Exam — November 2022
Optimization for Systems and Control (SC42056)

The exam consists of 2 main questions. For Question 1 the maximal score for each problem is 9; as there are 8 problems, this implies that the maximal total score for Question 1 is 72. The maximal total score for Question 2 is 28. The detailed subscores are marked in red next to each task.

QUESTION 1 ($8 \times 9 = 72$ points)

Consider the following optimization problems:

P1. $$\begin{aligned} \min_{x \in \mathbb{R}^4} \ & \sqrt[5]{(x_1 + 6x_2 + 8x_3 - 9x_4 - 10)^2} \\ \text{s.t. } & \cosh(x_1^2 + 2x_2^2 + 4x_1x_2 + x_3^4 + x_4^6) + 2x_4 + x_2^2 + (x_2^6 + 8x_3^2)^2 = 1600 \end{aligned}$$

P2. $$\begin{aligned} \min_{x \in \mathbb{Z}^4} \ & 7^{4x_1 + 2x_2 - 3x_3 - 8x_4 - 5} \\ \text{s.t. } & \|x\|_2 \leq 36 \\ & |7x_1 - 2x_2 + 3x_3 + 6x_4 - 25| \leq 175 \\ & 4x_1 - 3x_2 - x_3 + 7x_4 \geq -3 \end{aligned}$$

P3. $$\begin{aligned} \max_{x \in \mathbb{R}^4} \ & 1 - 2\exp(4x_1^2 + x_1x_2 + x_2^2 + x_3^2 - 8x_3x_4 + x_4^2) \\ \text{s.t. } & \exp(7)\exp(2x_1)\exp(4x_2)\exp(x_3)\exp(-2x_4) \leq \exp(21) \\ & \min(2x_1 + x_2 + 3x_3 + x_4 - 5,\, 7x_1 - x_2 + 6x_3 - 6x_4 - 7) \geq 0 \\ & \|x\|_\infty \leq 10 \end{aligned}$$

P4. $$\begin{aligned} \min_{x \in \mathbb{R}^5} \ & 3x_1^4 + x_2^2 + 2x_3^2 + 2x_1x_3 + 6x_4^2 + 4x_5^2 - 8x_4x_5 \\ \text{s.t. } & (x_1 + 2x_2 + 8x_3 - 9x_4 + 8x_5)^2 \leq 256 \\ & \sqrt{(-x_1 + 3x_2 - x_3 + 6x_4 + x_5)^2} \geq 2 \\ & |x_1| + |3x_2| + |x_3| + |8x_4| \leq 7 \end{aligned}$$

P5. $$\begin{aligned} \max_{x \in \mathbb{R}^4} \ & 1 - \|x\|_3 + \log_5\left(10^8 - \cosh\left(\max(x_1^2 + x_2^2, x_3^2 + x_4)\right)\right) \\ \text{s.t. } & 1 + \cosh(\max(x_1^2 + x_2^2, x_3^2 + x_4)) \leq 10^8 \\ & 5 - \max(x_1 + 2x_2 + 3x_3, 2x_1 + 8x_2 + 4x_4) \leq 7 \\ & \sinh\left((3x_1 + 5x_2 - 2x_3 + 5x_4^2 + 9)^3\right) \leq 2 \end{aligned}$$

$$\begin{aligned}
\text{P6.}\quad & \max_{x\in\mathbb{R}^4}\ 3-\exp(x_1+2x_2+6x_3+x_4+1)-\frac{2}{(x_1+x_2-4x_3-x_4+3)^4}\\
& \text{s.t. } x_1+x_2-4x_3-x_4\geq 1\\
& \qquad 1\leq \|x\|_2+\|x\|_\infty+2\leq 25
\end{aligned}$$

$$\begin{aligned}
\text{P7.}\quad & \min_{x\in\mathbb{Z}^4}\ -3+\arctan(7|x_1|+3|x_2|+|x_3|+2|x_4|+1)^2\\
& \text{s.t. } (6+3x_1+x_2-x_3-x_4)^4\leq 16000\\
& \qquad \min(3+2x_1+3x_2-x_4,4x_2+3x_4,2x_1-8)\geq 9\\
& \qquad \|x\|_1\leq 30
\end{aligned}$$

$$\begin{aligned}
\text{P8.}\quad & \max_{x\in\mathbb{R}^3}\ 3\sin\left(\frac{1}{1+4x_1^2+24x_2^2+2x_3^2-8x_1x_2-4x_1x_3+4x_2x_3}\right)\\
& \text{s.t. } |x_1+x_2-4x_3|\geq 1\\
& \qquad \|x\|_1+2x_1-4x_2+8x_3\leq 25\\
& \qquad \cosh(2^{x_1-x_2+x_3})\geq 10000
\end{aligned}$$

Important for all tasks:

* Give only one solution! If more than one solution is given, the worst one will be taken.
* In your answer do *not* refer to your answer to a previous or a subsequent problem. Such a statement will considered to be a wrong answer.
* A correct answer without the correct motivation will be considered to be a wrong answer.

<u>Tasks:</u> For each of the problems P1 up to P8:

1.5 (a) Reduce the given optimization problem to the *most simple type* of optimization problem that the given problem can be reduced to. Label the constraints using the **labels** (1), (2), (3), etc.
In your answer you can use f to denote the original objective function (in the current minimization or maximization format), g to refer to the inequality constraint function (in the format $g(x) \leq 0$), and h to the equality constraint function (in the format $h(x) = 0$).
Explain in Task (f) how you reduced the problem.

4 (b) For the *simplified* problem you obtained in Task (a), indicate whether the objective function is linear, convex quadratic, convex, or nonconvex over the feasible set (***<u>ex</u>cluding*** constraints that restrict the variables to a discrete set, as such constraints would in general make the objective function nonconvex). Select the most restrictive option.
Motivate your answer in Task (f).

Moreover, for ***each*** individual constraint of the *simplified* problem examine whether the constraint is linear/affine, convex, or nonconvex over the feasible set consisting of the other constraints (***<u>ex</u>cluding*** constraints that restrict the variables to a discrete set, as such constraints would in general make the constraint nonconvex). Select the most restrictive option. List the **labels** (as defined in Task (a)) of the linear, convex, and nonconvex constraints, if any.
Motivate your answer in Task (f).

1 (c) Characterize the type of the *simplified* optimization problem you obtained in Task (a), *irrespective* of the optimization algorithm used later on (so if you have a nonconvex constrained problem that you will solve using a barrier or penalty function method in item (d), you should still mark NCC here). Select one of the following types:

- LP: linear programming problem
- QP: convex quadratic programming problem
- CP: convex optimization problem
- NCU: nonconvex unconstrained optimization problem
- NCC: nonconvex constrained optimization problem
- MILP: mixed-integer linear programming problem

Select the most restrictive option.
The first checkbox serves to indicate that a problem can be recast as multiple, more simple problems. If you obtain, e.g., 4 linear programming problems, check — next to the LP checkbox — also the first checkbox and fill out 4 on the dots.
Motivate your answer in Task (f).

2 (d) Now determine the *best suited optimization method* for the given problem. Select one of the following methods:

M1 : Modified simplex algorithm for quadratic programming

M2 : Gradient projection method with variable step size line minimization

M3 : Davidon-Fletcher-Powell quasi-Newton algorithm

M4 : Line search method with the steepest descent direction and cubic line minimization

M5 : Lagrange method + Broyden-Fletcher-Goldfarb-Shanno quasi-Newton algorithm

M6 : Lagrange method + steepest descent method

M7 : Penalty function approach + Levenberg-Marquardt algorithm

M8 : Barrier function approach + steepest descent method

M9 : Barrier function approach + Nelder-Mead method

M10 : Sequential quadratic programming

M11 : Simulated annealing

M12 : Branch-and-bound method for mixed-integer linear programming

Motivate your answer in Task (f).

* If it is necessary to use the selected algorithm in a *multi-start or multi-run optimization* procedure, you have to ***indicate*** this clearly in your answer. Then also motivate why multi-start or multi-run optimization is needed.

0.5 (e) Give the most appropriate stopping criterion for the selected optimization method.
In your answer you can use f_s to denote the simplified objective function (in the current minimization or maximization format) of the reply to Task (a), g_s to refer to the inequality constraint function of Task (a) (in the format $g_s(x) \leq 0$), and h_s to the equality constraint function of Task (a) (in the format $h_s(x) = 0$).

(f) Motivate the answers to Tasks (a), (b), (c), and (d).

QUESTION 2 ($11 + 14 + 3 = 28$ points)

Question 2.1 (11 points)

Tasks:

8 a) Explain how the branch-and-bound method for mixed-integer linear programming works.

1 b) Will the branch-and-bound method always yield the globally optimal solution of a mixed-integer linear programming problem in a finite amount of time? If not, when not? Explain your answers.

1 c) Under which conditions is a mixed-integer linear optimization problem convex? Explain your answer.

1 c) For regular linear optimization problems one of the vertices of the feasible set (defined by the linear (in)equality constraints) will always be an optimal point. Does that also hold for mixed-integer linear optimization problems? Why (not)? Explain your answers.

Question 2.2 (14 points)

Consider the following optimization problem:

$$\begin{aligned} \max_{(x,y)\in\mathbb{R}^2} \ & x^2 + 2y^2 - xy - 4y + 9 \\ \text{s.t.} \ & x, y \geqslant 0 \\ & x + y \leqslant 2 \ . \end{aligned}$$

Tasks:

2 a) Is this optimization problem convex or not? Motivate your answer.

11 b) Now solve the given optimization problem using the gradient projection method and perform two iteration steps:
Compute the first two search directions for the gradient projection method, and perform the subsequent line minimizations. The starting point is equal to $(x_0, y_0) = (1, 0)$ and the line minimization method is assumed to find the *exact* line minimum.

1 c) Is the optimum you found a global optimum? Motivate your answer.

Question 2.3 (3 points)

For an optimization problem of the form

3

$$\max_{x\in S} f(x) \quad \text{subject to } g(x) = 0 \ ,$$

what are the *least restrictive* conditions on the functions f and g, and on the set S for the given problem to be convex? Motivate your answer.

End of the exam | End of the exam