Exam — November 2022 – Grading template
Optimization for Systems and Control (SC42056)

Important: Please recall the following instructions from the exam procedure:

- Note that – just as in previous years – **correct results without proper and correct motivation will not receive any marks.**

For an example on how proper and correct motivations look like, please consult the worked solutions for Sample Exams 1 and 2 and for the exams of October 2013 and October 2014

Additional scoring guidelines

S0: correct result without proper and correct motivation: 0
likewise: if in Question 1 the answer for (c) and/or (d) is formally correct, but an error is made in (a) or (b) that affects the result for (c) and/or (d): 0 for (c) and/or (d)

S1a: small computation error that *does* affect result: -0.5

S1b: small computation error that does *not* affect result: -0.25

S2: partially incomplete motivation for convexity or simplification: -50%

S3: missing, wrong, or not properly motivated $N\times$ in Question 1: -2

S4: multi-start listed when it is not needed: -0.5

S5: ∇f as row vector: -0.5

S6: redundant function in stopping criterion: 0

S7: introduction of redundant variables or constraints that are not needed at all: -1

S8: introduction of additional wrong constraints and/or wrong classification of extra/unsimplified constraint: -1

S9: if a gradient-based algorithm is selected in case the gradient is hard to compute and a gradient-free algorithm is available: 0 in (d) and -1 in (c)

S10: as indicated in instructions: if 2 or more solutions are given, the worst one is assumed to have been selected

CE: Even if the answers to (a)-(d) are wrong, you can still score marks for (e) if and only if (a)-(d) are internally consistent and all result in the answer given in (e) and if (e) is 100 % correct and complete.

On the next pages **concise** answers are given with scores marked in red. To earn the indicated score the corresponding answer has to be given completely, including the information inside the brackets; else the score is 0.

QUESTION 1 ($8 \times 9 = 72$ points)

- **P1**

1.5 (a) As the function $\sqrt[5]{\cdot}$ is a nondecreasing function, we can minimize the argument instead. The $(\cdot)^2$ function can stay, **or** it can be replaced by $|.|$; however, it cannot just be removed: 0.75

Option 1: The constraint can be kept (this is the most simple solution) : 0.75, **or**

Option 2: The constraint can used to express x_1 as a function of the other variables by solving a quadratic equation of the form $x_1^2 + 4x_1x_2 + (2x_2^2 + x_3^4 + x_4^5 \pm \text{acosh}(\ldots))$, but since cosh has a U-shape and is always larger than or equal to 1, this leads to 4 possible expressions for x_1 and additional constraints that the discrimant of the quadratic equation is nonnegative and that the argument of the inverse cosh is larger than or equal to 1, i.e., $1600 - 2x_4 - x_2^2 - (x_2^6 + 8x_3^2)^2 \geq 1$: 0.75

4 (b) Option 1: $(\cdot)^2$ is convex quadratic (as the argument of $(\cdot)^2$ is linear), **or** $|.|$ is convex: 1.5

In this case the constraint is nonconvex (as it is an equality constraint and as $h(\cdot)$ is not affine): 2.5

Option 2: Elimination of x_1 would result in a nonlinear and nonconvex objective function (+ explain why): 1.5

In this case the constraints are nonconvex (as $-x_2^2$ is a nonconvex term): 2.5

1 (c) for option 1: NCC: nonconvex constrained optimization problem

for option 2: 4 $\times$ NCC: nonconvex constrained optimization problem

2 (d) The gradient and the Hessian of the objective function can be computed analytically. So the best choice is:

for option 1: multi-start : 0.5 + M5: Lagrange + BFGS quasi-Newton algorithm: 1.5

for option 2: multi-start : 0.5 + M10: SQP: 1.5 **or**

multi-start : 0.5 + M7: penalty + Levenberg-Marquardt: 1.5

0.5 (e) for option 1: $\|\nabla f_s(x_k) + \lambda \nabla h_s(x_k)\|_2 \leq \varepsilon_1$ and $\|h_s(x_k)\|_2 \leq \varepsilon_2$ (note: λ is a scalar, so no transpose needed) **or**

for option 2: for M10 (SQP): KKT conditions with ε (list them, with f_{elim}, and g_s, no h_s!) **or**

for M7 (penalty+Levenberg-Marquardt) $\|\nabla f_{\text{penalty+}}(x_k)\|_2 \leq \varepsilon$ where $f_{\text{penalty+}}$ is the sum of the simplified objective function after elimination and the penalty function.

- **P2**

1.5 (a) As the function $7^{(\cdot)}$ is an increasing function, we can minimize the argument instead : 1. Moreover, the term -5 can be dropped. So we can minimize $f_s(x) = 4x_1 + 2x_2 - 3x_3 - 8x_4$: 0.5
For simplifications of the constraints, see (b).

4 (b) The objective function f_s is linear: 1
Constraint (1) is convex as a norm function is convex in its argument: 2
Constraint (2) can be written as a set (i.e., intersection) of 2 affine constraints: $-175 \leq 7x_1 2x_2 + 3x_3 + 6x_4 - 25 \leq 175$: 1
Constraint (3) is affine.

1 (c) NCC: nonconvex constrained optimization problem [as the feasible set $\mathbb{Z}^4$ is discrete]

2 (d) Since the feasible set $\mathbb{Z}^4$ is discrete and since (1) is not an affine constraint, the only choice is multi-run: 0.5 M11 simulated annealing: 1.5

0.5 (e) Temperature becomes less than some threshold ($T \leqslant T_{\text{final}}$)

- **P3**

1.5 (a) We first transform the maximization problem into a minimization problem with objective function $-1+2\exp(\dots)$. The constant -1 and the factor 2 can be omitted. As $\exp(\cdot)$ is an increasing function we can minimize its argument. So we finally get $\min_x f_s(x) := \min_{x\in\mathbb{R}^4} 4x_1^2 + x_1x_2 + x_2^2 + x_3^2 - 8x_3x_4 + x_4^2$: 1.5
For simplifications of the constraints, see (b).

4 (b) Although x_1x_2 can be absorbed into e.g. $(0.5x_1+x_2)^2$, the term $-8x_3x_4$ cannot be absorbed into a square of an affine expression. So the objective function is **nonconvex** quadratic: 1.75
As $e^x \cdot e^y = e^{x+y}$, constraint (1) can be rewritten as $7+2x_1+4x_2+x_3-2x_4 \leq 21$, which is an affine constraint: 0.75
Constraint (2) can be written as a set (intersection!) of 2 affine constraints $2x_1+x_2+3x_3+x_4 \geq 5$ **and** $7x_1-x_2+6x_3-6x_4 \geq 7$: 0.75
Constraint (3) can be written as $\max_{i=1,2,3,4} |x_i| \leq 10$ and thus $|x_i| \leq 10$ for $i=1,2,3,4$, which in its turn is equivalent to a set (intersection!) of 8 affine constraints: $-10 \leq x_i \leq 10$ for $i=1,2,3,4$: 0.75

1 (c) NCC: nonconvex constrained optimization problem

2 (d) multi-start : 0.5 + M10: SQP: 1.5 **or**
multi-start : 0.5 + M7: Penalty + Levenberg-Marquardt: 1.5
(as these use 2nd-order information and as gradient and Hessian are easy to compute)

0.5 (e) for M10 (SQP): KKT conditions with ε (list them!) **or**
for M7 (penalty+Levenberg-Marquardt): $\|\nabla f_{\text{penalty+}}(x_k)\|_2 \leq \varepsilon$ (where $f_{\text{penalty+}}$ is the sum of the simplified objective function and the penalty function)

- **P4**

1.5 (a) The objective function cannot be simplified : 1.5
For simplifications of the constraints, see (b).

4 (b) Although $-8x_4x_5$ can be absorbed into e.g. $(2x_4 - 2x_5)^2$, the term $2x_1x_3$ cannot be absorbed into a square of an affine expression. So the objective function is **nonconvex**: 1
(1) can be rewritten as a set (intersection!) of 2 affine constraints: $-16 \leq \ldots \leq 16$: 0.75
Constraint (2) can be written as $|-x_1 + 3x_2 - x_3 + 6x_4 + x_5| \geq 2$, which is equivalent to the **union** of 2 affine constraints: $-x_1 + 3x_2 - x_3 + 6x_4 + x_5 \leq -2$ **or** $-x_1 + 3x_2 - x_3 + 6x_4 + x_5 \geq 2$: 1.5
Constraint (3) can be written as a set (intersection) of $2^4 = 16$ affine constraints: $\pm x_1 \pm 3x_2 \pm x_3 \pm 8x_4 \leq 7$: 0.75

1 (c) NCC: nonconvex constrained optimization problem

2 (d) multi-start : 0.5 + M10: SQP: 1.5 **or**
multi-start : 0.5 + M7: Penalty + Levenberg-Marquardt: 1.5
(as these use 2nd-order information and as gradient and Hessian are easy to compute)

0.5 (e) for M10 (SQP): KKT conditions with ε (list them!) **or**
for M7 (penalty+Levenberg-Marquardt): $\|\nabla f_{\text{penalty+}}(x_k)\|_2 \leq \varepsilon$ (where $f_{\text{penalty+}}$ is the sum of the simplified objective function and the penalty function)

- **P5**

1.5 (a) The constant term 1 can be dropped and the problem can be written as a minimization problem:

$$\min_x \|x\|_3 - \log_5(10^8 - \cosh(\max(x_1^2 + x_2^2, x_3^2 + x_4))) \quad : 1.5$$

Also note that the first constraint guarantees that the argument of the $\log_5$ function is always larger than or equal to 1
For simplifications of the constraints, see (b).

4 (b) The first term of the simplified objective function is a norm, which is convex in its argument
The second term of the simplified objective function is a composite function, where $-\log_5(\cdot)$ is convex and decreasing, and the negative of the argument (i.e., $\cosh(\max(x_1^2 + x_2^2, x_3^2 + x_4)) - 10^8$) is convex (see below), so the composite function is also convex
Square and linear functions are convex functions, the sum of convex functions is convex, and the max of two convex functions is also convex; moreover, $\max(x_1^2 + x_2^2, x_3^2 + x_4)$ will always be nonnegative as the first argument of the max function is always nonnegative; so we get a $\cosh(\cdot)$ function with a nonnegative argument, so $\cosh(\cdot)$ is convex and increasing, and its argument is convex
so the objective function is convex: 2
Constraint(1) is convex, following the same reasoning as above
Constraint (2) can be rewritten as $\max(\dots,\dots) \geq -2$, which results in the **union** of two affine constraints: $x_1 + 2x_2 + 3x_3 \geq -2$ **or** $2x_1 + 8x_2 + 4x_4 \geq -22$: 1
As $\sinh(\cdot)$ and $(\cdot)^3$ are increasing functions, constraint(3) results in a constraint of the form $3x_1 + 5x_2 - 2x_3 + 5x_4^2 + 9 \leq \sqrt[3]{\text{asinh}(2)}$, which is a convex constraint as a square and a linear function are both convex and a sum of convex functions is convex): 1

1 (c) $2 \times$ CP: convex optimization problem

2 (d) There is no dedicated convex optimization algorithm available, but the subgradient is easy to compute while formally speaking the Hessian is not defined everywhere as the objective function is non-smooth due to the max operator; hence, we can use M8: barrier + steepest descent: 1.5
[exceptionally, we also accept M10: SQP: 1.5 **or** M7: Penalty + Levenberg-Marquardt: 1.5]
Multi-start is not required as the problem is convex: 0.5

0.5 (e) for M8 (barrier + steepest descent): $\|\nabla f_{\text{barrier+}}(x_k)\|_2 \leq \varepsilon$ (where $f_{\text{barrier+}}$ is the sum of the objective function and the barrier function) **or**
for M10 (SQP): KKT conditions with ε (list them!) **or**
or M7 (penalty+Levenberg-Marquardt): $\|\nabla f_{\text{penalty+}}(x_k)\|_2 \leq \varepsilon$ (where $f_{\text{penalty+}}$ is the sum of the objective function and the penalty function)

- **P6**

1.5 (a) The term 3 can be dropped and the maximization problem can then be rewritten as a minimization problem with objective function $f_s(x) = \exp(\ldots) + \frac{2}{(\ldots)^4}$. The constraints should not/cannot be simplified.

4 (b) The objective function is convex as the function $\exp(\cdot)$ is convex and it has an affine argument, as the function $1/(\cdot)^4$ is convex for strictly positive arguments (which is guaranteed by constraint (1)) and it has an affine argument, and as the sum of two convex functions is convex: 2.5
Constraint (1) is affine.
The first part of constraint (2): $1 \leq \|x\|_2 + \|x\|_\infty + 2$ always holds as norm functions are nonnegative. The second part of (2) $\|x\|_2 + \|x\|_\infty + 2 \leq 25$ is convex as norm functions are convex in their argument and as a sum of convex functions is also convex: 1.5 if **both** are classified and motivated correctly

1 (c) CP: convex optimization problem

2 (d) There is no dedicated convex optimization algorithm available, but the subgradient and Hessian are easy to compute; so we can use M10: SQP: 1.5 **or** M7: Penalty + Levenberg-Marquardt: 1.5
Multi-start is not required as the problem is convex: 0.5

0.5 (e) for M10 (SQP): KKT conditions with ε (list them!) **or**
for M7 (penalty+Levenberg-Marquardt): $\|\nabla f_{\text{penalty+}}(x_k)\|_2 \leq \varepsilon$ (where $f_{\text{penalty+}}$ is the sum of the objective function and the penalty function)

- **P7**

1.5 (a) The term -3 can be omitted as it does not influence the location of the optimum. Since $\arctan(\cdot)$ is a nondecreasing function, we can also minimize the argument instead: 0.5
Since $(\cdot)^2$ is an increasing function for positive arguments (which is the case here), we can just minimize the argument. The constant 1 can be dropped: 0.5
So the objective function to be minimized is $7|x_1|+3|x_2|+|x_3|+2|x_4$, which is a non-negative sum of absolute values. By introducing dummy variables $\alpha_i \geq |x_i|$ (which can be rewritten as affine constraints $\alpha_i \geq x_i$, $\alpha_i \geq -x_i$), we can instead consider $\min_{\alpha,x} 7\alpha_1 + 3\alpha_2 + \alpha_3 + 2\alpha_4$: 0.5
For simplifications of the constraints, see (b).

4 (b) The simplified objective function is linear: 1
Since an even power is U-shaped, constraint (1) can be rewritten is $-\sqrt[4]{16000} \leq 3x_1 + x_2 - x_3 - x_4 + 6 \leq \sqrt[4]{16000}$, which is an affine constraint: 1
Constraint (2) can be rewritten as a set (intersection!) of 3 affine constraints: $2x_1 + 3x_2 - x_4 + 3 \geq 9$, $4x_2 + 3x_4 \geq 9$, $2x_1 - 8 \geq 9$: 1
The third constraint can be rewritten as $|x_1|+|x_2|+|x_3|+|x_4| \leq 30$, which can be rewritten as a set (intersection!) of $2^4 = 16$ affine constraints: $\pm x_1 \pm x_2 \pm x_3 \pm x_4 \leq 30$ or equivalently 1 affine constraint $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 30$: 1

1 (c) MILP: mixed-integer linear programming problem

2 (d) M12: Branch-and-bound method for mixed-integer linear programming

0.5 (e) Optimum is obtained once the entire search tree has been explored

- **P8**

1.5 (a) The objective function has the form $3\sin\left(\dfrac{1}{1+f_s(x)}\right)$ where f_s can be rewritten as a sum of squares: e.g., $f_s(x) = (-2x_1 + 2x_2 + x_3)^2 + 20x_2^2 + x_3^2$ or $f_s(x) = 2(-x_1 + x_2 + x_3)^2 + (x_1 - 2x_2)^2 + x_1^2 + 18x_2^2$ (alternatively, one can verify that $H = \begin{bmatrix} 8 & -8 & -4 \\ -8 & 48 & 4 \\ -4 & 4 & 4 \end{bmatrix}$ is a positive definite matrix.
Since f_s is nonnegative, the argument of the sin function is in the interval $[0,1]$, and $3\sin$ is increasing on this interval. Hence, we can maximize $\dfrac{1}{1+f_s(x)}$ instead: 0.5
The function $\dfrac{1}{(\cdot)}$ is decreasing for positive arguments (which is the case here). So we can minimize f_s instead (not that the term 1 can also be dropped): 1
For simplifications of the constraints, see (b).

4 (b) The simplified objective function is convex quadratic as it is a sum of squares of affine function or since H is positive definite: 1
Constraint (1) can be rewritten as a **union** of affine constraints: $x_1 + x_2 - 4x_3 \leq -1$ **or** $x_1 + x_2 - 4x_3 \geq 1$: 1
Constraint (2) can be rewritten as a set (intersection!) of $2^3 = 8$ affine constraints: $|x_1| + |x_2| + |x_3| + 2x_1 - 4x_2 + 8x_3 \leq 25$ or $\pm x_1 \pm x_2 \pm x_3 + 2x_1 - 4x_2 + 8x_3 \leq 25$: 1
Since cosh is increasing for nonnegative arguments (which is the case here), and since $2^{\cdot}$ is also increasing, constraint(3) can be rewritten as $x_1 - x_2 + x_3 \geq \log_2(\text{acosh}(10000))$, which is an affine constraint: 1

1 (c) $2 \times$ QP: convex quadratic programming problem

2 (d) M1: Modified simplex algorithm for quadratic programming

0.5 (e) The optimum is obtained in a finite number of steps.

QUESTION 2 $(11 + 14 + 3 = 28$ points$)$

- **Question 2.1**

8 (a) Mention/provide at least the following:

- search tree (as word or as figure): 1
- in each node a regular LP problem (relaxation) is solved, giving a lower bound for the MILP solution: 1
- branching by selecting a x_i that should become integer, and that currently has a non-integer value (say θ_i) and by adding two constraints $x_i \leq \text{floor}(\theta_i)$ and $x_i \geq \text{ceil}(\theta_i)$: 2
- pruning of infeasible branches: 1
- pruning of branches that would not lead to a lower objective function: 1
- stop exploring current branch if feasible mixed-integer solution is found: 1
- stop criterion (no more unexplored nodes): 1

In case of wrong statements, a penalty of -1 applies for each wrong statement.

1 (b) If one or more of the components of the optimal solution of the given MILP problem tend to $-\infty$ or $+\infty$, then for those components it does not matter whether the corresponding variables are integer or not, which means that these cases will be detected as solution of one of the LP relaxations within the tree. So we can focus the rest of our analysis on the case where all components of the optimal solution are finite (the latter will also directly be the case if the feasible set is bounded).
As in each branching step we are moving to the nearest integers smaller than and larger than the current non-integer value and as the number of integer variables is finite, we will always have a finite search tree if the optimal solution of the MILP problem is finite.
As the search tree is finite, the number of nodes that is explored will also be finite and then the branch-and-bound algorithm will yield the local optimum in a finite amount of time.

1 (c) A mixed-integer linear programming problem is convex if there are no integer variables: 0.5 or if there are integer variables but their value is directly or indirectly fixed by the constraints: 0.5

1 (d) Since a vertex does not necessarily have integer values for all components that should be integer, we can easily construct a counter example where the optimal solution is not a vertex of the feasible set. Consider, e.g., $\min_{x \in \mathbb{Z}} x$ subject to $-0.5 \leq x \leq 0.5$, which has $x^* = 0$ as optimal integer solution, while the vertices are $x = -0.5$ and $x = 0.5$

- **Question 2.2**

2 (a) If the problem is characterized as convex, the score for the entire subquestion (a) will be 0

First of all we transform the maximization problem into a minimization problem: 0.75

$$\min_{(x,y) \in \mathbb{R}^2} \; -9 - x^2 - 2y^2 + xy + 4yy$$
$$\text{s.t.} \;\; x, y \geq 0$$
$$x + y \leq 2$$

Due to the term $-x^2$ the objective function is not convex: 1
Hence, the given problem is nonconvex: 0.25

11 (b) After writing the optimization problem in the standard form (with minimization instead of maximization), the gradient of the objective function is given by
$\nabla f(x,y) = \begin{bmatrix} -2x+y & -4y+x+4 \end{bmatrix}^{\mathrm{T}}$: 1
The point $(1,0)$ is on the boundary of the feasible set. We have $\nabla f(1,0) = \begin{bmatrix} -2 & 5 \end{bmatrix}^{\mathrm{T}}$. The *negative* gradient $-\nabla f(1,0) = \begin{bmatrix} 2 & -5 \end{bmatrix}^{\mathrm{T}}$: 1 points towards the infeasible region: 1; so we have to project $-\nabla f(1,0)c = \begin{bmatrix} 2 & -5 \end{bmatrix}^{\mathrm{T}}$ on the x-axis, which yields $\begin{bmatrix} 2 & 0 \end{bmatrix}^{\mathrm{T}}$ as the search direction: 1
So we set $x = 1+2s = 1+t$ and $y = 0$ and we optimize

$$F(s) = f(x(t), y(t)) = -9-(1+t)^2 = 9-1-t^2-2t = 9-t^2-2t$$

over the feasible region $t \in [-1,1]$: 1 Just optimizing the step size yields the line *maximum* in $(0,0)$ for $t=-1$. The line *minimum* $(2,0)$ is found on the right-most boundary of the feasible set: 2
In $(2,0)$ the *negative* gradient is equal to $\begin{bmatrix} 4 & -6 \end{bmatrix}^{\mathrm{T}}$ and it points towards the infeasible region: 2, as the constraints $y=0$ and $x+y=2$ are both active, the projection of $-\nabla f(2,0)$ on these constraints is $(0,0)$; so $(2,0)$ is a local optimum: 2

1 (c) Since the problem is nonconvex, a local minimum is not necessarily a global optimum: 1
Alternatively, we can determine the global optima of the given optimization problem by using the KKT conditions or by computing both the unconstrained optimum (and check whether it is feasible) and the constrained optima on each of the edges of the feasible set, and next select the overall minimum and compare its function value with $(2,0)$. Then we find that $(2,0)$ is indeed a global minimum: 1

- **Question 2.3**

3 First we rewrite the optimization problem in standard form. This yields

$$\min_{x \in S} \big(-f(x)\big) \quad \text{subject to } g(x) = 0 \ .$$

This problem is convex if the following conditions are satisfied (we assume that none of the constraints is used to eliminate variables):

* $-f$ is convex : 0.5 over the feasible set $\tilde{S}$ defined by the constraints, i.e., $\tilde{S} = \{x \in S | g(x) = 0\}$: 0.5,
  or equivalently f is concave over the feasible set $\tilde{S}$
* g is affine over the set S: 0.5
  The motivation for this is that $g(x) = 0$ can be written as $g(x) \leq 0$ and $-g(x) \leq 0$, so both g and $-g$ should be convex over the set S, which implies that g should be affine over S: 0.75
* the restriction of S to the set of points for which $g(x) = 0$, i.e., the set $\tilde{S}$ defined above, is a convex set: 0.75