Exam — October 2021
Optimization for Systems and Control (SC42056)

The exam consists of 2 main questions. For Question 1 the maximal score for each problem is 8; as there are 9 problems, this implies that the maximal total score for Question 1 is 72. The maximal total score for Question 2 is 28. The detailed subscores are marked in red next to each task.

QUESTION 1 $(9 \times 8 = 72$ points)

Consider the following optimization problems:

P1. $$\min_{x \in \mathbb{R}^{10}} \max(A_1(3x), 2A_2(x), 3^{A_3(6x)+8A_4(2x)}, 6)$$
$$\text{s.t. } \|x\|_2^4 \leq 1000$$
$$(x_1^6 + x_2^2 - 3x_3 + 8x_4^2 - x_5x_7 + 4x_5^2 + 8x_7^2 + x_9 - x_{10})^5 \leq 32$$
where the functions A_1, A_2, A_3, and A_4 correspond to the outcomes of physical experiments that take about 60 seconds to carry out and for which it is known that A_1, A_2, A_3, and A_4 are convex in their argument.

P2. $$\min_{x \in \mathbb{R}^5} \cosh(x_1^2 + x_2^2 + x_3^2 + 2x_5) + \max\left(|2x_1 - 5x_2 + 8x_4 - 2|, (4x_1 - x_2 - 2x_3 - 7x_5)^{-2}\right)$$
$$\text{s.t. } 4x_1 - x_2 - 2x_3 - 7x_5 \geq 1$$
$$\log(20 - 8x_1^2 - 6x_2^2 - x_3 - x_4 + 2x_5) \geq 9$$
$$8x_1^2 + 6x_2^2 + x_3 + x_4 - 2x_5 \leq 12$$
$$2(x_1^2 + x_2^2)^4 + 4(x_3 - x_4 + x_5) \leq 10000$$

P3. $$\min_{x \in \mathbb{Z}^5} (3|x_1| + 2|x_2| + 8|x_3| + 6|x_4| + 3|x_5| + 7)^{-5}$$
$$\text{s.t. } 6 \leq 2 + \|x\|_\infty$$
$$(3 + 2x_1 + x_2 - 5x_3 + 9x_4 - 2x_5)^3 \geq 1000$$
$$5|x_1| + 4|x_2| + 3|x_3| + 3|x_4| + |x_5| \leq 50$$

P4. $$\min_{x \in \mathbb{R}^4} \sqrt[7]{(x_1 + 6x_2 + 8x_3 - 9x_4 - 10)^2}$$
$$\text{s.t. } \exp(2x_1^2 + 2x_2^2 + 2x_1^2x_2^2 + x_3^4) + 2x_1^2 + x_2^4 + x_3^6 + 8x_4^2 = 100$$

P5. $$\max_{x \in \mathbb{R}^4} \sinh\left(1 - \cosh(2x_1^2 + 4x_2^2 + 2x_3^2 - 8x_3x_4 + 16x_4^2)\right)$$
$$\text{s.t. } 1 \leq |x_1 + 2x_2 + 3x_3 + 4x_4| \leq 25$$
$$(7x_1 + x_2 - 2x_3 + 5x_4)^4 \geq 16$$

P6. $\min_{\theta \in \mathbb{R}^6} \sum_{k=1}^{7500} \varepsilon^2(k,\theta)$ s.t. $\theta_i \in [0,\pi/2]$ for $i = 1,2,\ldots,6$

where the signal $\{\varepsilon(k,\theta)\}_{k=1}^{7500}$ is the 1-step ahead prediction error for a discrete-time system (with a sample time step of 1 s) for a zero-mean noisy input signal $\{u(k)\}_{k=1}^{7500}$ with a uniform distribution in the interval $[-1,1]$ and where a discrete-time model of the form

$$y(k) = (\cos\theta_1)\cdot e^{2y(k-1)} + (\sin\theta_2)\cdot y(k-3)\cos(u(k)) + \theta_3|u(k-3)| + \\ (\sin\theta_4)u^4(k-2) + \theta_5 y(k-4)\sin(y(k-1)) + (\sin\theta_6)\cdot \frac{u^2(k-1)}{1+y^2(k-2)u(k-2)}$$

with initial conditions $y(i) = \exp(i)$ and $u(j) = 0$ for $i = 0,-1,-2,-3$ and $j = 0,-1,-2$ is considered.

P7. $\max_{x \in \mathbb{R}^4} \Big(2 - \arctan\big(\cosh(3x_1^2 + x_1x_2 + x_2^2 + 4x_3^2 - 8x_3x_4 + 5x_4^2)\big)\Big)$

$$\begin{aligned} \text{s.t. } & \exp(3)\exp(2x_1)\exp(4x_2)\exp(x_3)\exp(-2x_4) \le \exp(17) \\ & \min(2x_1 + x_2 + 3x_3 + x_4,\, 7x_1 - x_2 + 6x_3 - 6x_4) \ge 12 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 \le 6^2 \end{aligned}$$

P8. $\min_{\gamma \in \mathbb{R}^{10}} I(\gamma)$

s.t. $0 \le \gamma_i \le 1$ for all i

where

$$I(\gamma) = \int_0^{\gamma_1}\int_{\gamma_2}^{1}\int_0^{\gamma_3} \frac{\cos(\gamma_4 t_1) + \gamma_5(1+t_2)^{\gamma_6} + \exp(2t_3)(1+\sinh(\gamma_7 t_1^2))}{|2 + t_2\cosh(\gamma_6 t_2) + \sin(\gamma_8 t_1 + \gamma_9 t_3)^2 + \exp(\gamma_{10} t_2)|}\, dt_1 dt_2 dt_3$$

P9. $\min_{x \in \mathbb{R}^4} (4|x_1| + 3|x_2| + |x_3| + 2|x_4| + 1)^4$

$$\begin{aligned} \text{s.t. } & \exp(6 + 3x_1 + x_2 - x_3 - x_4)^4 \le 16000 \\ & \max(3 + 2x_1 + 3x_2 - x_4, 4x_2 + 3x_4, 2x_1 - 8) \le 9 \\ & \|x\|_\infty + \|x\|_1 \ge 2 \end{aligned}$$

Important for all tasks:

* Give only one solution! If more than one solution is given, the worst one will be taken.
* In your answer do *not* refer to your answer to a previous or a subsequent problem. Such a statement will considered to be a wrong answer.
* A correct answer without the correct motivation will be considered to be a wrong answer.

Tasks: For each of the problems P1 up to P9:

1 (a) Reduce the given optimization problem to the *most simple type* of optimization problem that the given problem can be reduced to. Label the constraints using the labels (1), (2), (3), etc.
In your answer you can use f to denote the original objective function (in the current minimization or maximization format), g to refer to the inequality constraint function (in the format $g(x) \leq 0$), and h to the equality constraint function (in the format $h(x) = 0$).
Explain in Task (f) how you reduced the problem.

4 (b) For the *simplified* problem you obtained in Task (a), indicate whether the objective function is linear, convex quadratic, convex, or nonconvex over the feasible set (***excluding*** constraints that restrict the variables to a discrete set, as such constraints would in general make the objective function nonconvex). Select the most restrictive option.
Motivate your answer in (f).

Moreover, for ***each*** individual constraint of the *simplified* problem examine whether the constraint is linear/affine, convex, or nonconvex over the feasible set consisting of the other constraints (***excluding*** constraints that restrict the variables to a discrete set, as such constraints would in general make the constraint nonconvex). Select the most restrictive option. List the labels (as defined in Task (a)) of the linear, convex, and nonconvex constraints, if any.
Motivate your answer in Task (f).

1 (c) Characterize the type of the *simplified* optimization problem you obtained in Task (a), *irrespective* of the optimization algorithm used later on (so if you have a nonconvex constrained problem that you will solve using a barrier or penalty function method in item (d), you should still mark NCC here). Select one of the following types:

- LP: linear programming problem
- QP: convex quadratic programming problem
- CP: convex optimization problem
- NCU: nonconvex unconstrained optimization problem
- NCC: nonconvex constrained optimization problem
- MILP: mixed-integer linear programming problem

Select the most restrictive option.
The first checkbox serves to indicate that a problem can be recast as multiple, more simple problems. If you obtain, e.g., 4 linear programming problems, check — next to the LP checkbox — also the first checkbox and fill out 4 on the dots.
Motivate your answer in Task (f).

1.5 (d) Now determine the *best suited optimization method* for the given problem. Select one of the following methods:

M1 : Simplex algorithm for linear programming

M2 : Gradient projection method with variable step size line minimization

M3 : Interior point algorithm

M4 : Gauss-Newton least squares algorithm

M5 : Davidon-Fletcher-Powell quasi-Newton algorithm

M6 : Levenberg-Marquardt algorithm

M7 : Line search method with the steepest descent direction and cubic line minimization

M8 : Lagrange method + Broyden-Fletcher-Goldfarb-Shanno quasi-Newton algorithm

M9 : Barrier function approach + steepest descent method

M10 : Penalty function approach + line search method with Powell directions and variable step size line minimization

M11 : Simulated annealing

M12 : Branch-and-bound method for mixed-integer linear programming

Motivate your answer in (f).

* If it is necessary to use the selected algorithm in a *multi-start or multi-run optimization* procedure, you have to ***indicate*** this clearly in your answer. Then also motivate why multi-start or multi-run optimization is needed.

0.5 (e) Give the most appropriate stopping criterion for the selected optimization method.
In your answer you can use f_s to denote the simplified objective function (in the current minimization or maximization format) of the reply to Task (a), g_s to refer to the inequality constraint function of Task (a) (in the format $g_s(x) \leq 0$), and h_s to the equality constraint function of Task (a) (in the format $h_s(x) = 0$).

(f) Motivate the answers to Tasks (a), (b), (c), and (d).

QUESTION 2 $(9+16+3=28$ points)

Question 2.1 (9 points)

Tasks:

7 a) Explain how the ellipsoid algorithm works in case the number of optimization variables is larger than 1. Illustrate your explanation by drawing one or more pictures.
Note: It is not needed to provide the exact expressions for the iteration formulas. However, the main ingredients of the iteration formulas should be listed.

2 b) What are the two main advantages of the ellipsoid algorithm compared to the cutting-plane algorithm?
Motivate your answer.

Question 2.2 (16 points)

Consider the following optimization problem:

$$\begin{aligned}
&\max_{(x,y)\in\mathbb{R}^2} \ 10-4x^2-y^2+2xy+50x-20y \\
&\text{s.t. } |x|+|2y|\leq 8 \\
&\quad\ \ 1\leq 2^{(y-1)^2}\leq 2^{16}
\end{aligned}$$

Tasks:

2 a) Is this optimization problem convex or not? Motivate your answer.

12 b) Solve the given optimization problem using the gradient projection method and perform **two** iteration steps:
Take $(x_0,y_0)=(5,0)$ as starting point, compute two consecutive search directions for the gradient projection method, and each time perform the corresponding line minimization, where for the line minimization you should determine the *exact* line minimum.
Explain your answer.

2 c) Is the point (x_2,y_2) found as a result of the second iteration of the gradient projection method above, a global optimum, a local optimum, or no optimum at all of the given optimization problem?
Motivate your answer.

Question 2.3 (3 points)

For an optimization problem of the form $\min_{x\in\mathbb{R}^n} f(x)$ subject to $g(x)\leq 0$, what is the interpretation of $\mu_i>0$ in the (partial) Karush-Kuhn-Tucker conditions $\mu^{\mathrm{T}}g(x)=0$, $\mu\geq 0$, $g(x)\leq 0$.
3 Motivate your answer.

End of the exam

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