Exam — November 2020
Optimization in Systems and Control (SC42055/6)

- This exam is an online open-book take-home exam using the Brightspace Assignments function.
- Make sure to upload a ***single-file pdf file*** with the scans or pictures of your handwritten notes (other formats will not be allowed) by 15.00 on November 6, 2020.
- If you are a registered extra-time student, you get until 15.15 on November 6, 2020 to upload your solution.
- Please recall that the number of questions is larger than what the majority of students will be able to answer within the allocated time span. So there is ***no need at all to worry if you cannot answer all questions***. Just start with the questions that you feel most familiar with and try to answer as many questions correctly as possible.
- The exam consists of 2 main questions. For Question 1 the maximal score for each problem is 7.5; the maximal score for Question 2.a is 10 and for Question 2.b it is 12. The detailed subscores are marked in red next to each task.
- Make sure to clearly motivate your answers; so make sure to provide both the final results as well as important intermediate steps and the procedure followed to reach the results. Just listing the final answer without any explanation is not sufficient.
- It is also important to note that similarities in replies among different students will be penalized, and in the worst case reported to the Board of Examiners.
- Matlab and other computation tools can be used for verification purposes only; they cannot be used to replace calculations by hand.
- Put your name and your student ID on *each* page of your solutions.
- Copy the following text on the first page of your solution in your own handwriting:

  > I declare that I will make this take-home exam all by myself without any support from or direct or indirect contact with anybody else, that I will not cheat in any way, and that I will not help other students with their exam. I am aware that fraud can lead to this exam being declared invalid.

- Also copy the following text on the first page of your solution in your own handwriting:

  > I have also read and understood the document of September 28, 2020 that explains the procedure for the SC42056 Q1 2020-2021 exam and that I am aware of and agree with the possible consequences of not following the procedure.

  For your information this document is attached once more at the end of the exam.

- **Communication during the exam**:

  - For very urgent questions during the exam, you can contact me by email at ee2s21-DCSC@tudelft.nl
  - This email address can also be used as a backup in case of upload problems via Brightspace.
  - If there are very urgent messages towards all the students during the exam, I will use the Announcement function of Brightspace.

- I wish all of you good luck with the exam!

QUESTION 1: Optimization methods I ($8 \times 7.5 = 60$ points)

Throughout this question let α be the last **non-zero** digit of your student ID, and let β be the one but last **non-zero and non-α** digit of your student ID. So if your student ID is 12490090, then $\alpha = 9$ and $\beta = 4$. Moreover, we always have $\alpha \neq \beta$.
When applicable (i.e., when α or β are mentioned), list the values of α and β in your answers, and use their *numerical* values in the problems below – so do not work with symbolic expressions.

Consider the following optimization problems:

P1.
$$\min_{x\in\mathbb{R}^5} \sqrt[3]{x_1^6 + 7x_2^4 + x_3^2 - x_3x_4 + x_4^2 - 9x_5 + 6}$$
$$\text{s.t. } (1 + x_1^2 + x_2^4 + x_3^2)^4 + 3x_2 + 7x_3^2 + 3x_4^6 + x_5^2 + (\alpha+1)^{x_1+2x_2+4x_3-x_4+x_5} = 80$$

P2.
$$\max_{x\in\mathbb{Z}^4} (5x_1 + 4x_2 - 2x_3 - 8x_4 - 7)^{-5}$$
$$\begin{aligned} \text{s.t. } & 5x_1 + 4x_2 - 2x_3 - 8x_4 - 7 \geq \alpha \\ & \|x\|_1 \leq 121 \\ & |7x_1 - 8x_2 + 3x_3 + 6x_4 - 15| \leq 196 \\ & 2x_1 - 3x_2 - x_3 + 7x_4 + 13 \geq 0 \end{aligned}$$

P3.
$$\min_{\theta\in\mathbb{R}^7} \sum_{k=1}^{5000} z^2(k,\theta) \qquad \text{s.t. } \|\theta\|_1 \leq 10 + \alpha$$
where the signal $\{z(k,\theta)\}_{k=1}^{5000}$ is the 1-step ahead prediction error for a discrete-time system (with a sample time step of 0.1 s) for a white-noise input signal $\{u(k)\}_{k=1}^{5000}$ and where a discrete-time model of the form
$$\begin{aligned} y(k) = {} & \theta_1 e^{\alpha y(k-1)} + \theta_2 y(k-3)\cos(u(k)) + \theta_3 \frac{u^2(k-1)}{1 + y^2(k-2)u(k-2)} + \\ & \theta_4 u^4(k-2) + \theta_5 y(k-4)\sin(y(k-1)) + \theta_6 |u(k-3)| + \beta\theta_7 \end{aligned}$$
with initial conditions $y(i) = \sqrt{\alpha}$ and $u(j) = 1$ for $i = 0, -1, -2, -3$ and $j = 0, -1, -2$ is considered.

P4.
$$\min_{x\in\mathbb{R}^5} (13|x_1| + 2|x_2| + 8|x_3| + 6|x_4| + 3|x_5| + 7)^{3+\alpha}$$
$$\begin{aligned} \text{s.t. } & (5 + 7x_1 + 3x_2 - 6x_3 + x_4 - x_5)^2 \leq 256 \\ & \max(-8 + 3x_5, 4 + 7x_1 + 3x_2 - x_4, 4x_3 + 3x_4, 2x_1 + x_5 - 8) \leq 9 \\ & \|x\|_\infty \geq 2 + \beta \end{aligned}$$

P5.
$$\min_{x\in\mathbb{R}^4} 3\exp\big((x_1 + x_2 - 4x_3 - x_4 + 3)^4 - 1\big) + (-1)^\beta \log_3(x_1 + 2x_2 + 6x_3 + x_4 + 1)$$
$$\begin{aligned} \text{s.t. } & \min(7x_1 + 9x_2, 8x_3 + x_4, x_2 + 4x_3 + x_4) \geq 1 \\ & 9 + (-1)^{\alpha+1}(x_1^2 + x_2^4 + x_3^6 + x_4^8) \leq 36 \end{aligned}$$

P6. $\max_{x\in\mathbb{R}^4} \; -14x_1^4+\alpha x_1x_2-13x_2^4-7x_3^2-2x_3x_4-6x_4^2+2x_1-5x_2+18x_3+7\beta$

$$\begin{aligned}\text{s.t. }\; & \cosh(3)\le\cosh(2x_1)\cosh(4x_2)\cosh(x_3)\cosh(-2x_4)\le\cosh(15)\\ & \max(x_1+x_2+2x_3+x_4,\, 4x_1-x_2+6x_3-5x_4)\ge 4\\ & x_1^{2\alpha}+x_2^{2\alpha}+x_3^{2\alpha}+x_4^{2\beta}\le e^9\end{aligned}$$

P7. $\max_{x\in\mathbb{R}^4} \; \sinh\left(1-\cosh(2x_1^2+4x_2^2+2x_3^2-8x_3x_4+16x_4^2)\right)$

$$\begin{aligned}\text{s.t. }\; & x_1+2x_2+3x_3+4x_4\ge 5\\ & (7x_1+x_2-2x_3+5x_4)^{2\beta}\ge\alpha\end{aligned}$$

P8. $$\min_{\theta\in\mathbb{R}^{10}} \int_0^{\theta_1}\int_{\theta_2}^{\beta}\int_0^{\theta_3} \frac{\sinh(\theta_4t_1)+\theta_5^2(1+t_2)^{\theta_6}+\exp(7t_3)(1+\cos(\theta_7t_1^2))}{|2+t_2\cos(\theta_6t_2)+\sin(\theta_8t_1+\theta_9t_3)^2+\exp(\theta_{10}t_2)|}\,dt_1dt_2dt_3$$

s.t. $0\leqslant\theta_i\leqslant\min(\alpha,\beta)$ for all i

Important for all tasks:

* Give only one solution! If more than one solution is given, the worst one will be taken.

* In your answer do *not* refer to your answer to a previous or a subsequent problem. Such a statement will considered to be a wrong answer.

* A correct answer without the correct motivation will be considered to be a wrong answer.

Tasks: For each of the problems P1 up to P8:

1 (a) Replace α and β by their numerical value, and reduce the given optimization problem to the *most simple type* of optimization problem that the given problem can be reduced to. Explain how you reduced the problem.

3 (b) For the *simplified* problem you obtained in Task (a), indicate whether the objective function is linear, convex quadratic, convex, or nonconvex over the feasible set (***excluding*** constraints that restrict the variables to a discrete set, as such constraints would in general make the objective function nonconvex). Select the most restrictive option.
Motivate your answer.

Moreover, for ***each*** individual constraint of the *simplified* problem indicate whether the constraint is linear, convex, or nonconvex over the feasible set consisting of the other constraints (***excluding*** constraints that restrict the variables to a discrete set, as such constraints would in general make the constraint nonconvex). Select the most restrictive option.
Motivate your answer.

1 (c) Characterize the type of the *simplified* optimization problem you obtained in Task (a). Select one of the following types:

- LP: linear programming problem
- QP: convex quadratic programming problem
- CP: convex optimization problem
- NCU: nonconvex unconstrained optimization problem
- NCC: nonconvex constrained optimization problem
- MILP: mixed-integer linear programming problem

Select the most restrictive option.
If the simplified optimization problem is a multiple one, then indicate that explicitly. So if you obtain, e.g., 4 linear programming problems, then indicate "4 $\times$ LP".
Motivate your answer.

[**Important**: The list of tasks is continued on the next page.]

2 (d) Now determine the *best suited optimization method* for the given problem. Select one of the following methods:

M1 : Simplex algorithm for linear programming

M2 : Gradient projection method with variable step size line minimization

M3 : Ellipsoid algorithm

M4 : Gauss-Newton least squares algorithm

M5 : Davidon-Fletcher-Powell quasi-Newton algorithm

M6 : Levenberg-Marquardt algorithm

M7 : Line search method with the steepest descent direction and cubic line minimization

M8 : Lagrange method + Davidon-Fletcher-Powell quasi-Newton algorithm

M9 : Lagrange method + Nelder-Mead method

M10 : Barrier function approach + steepest descent method

M11 : Branch-and-bound method for mixed-integer linear programming

Motivate your answer.

* If it is necessary to use the selected algorithm in a *multi-start or multi-run optimization* procedure, you have to ***indicate*** this clearly in your answer. Then also motivate why multi-start or multi-run optimization is needed.

0.5 (e) Give the most appropriate stopping criterion for the selected optimization method.

QUESTION 2: Optimization methods II $(10+12=22$ points)

Throughout this question let α be the last **non-zero** digit of your student ID, and let β be the one but last **non-zero and non-α** digit of your student ID. So if your student ID is 12490090, then $\alpha = 9$ and $\beta = 4$. Moreover, we always have $\alpha \neq \beta$.
When applicable (i.e., when α or β are mentioned), list the values of α and β in your answers, and use their *numerical* values in the problems below – so do not work with symbolic expressions.

Question 2.a (10 points)

Consider the following optimization problem:

$$\begin{aligned}
\min_{x \in \mathbb{R}^2} \quad & 3^{\alpha x_1 - x_2 - \beta} \\
\text{s.t.} \quad & \|x\|_1 \leq 20 \\
& 0.5x_1 + \beta x_2 \geq \alpha
\end{aligned}$$

Tasks:

2 (a) Is this optimization problem convex or not? Motivate your answer.

7 (b) Determine an optimum of the given optimization problem. Explain your answer.

1 (c) Is the optimum you found a global optimum? Motivate your answer.

Question 2.b (12 points)

Consider the following optimization problem:

$$\max_{(x,y) \in \mathbb{R}^2} \alpha + (-1)^{\alpha}(x^2 + 2y^2 - xy - \beta y)$$

Tasks:

2 (a) Is this optimization problem convex or not? Motivate your answer.

10 b) Solve the given optimization problem using the steepest descent method and perform one iteration step:
Take $(x_0, y_0) = (\alpha, 0)$ as starting point, compute the search direction for the steepest descent method, and perform the corresponding line minimization, where for the line minimization you should determine the *exact* line minimum.
Explain your answer.

End of the exam

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