the input signal for system P_1.
The signals r and d are considered to be the input signals of the overall closed-loop system; the signals y and u are considered to be the output signals of the overall closed-loop system.

Tasks:

a) Determine the transfer function matrix M in terms of P_1, P_2, K_1, and K_2 such that

$$\begin{bmatrix} u(k) \\ y(k) \end{bmatrix} = M(q) \begin{bmatrix} r(k) \\ d(k) \end{bmatrix}$$

where q denotes the forward shift operator.

b) Show that any controller parameterized by

$$K_1(q) = \frac{Q_1(q)}{1 - P_1(q)Q_1(q) - P_1(q)P_2(q)Q_2(q)}$$
$$K_2(q) = \frac{Q_2(q)}{1 - P_1(q)Q_1(q) - P_1(q)P_2(q)Q_2(q)}$$

where Q_1 and Q_2 are rational stable transfer functions, will stabilize the overall closed-loop system.

End of the exam