Exam — November 2023
Optimization for Systems and Control (SC42056)

The exam consists of 2 main questions. For Question 1 the maximal score for each problem is 9; as there are 8 problems, this implies that the maximal total score for Question 1 is 72. The maximal total score for Question 2 is 28. The detailed subscores are marked in red next to each task.

QUESTION 1 ($8 \times 9 = 72$ points)

Consider the following optimization problems:

P1. $$\max_{x \in \mathbb{Z}^3} \exp(x_1 + 5x_2 + x_3 - 8) + \sinh(x_1 + 5x_2 + x_3 - 8)$$
$$\text{s.t. } \|x\|_1 \leqslant 12$$
$$(5 + 3x_1 + x_2 - x_3)^3 \leqslant 81$$
$$\max(3 + 2x_1 + 3x_2, 4x_2 + 3x_3, 8 - x_1 + x_2 - x_3) \geqslant 5$$

P2. $$\min_{x \in \mathbb{R}^4} \sqrt[5]{(x_1 + 6x_2 + 8x_3 - 9x_4 - 10)^2}$$
$$\text{s.t. } 3^{x_1^2 + x_2^2 + 2x_1x_2 + x_3^6} + 3x_1 + 2x_4 + x_2^2 - \log(1 + x_2^6 + 8x_3^2) = 123$$

P3. $$\max_{x \in \mathbb{R}^4} \arctan\left(1 - (2x_1^2 + 4x_2^2 + 2x_3^2 - 8x_3x_4 + 16x_4^2)^4\right)$$
$$\text{s.t. } 2 \leqslant (x_1 + 2x_2 + 3x_3 + 4x_4)^2 \leqslant 144$$
$$15 - 3^{x_1} - |3x_2 - 2x_3 + 5x_4|^3 \geqslant 3$$

P4. $$\min_{x \in \mathbb{R}^4} \cosh(7|x_1| + 4|x_2| + |x_3| + 2|x_4| + 2)$$
$$\text{s.t. } (6 + 3x_1 + x_2 - x_3 - x_4)^2 \leqslant 225$$
$$\min(3 + 2x_1 + 3x_2 - x_4, 4x_2 + 3x_4, 2x_1 - 8) \geqslant 9$$
$$\|x\|_\infty \geqslant \pi$$

P5. $$\min_{v \in \mathbb{R}^{10}} I(v)$$
$$\text{s.t. } v_i^2 \leqslant 4 \quad \text{for } i = 1, 2, \ldots, 10$$
where
$$I(v) = \int_{-5}^{v_1} \int_{v_2}^{5} \int_{-5}^{v_3} \frac{\exp(v_4 x_1) + v_5(1 + x_2)^{v_6^2} + 2^{x_3}(1 + \sin(v_7 x_1^2))}{3 + |x_2 \cos(v_6 x_2) + \sin(v_8 x_1 + v_9 x_3)^4 + \exp(v_{10} x_2)|} \, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$

P6. $$\min_{x\in\mathbb{R}^5} \; 3x_1^4 + x_2^2 + 2x_3^2 + 2x_1x_3 + 2x_4x_5$$
$$\begin{aligned} \text{s.t.}\;\; & (x_1 + 2x_2 + 8x_3 - 9x_4 + 8x_5)^4 \leqslant 16 \\ & 2^{-x_1+3x_2-x_3+6x_4+x_5} \geqslant 4 \\ & \|x\|_1 + \|x\|_\infty \leqslant 9 \end{aligned}$$

P7. $$\max_{x\in\mathbb{R}^4} \; 3\log(x_1 + 2x_2 + 6x_3 + x_4 + 1) - (|x_1 + x_2 - 4x_3 - x_4 + 3| - 1)^4$$
$$\begin{aligned} \text{s.t.}\;\; & \min(x_1 + 2x_2, 6x_3 + x_4) \geqslant 1 \\ & x_1^2 + x_2^4 + x_3^6 + x_4^8 \leqslant 25 \\ & (2^{x_1} + 3^{x_2})^2 + |4x_3 + 3x_4| \leqslant 256 \end{aligned}$$

P8. $$\max_{x\in\mathbb{R}^3} \; 1 - \exp\left(5x_1^2 + 2x_2^2 - 8x_2x_3 + 2x_3^2 - 5x_1 - 6x_2 + 2x_3 - 8\right)$$
$$\begin{aligned} \text{s.t.}\;\; & \big(\cosh(x_1 + x_2 - x_3)\big)^2 \leqslant 10^6 \\ & \min\big(|x_1 + 20x_2 + 30x_3 - 6|, (x_1 + 8x_2 - x_3 - 5)^3\big) \leqslant 625 \end{aligned}$$

Important for all tasks:

* Give only one solution! If more than one solution is given, the worst one will be taken.
* In your answer do *not* refer to your answer to a previous or a subsequent problem. Such a statement will considered to be a wrong answer.
* A correct answer without the correct motivation will be considered to be a wrong answer.

Tasks: For each of the problems P1 up to P8:

1 (a) Reduce the given optimization problem to the *most simple type* of optimization problem that the given problem can be reduced to. Label the constraints using the **labels** (1), (2), (3), etc.
In your answer you can use f to denote the original objective function (in the current minimization or maximization format), g to refer to the inequality constraint function (in the format $g(x) \leqslant 0$), and h to the equality constraint function (in the format $h(x) = 0$).
Explain in Task (f) how you reduced the problem.

4 (b) For the *simplified* problem you obtained in Task (a), indicate whether the objective function is linear, convex quadratic, convex, or nonconvex over the feasible set (***excluding*** constraints that restrict the variables to a discrete set, as such constraints would in general make the objective function nonconvex). Select the most restrictive option.
Motivate your answer in Task (f).

Moreover, for ***each*** individual constraint of the *simplified* problem examine whether the constraint is linear/affine, convex, or nonconvex over the feasible set consisting of the other constraints (***excluding*** constraints that restrict the variables to a discrete set, as such constraints would in general make the constraint nonconvex). Select the most restrictive option. List the **labels** (as defined in Task (a)) of the linear, convex, and nonconvex constraints, if any.
Motivate your answer in Task (f).

1 (c) Characterize the type of the *simplified* optimization problem you obtained in Task (a), *irrespective* of the optimization algorithm used later on (so if you have a nonconvex constrained problem that you will solve using a barrier or penalty function method in item (d), you should still mark NCC here). Select one of the following types:

- LP: linear programming problem
- QP: convex quadratic programming problem
- CP: convex optimization problem
- NCU: nonconvex unconstrained optimization problem
- NCC: nonconvex constrained optimization problem
- MILP: mixed-integer linear programming problem

Select the most restrictive option.
The first checkbox serves to indicate that a problem can be recast as multiple, more simple problems. If you obtain, e.g., 4 linear programming problems, check — next to the LP checkbox — also the first checkbox and fill out 4 on the dots.
Motivate your answer in Task (f).

2.5 (d) Now determine the *best suited optimization method* for the given problem. Select one of the following methods:

M1 : Simplex algorithm for linear programming

M2 : Gradient projection method with variable step size line minimization

M3 : Ellipsoid algorithm

M4 : Gauss-Newton least squares algorithm

M5 : Broyden-Fletcher-Goldfarb-Shanno quasi-Newton algorithm

M6 : Line search method with the steepest descent direction and golden section line minimization

M7 : Lagrange method + steepest descent method

M8 : Lagrange method + line search method with Powell directions

M9 : Penalty function approach + steepest descent method

M10 : Barrier function approach + Nelder-Mead method

M11 : Simulated annealing

M12 : Branch-and-bound method for mixed-integer linear programming

Motivate your answer in Task (f).

* If it is necessary to use the selected algorithm in a *multi-start or multi-run optimization* procedure, you have to ***indicate*** this clearly in your answer. Then also motivate why multi-start or multi-run optimization is needed.

0.5 (e) Give the most appropriate stopping criterion for the selected optimization method.
In your answer you can use f_s to denote the simplified objective function (in the current minimization or maximization format) of the reply to Task (a), g_s to refer to the inequality constraint function of Task (a) (in the format $g_s(x) \leqslant 0$), and h_s to the equality constraint function of Task (a) (in the format $h_s(x) = 0$).

(f) Motivate the answers to Tasks (a), (b), (c), and (d).

QUESTION 2 $(9+15+4=28$ points)

Question 2.1 (9 points)

Tasks:

7 a) Explain how the interior-point algorithm works. Illustrate your explanation by drawing one or more pictures.
Note: It is not needed to provide the exact expressions for the formulas for, e.g., barrier or penalty functions, nor for the derivation of the stopping criterion. However, the main features of the formulas should be indicated.

1 b) Can any convex optimization problem be solved using the interior-point algorithm?

1 c) If one wants to solve a convex optimization problem using a barrier or penalty function approach combined with a non-convex unconstrained local optimization algorithm, do we then need multi-start? If so, when? Motivate your answer.

Question 2.2 (15 points)

Consider the following optimization problem:

$$\begin{aligned} \max_{x\in\mathbb{R}^2} \ & |x_1+x_2-2|^4 \\ \text{s.t.} \ & \|x\|_\infty \leqslant 4 \\ & \max(x_1+x_2, -1-x_1-x_2) \leqslant 6 \ . \end{aligned}$$

Tasks:

2 a) Is this optimization problem convex or not? Motivate your answer.

12 b) Find *all* optimal solutions of the given optimization problem.
Explain your answer.

1 c) Are the optima you found globally optimal? Motivate your answer.

Question 2.3 (4 points)

Task:

4 • Briefly define and explain the concept of Pareto optimal solutions. Illustrate your explanation with one or more pictures.

End of the exam End of the exam