① 2023-9-28

$$\min_{x} \; f(x)$$

$$g(x) \le 0$$

$$h(x) = 0$$

✓ LP

✓ QP

convex

✓ unconstrained

→ constrained

nonlinear, nonconvex

$$\min_{x,y} \; e^{x} + y^{2} + 3x^{2}y^{2}$$

$$2x + y = 6 \longrightarrow y = 6 - 2x$$

$$\rightarrow \min_{x} \; e^{x} + (6-2x)^{2} + 3x^{2}(6-2x)^{2} \rightarrow$$ unconstrained.

①

$$Ax = b \qquad x = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[2 \quad 1]\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\in \mathbb{R}^2} = 6$$

$n = 2$

$m = 1$

$\downarrow$

$n - m = 1$

$$A x_0 = b$$

$$A \bar{A}^T = 0$$

$$2x + y = 6$$

$$x_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$[2 \ 1] \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{= \bar{A}^T} = 0$$

$$\alpha = 1, \ \beta = -2$$

$$x = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot s = \begin{bmatrix} 3 + s \\ -2s \end{bmatrix} \qquad s \in \mathbb{R}$$

$$x = \begin{bmatrix} 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot t = \begin{bmatrix} t \\ 6 - 2t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad t \in \mathbb{R}$$

$$Ax = b$$

$$\downarrow$$

$$\boxed{x = x_0 + \bar{A}^T \cdot \underbrace{\bar{x}}_{\text{free vector}}}$$

$$Ax = \underbrace{Ax_0}_{b} + \underbrace{A \cdot \bar{A}^T}_{0} \cdot \bar{x} = b \ !$$

$$\min_{\bar{x}} f(x_0 + \bar{A}^T \cdot \bar{x}) \rightarrow \text{unconstrained problem}$$

2h10

$$\min_{x,y} \; e^x + y^2 + 3x^2y^2$$

$$x = 3+s$$
$$y = -2s$$

$$\min_{s \in \mathbb{R}} \; e^{3+s} + (-2s)^2 + 3(3+s)^2(-2s)^2$$

③ SVD

$U\cdot U^T = I = U^T U$

$V\cdot V^T = I = V^T V$

$$\underset{m\times n}{A} = \underset{m\times m}{U}\cdot \overset{m\times n}{\underset{\downarrow}{[\Sigma \quad 0]}}\cdot \underset{n\times n}{V^T}$$

$$= U\cdot[\Sigma \quad 0]\begin{bmatrix}V_1^T\\ V_2^T\end{bmatrix} = U\Sigma V_1^T$$

diagonal

diagonal always ≥ 0

A full rank $\to$ all $\Sigma_{ii} > 0$

$V_1 V_1^T = I = V_1^T V_1$

$V_2 V_2^T = I = V_2^T V_2$

$V_1 V_2^T = 0 = V_2^T V_1$

$\bar{A} = V_2^T$

$x_0 = V_1 \Sigma^{-1} U^T b$

? $A x_0 = b$ ✓

$$(U\Sigma V_1^T)(V_1\Sigma^{-1}U^T b) \overset{?}{=} b$$

$$\underbrace{U\Sigma \underbrace{V_1^T V_1}_{I}\Sigma^{-1}U^T}_{I} \cdot b \overset{!}{=} b$$

? $A\cdot \bar{A}^T = 0$

$$U\Sigma \underbrace{V_1^T\cdot V_2}_{0} \overset{!}{=} 0$$

$V\cdot V^T = I$

$$\underset{\substack{\downarrow \\ m\times n}}{[V_1} \ \underset{\substack{\downarrow \\ (n-m)\times n}}{V_2]}\begin{bmatrix}V_1^T\\ V_2^T\end{bmatrix} = \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}$$

$$\begin{pmatrix}V_1 V_1^T & V_2 V_1^T\\ V_1 V_2^T & V_2 V_2^T\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & I\end{pmatrix}$$

(4)

$x^2 + y^2 = 5 \rightarrow y = +\sqrt{5-x^2}$ → eliminate $\quad -\sqrt{5} \le x \le \sqrt{5}$

& $y = -\sqrt{5-x^2}$

$e^x + y = 5 \rightarrow y = 5 - e^x$ → eliminate

$\cos e^{x+y} + \frac{y^2}{x^2+1} + \tanh \cosh(x^2+y^3) = 5 \rightarrow x = ?\ F(y)$

$y = ?\ G(x)$

eliminate if possible

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inequality constraints → eliminate if possible

$x \ge 0 \rightarrow x = e^u \quad u \in \mathbb{R}$

$x = v^2 \quad v \in \mathbb{R}$

[redacted] | polar coordinates

$\cos^2\varphi + \sin^2\varphi = 1$

$x^2 + y^2 \le 4 \rightarrow x = r \cdot \cos\varphi$

$y = r \cdot \sin\varphi$

$0 \le r \le 2 \quad \varphi \in [-\pi/$[redacted]$, \pi/$[redacted]$]$

or

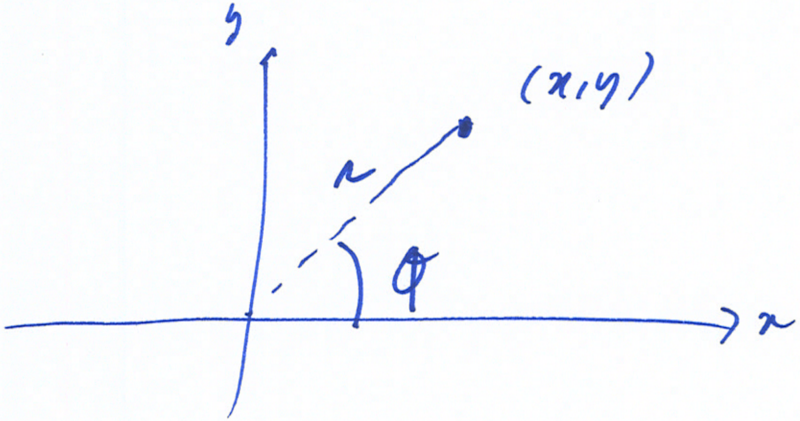
$x = 2\cos\theta \cdot \cos\varphi$

$y = 2\cos\theta \sin\varphi$

$\theta \in [0, \frac{\pi}{2}], \ \varphi \in [-\pi/$[redacted]$, \pi/$[redacted]$]$

(4bis)

polar coordinates



$x = r \cdot \cos\varphi$

$y = r \cdot \sin\varphi$

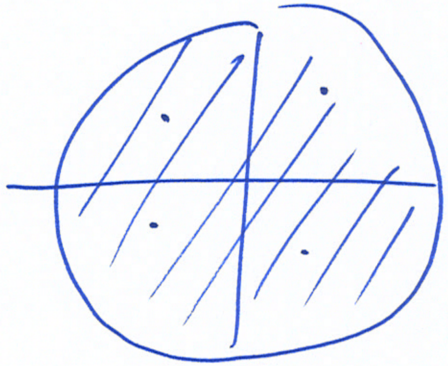
$0 \le r \le 2$ $\quad \theta \in [0, \frac{\pi}{2}]$

in general $0 \le \cos\theta \le 1$

&

$0 \le \underbrace{2\cos\theta}_{= r} \le 2$

$x^2+y^2 \le 4$



$0 \le r \le 2 \qquad \varphi \in [0, 2\pi]$

or $\varphi \in [-\pi, \pi]$

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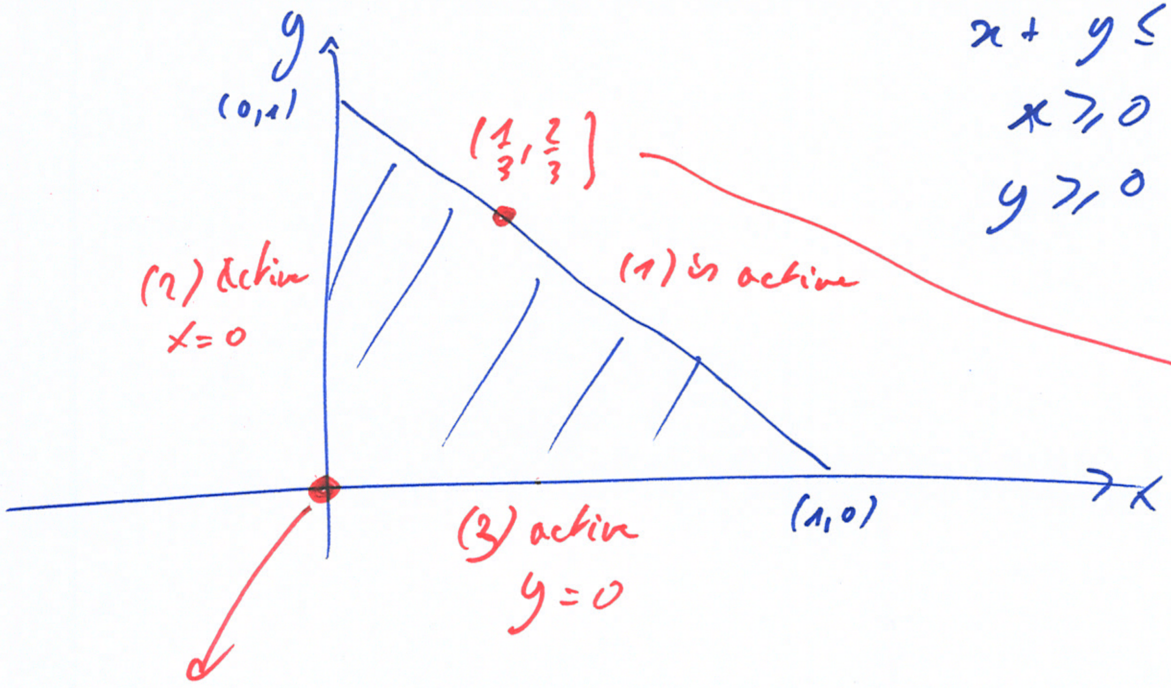

$$x + y \le 1 \qquad (1)$$
$$x \ge 0 \qquad (2)$$
$$y \ge 0 \qquad (3)$$

(1): $\frac{1}{3} + \frac{2}{3} = 1$
active

(2) $\frac{1}{3} > 0$
inactive

(3) $\frac{2}{3} > 0$
inactive

(0,0)
(2) & (3) active, (1) inactive

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$(\frac{1}{3}, \frac{2}{3})$ $\quad A_a = \begin{bmatrix} 1 & 1 \end{bmatrix}$ $\quad b_a = 1$

$(0, 0)$ $\quad A_a = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad b_a = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

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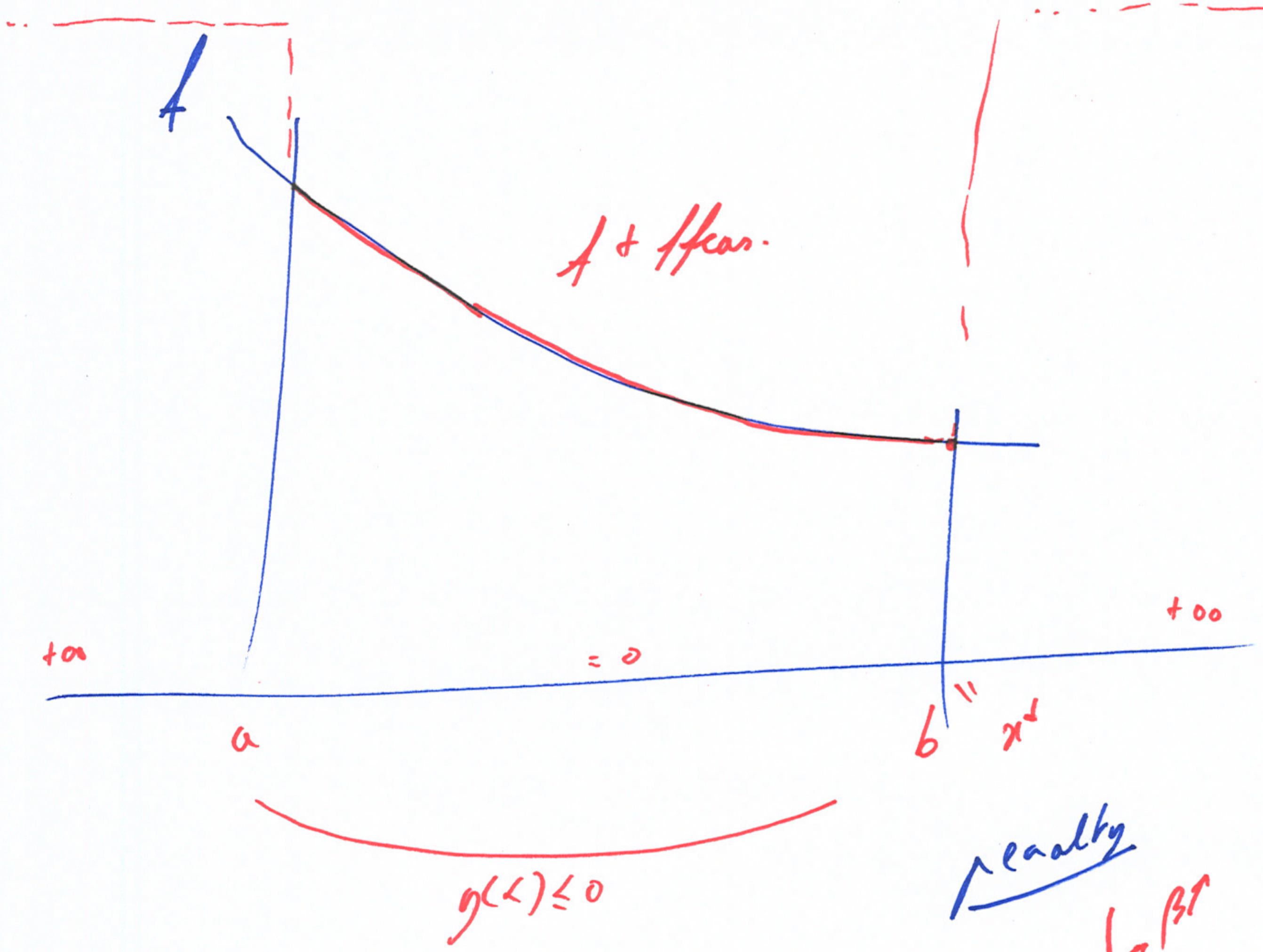
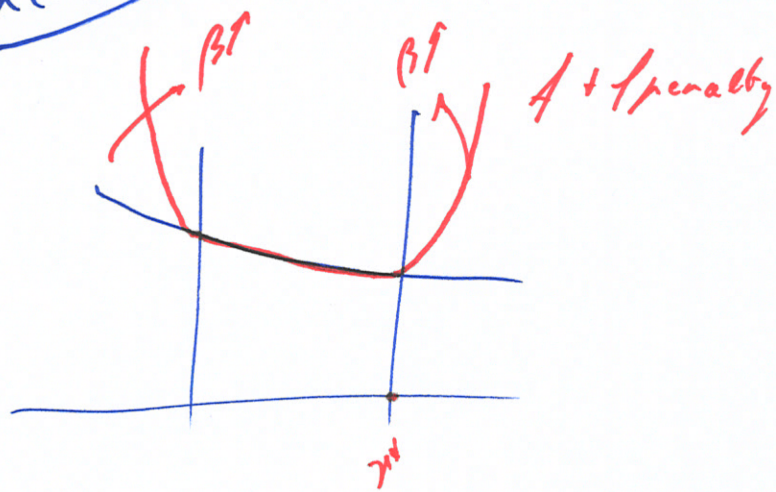


$$\min_x f(x)$$
$$g(x) \le 0$$
$$\downarrow$$
$$\min_x f(x) + 1_{feas}(x)$$
unconstrained

penalty



barrier