① Convex   2023 - 10 - 5

$$\min_x f(x) \qquad f, g \text{ convex}$$

$$g(x) \leq 0$$

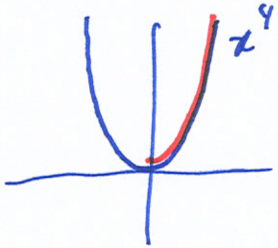
$$h(x) = 0 \rightarrow \text{conditions on } h \text{ for convexity of optimization problem?}$$

$$\begin{matrix} h(x) \leq 0 \\ h(x) \geq 0 \end{matrix} \rightarrow \begin{matrix} h(x) \leq 0 \\ -h(x) \leq 0 \end{matrix} \rightarrow \begin{matrix} h, -h \text{ convex} \\ \downarrow \\ h \text{ is affine.} \end{matrix} \quad [\forall x: g(x) \leq 0]$$

$h(g(x))$ $\qquad$ g, h convex, $h \uparrow$ $\rightarrow$ $h(g(\cdot))$ is convex

x^4, $3|x|+5$ $\rightarrow$ $(3|x|+5)^4$ is convex. ✓

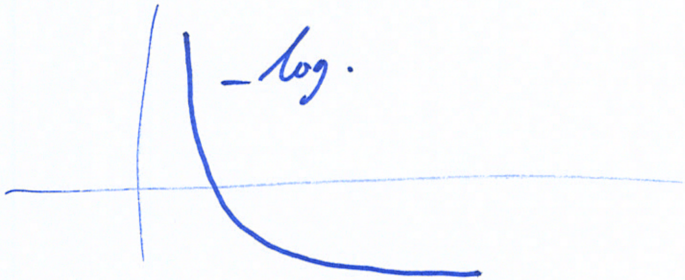
$(3|x|-5)^4$ is convex? not applicable



$\searrow$ increasing for positive arguments

(2)

$-g, h$ convex, $h \downarrow$ → $h(g(\cdot))$ is convex

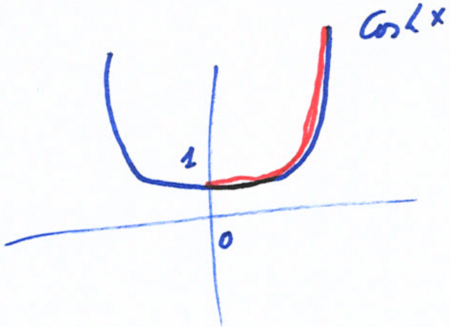


$\underbrace{-\log}_{h}\left(\underbrace{4-x^2}_{g}\right)$ → convex $x \in [-1, 1]$

h ↓ convex ↓

↳ $-g = -4 + x^2$ → convex.

$\cosh x = \dfrac{e^x + e^{-x}}{2}$



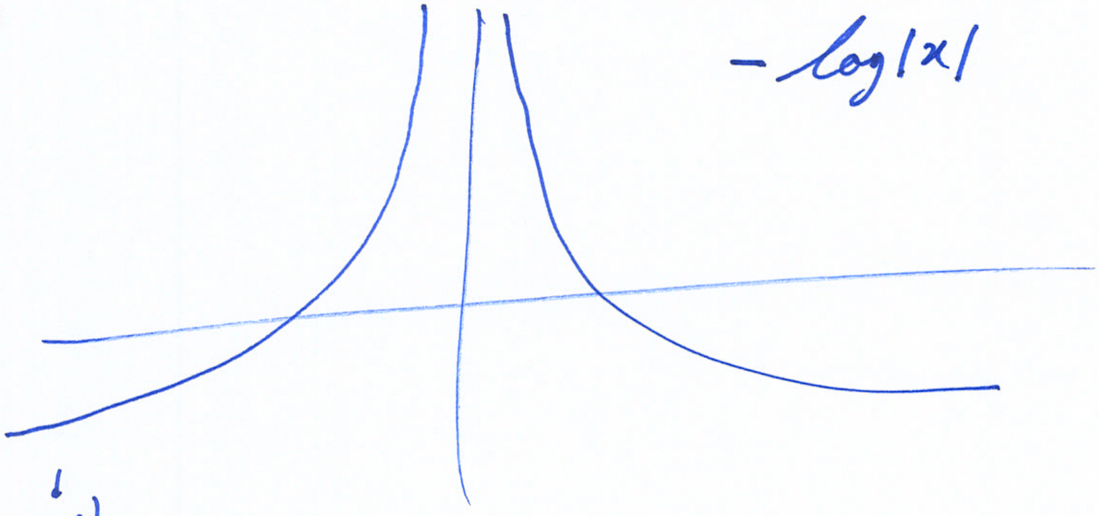
→ convex

$\cosh(x^2 + y^2)$ → convex!

$\cosh$ ↓ convex ↓ ↑ for pos. arguments

x^2+y^2 ↓ convex ≥ 0

(3)



$-\log|x| \rightarrow$ domain: $\mathbb{R} \setminus \{0\}$

$\downarrow$

not convex

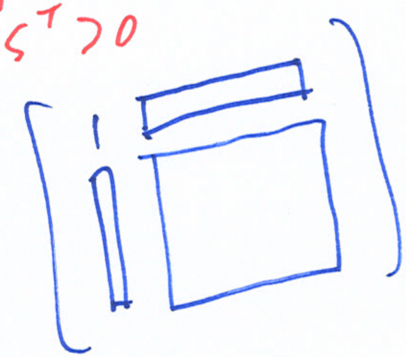
$\rightarrow$ function is <u>not</u> convex

$\|x\| \leq r \rightarrow x^T x \leq r^2 \rightarrow x^T \frac{1}{r^2} x \leq 1 \rightarrow \underset{Q}{1} - \underset{S}{x^T} \underset{R^{-1}}{\frac{1}{r^2}} \underset{S^T}{x} \geq 0$

$R = r^2 I$

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0$$

$R > 0, \quad Q - S R^{-1} S^T > 0$



$$\rightarrow \begin{bmatrix} 1 & x^T \\ x & r^2 I \end{bmatrix} \geq 0$$

linear matrix inequality in X

(3bis)

example $n=2$

$$\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & r^2 & 0 \\ x_2 & 0 & r^2 \end{bmatrix} = F_0 + F_1 x_1 + F_2 x_2$$

x^T, x

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \end{bmatrix}}_{= F_0^T} \qquad \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{= F_1^T} \qquad \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{= F_2^T}$$

(4bis)

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = p_{11} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{= E_{11}} + p_{12} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{= E_{12}} + p_{22} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}}$$

$p_{12} = p_{21}$

$P^T = P$

$$x_1 F_1 + x_2 F_2 + x_3 F_3$$

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$$F_0 + F_1 x_1 + F_2 x_2 + \dots + F_n x_n > 0$$

$F_i = F_i^T$

$x \in \mathbb{R}^n$

$A^T P + PA < 0 \qquad P > 0$

$P \in \mathbb{R}^{n \times n}$, P symmetrix

$$E_{ij} = \begin{array}{c} \begin{array}{ccc} & i & j \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ i \\ j \\ \\ \end{array}$$

$$E_{ii} = \begin{array}{c} i \\ \begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix} \end{array} \; i$$

$\rightarrow$ $\boxed{P = \sum_{i \le j} \mu_{ij} E_{ij} > 0}$ $\rightarrow$ F_i + symmetric $E_{ij} = E_{ij}^T$

$$x = \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{1n} \\ \mu_{22} \\ \mu_{21} \\ \vdots \\ \mu_{2n} \\ \vdots \\ \mu_{nn} \end{bmatrix}$$

LMI

$\mu_{11} \; \mu_{12} \dots \mu_{1n}$

$\mu_{21} \; \mu_{22} \dots \mu_{2n}$

μ_{nn}

$\mu \le$

see (4bis) for a larger copy

$$A^T\left(\sum \mu_{ij} E_{ij}\right) + \left(\sum \mu_{ij} E_{ij}\right) A = \boxed{\sum \mu_{ij} \overbrace{\underbrace{\left(A^T E_{ij} + E_{ij} A\right)}_{G_i = G_i^T}}^{-F_i} > 0}$$

$$G_i^T = (A^T E_{ij} + E_{ij} A)^T = (A^T E_{ij})^T + (E_{ij} A)^T$$
$$= E_{ij} A + A^T E_{ij} = G_i!$$

4bis

ρ_{ij}

$i \le j$

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \rho_{23} & \cdots & \rho_{2n} \\ & & & & \\ & & & \rho_{n-1,n-1} & \rho_{n-1\,n} \\ & & & \rho_{n,n-1} & \rho_{nn} \end{bmatrix}$$

$$x = \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{13} \\ \vdots \\ \rho_{1n} \\ \hline \rho_{22} \\ \rho_{23} \\ \vdots \\ \rho_{2n} \\ \hline \vdots \\ \hline \rho_{n-1,n-1} \\ \rho_{n-1,n} \\ \hline \rho_{nn} \end{pmatrix}$$

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$$
\begin{array}{ccc}
E_{11} & \to & F_1 \\
E_{12} & \to & F_2 \\
\vdots & & \\
E_{1n} & \to & F_n \\
E_{22} & \to & F_{n+1} \\
\vdots & & \vdots \\
E_{2n} & &
\end{array}
$$

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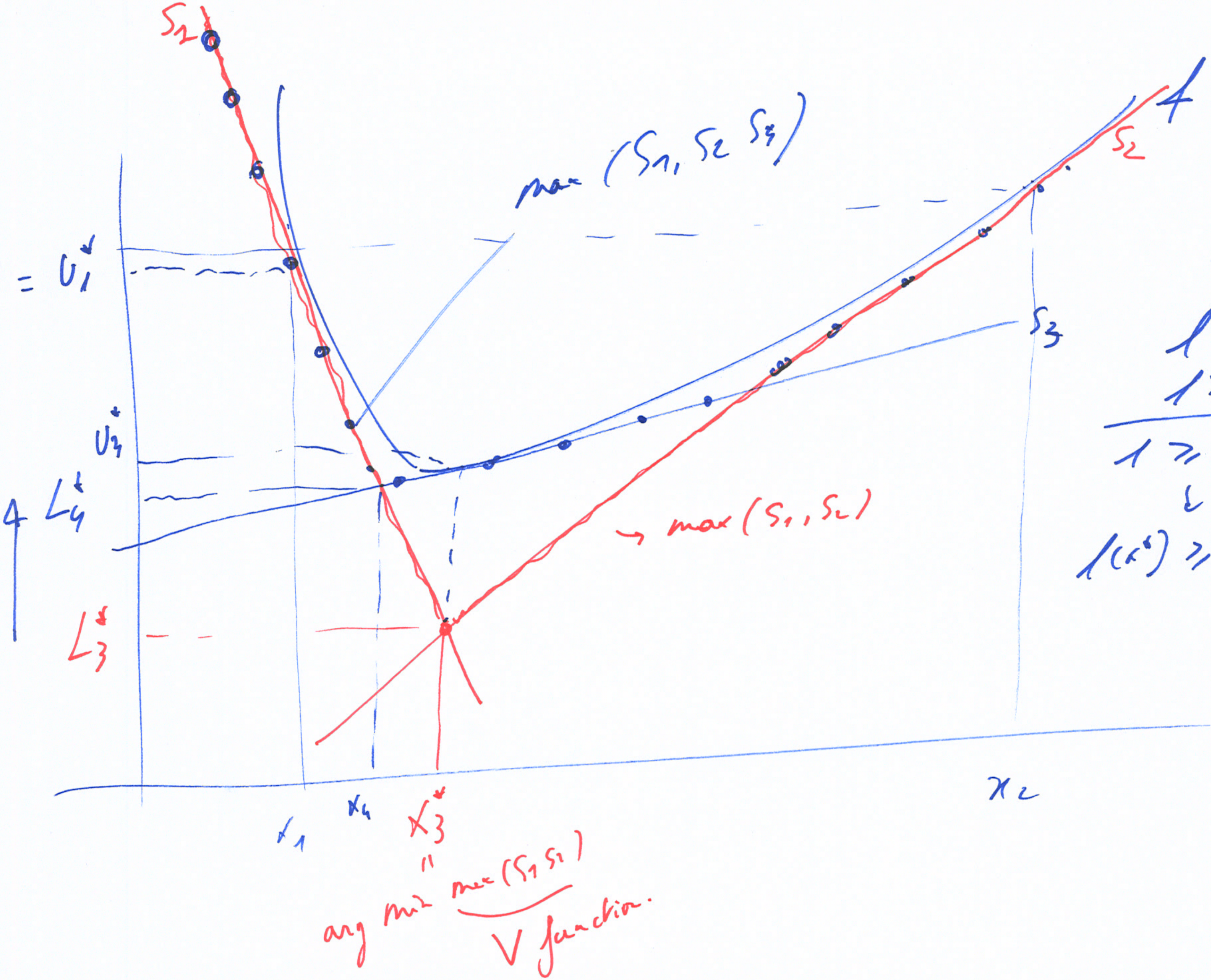


$$l \geq s_1$$
$$l \geq s_2$$

$$l \geq \max(s_1, s_2)$$
$$\downarrow$$
$$l(x^*) \geq \max(s_1(x^*), s(x^*))$$

(2)

$$\min_x \max(S_1\ S_2\ \dots\ S_k)$$

S_i : offline

$$\min_{x, L_k} L_k$$

offsue in x

$$L_k \geq S_1$$

$$L_k \geq S_2$$

$$\vdots$$

$$L_k \geq S_k$$

LP

$$L_k \geq \max(S_1, S_2 \dots, S_k)$$

$$\min_x L_k$$

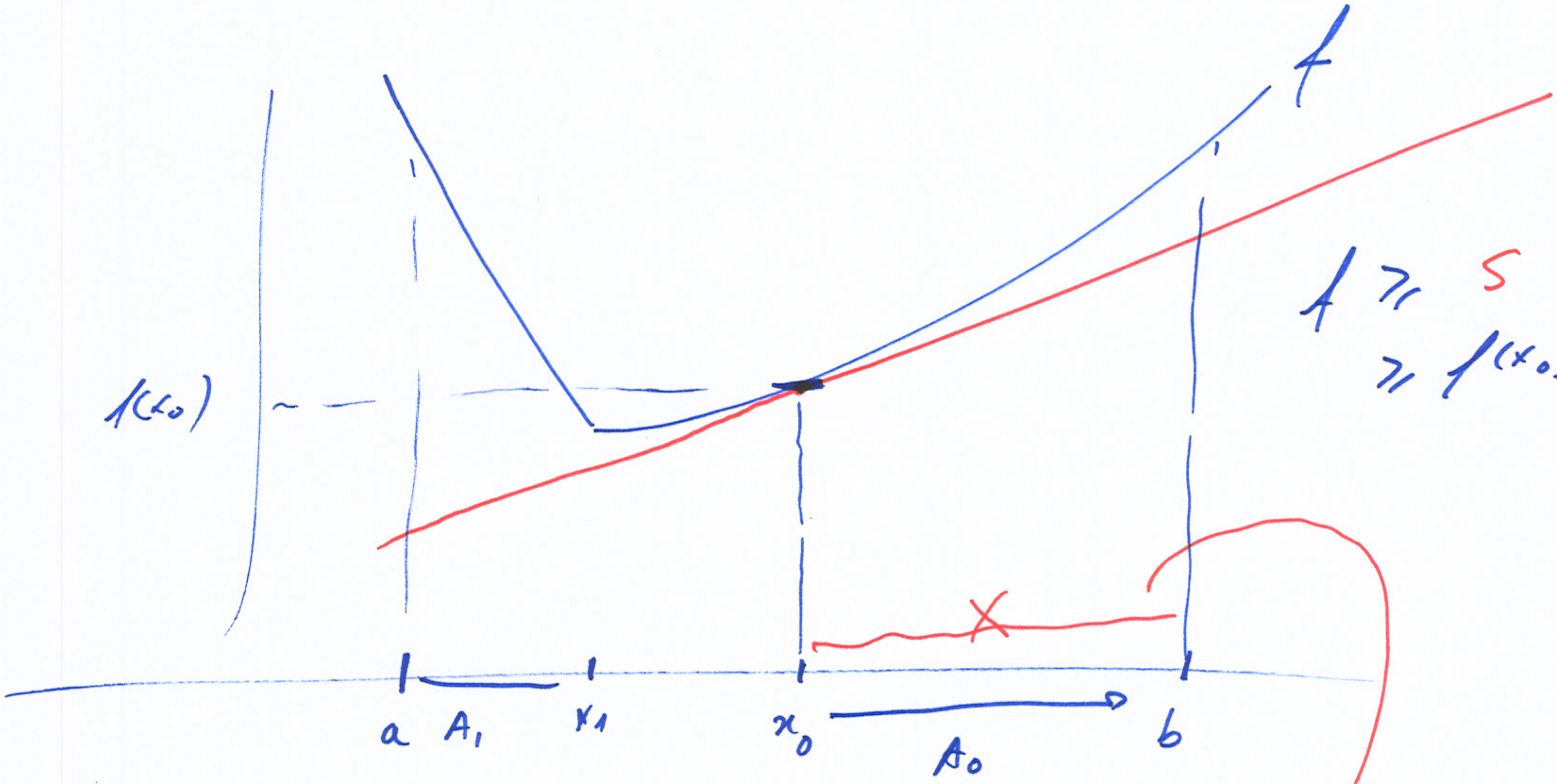
$$\Downarrow$$

$$L_k^* = \max(S_1^*, S_2^*, \dots S_k^*)$$

$$\max(4, 5)$$

$$\left.\begin{array}{l} \min L \\ L \geq 4 \\ L \geq 5 \end{array}\right\} L^* = 5$$

(8)



$f \geq s$
$\geq f(x_0) + D^+f(x_0) \cdot (x - x_0)$

Here $s \uparrow$ $\rightarrow$ to right of x_0: $s \geq f(x_0)$

$f(x) \geq f(x_0)$

$\rightarrow$ minimum will not be to right of x_0

$A_1 = \frac{A_0}{2}$ $\quad + \operatorname{sgn}(Df(x_0))$

$x_1 = x_0 - A_1$

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$\| x - x_{(0)} \| \leq r \longrightarrow \|\ \ \ \|^2 \leq r^2$

$(x - x_{(0)})^T (x - x_{(0)}) \leq r^2$

$(x - x_{(0)})^T \frac{1}{r^2} (x - x_0) \leq 1 \rightarrow$ hyper~~...~~ ball

$\frac{1}{r^2} \downarrow A_0^{-1}$