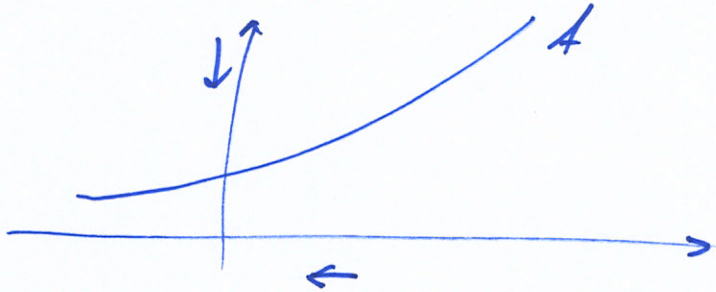
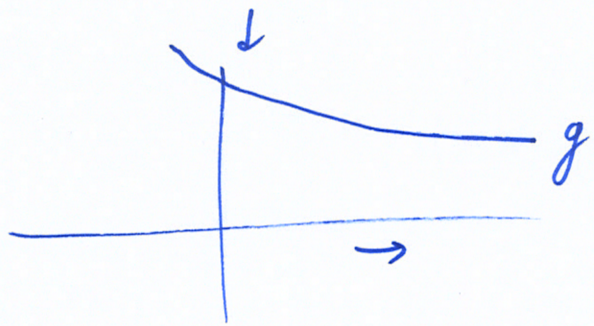
2023-10-19

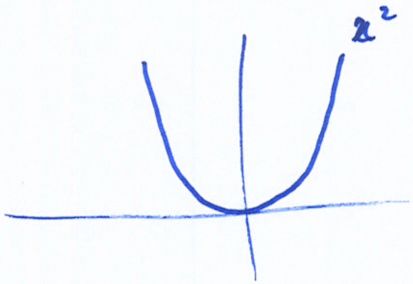
simplificare



$$\min_x f(x) \to \min x$$



$$\min_x g(x) \to \max x$$



$$\min_{x \geq 0} x^2 \to \min x$$

$$\min_x e^x + x^3 \to \min_x \cancel{e^{x^2-1} + (2x^4+x+5)^3}$$

$$\cancel{x^2-1 + 2x^4+x+5}$$

$$\min_x e^{\underbrace{x^2+1}} + (\underbrace{x^2+1})^3 \to \min_x x^2+1$$

$=$

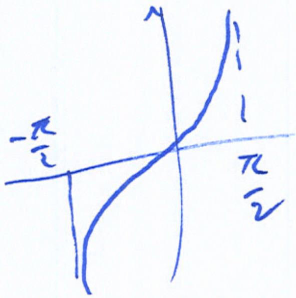
$$\min \quad x_1^2 + x_1x_2 + 2x_2^2 + x_3^2$$

t

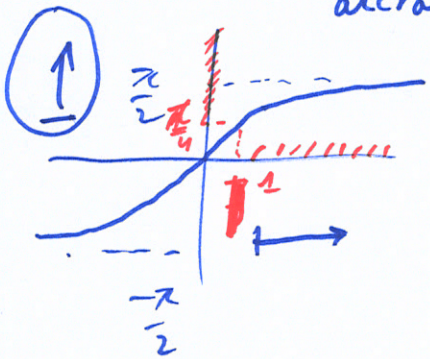
$$|x_1 + x_2 + x_3| \leq 3 \;\rightarrow\; \boxed{-3 \leq x_1 + x_2 + x_3 \leq 3}$$

linear

tan



arctan



$$\arctan \geq \frac{\pi}{4}$$

$$\downarrow$$

$$\text{arg.} \geq 1$$

- ~~LP~~
- ~~QP~~
- convex
- $\boxed{\text{non-convex}}$

$$\boxed{x_1 - x_2x_3 \geq 1}$$

↳ not linear

$$-x_1 + x_2x_3 \leq -1$$

$$\underbrace{1 - x_1 + x_2x_3 \leq 0}_{\text{convex?}} \rightarrow x_2x_3$$

↓ no

$$H = \begin{array}{c|cc} & x_2 & x_3 \\ \hline x_2 & 0 & 1 \\ x_3 & 1 & 0 \end{array}$$

$|H_1| = 0$

$|H| = -1$

~~pos. def.~~

* Convex quadratic

$$x_1^2 + \underline{x_1 x_2} + 2x_2^2 + x_3^2$$

① analyze H

② try to write as positive sum of squares

③ $\frac{1}{2} x^T H x$

①

$$H = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & \boxed{2} & 1 & 0 \\ x_2 & 1 & 4 & 0 \\ x_3 & 0 & 0 & 2 \end{array}$$

$$|H_{\{1\}\{1\}}| = |2| = 2 > 0$$

$$|H_{\{1,2\}\{1,2\}}| = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 1 = 7 > 0$$

$$|H| = 16 - 2 = 14 > 0$$

$\Rightarrow H > 0$ positive definite

$\rightarrow$ Convex obj. function

② sum of squares

$$\underline{\left(x_1 + \tfrac{1}{2}x_2\right)^2} \ldots = \underline{x_1^2 + x_1 x_2 + \tfrac{1}{4}x_2^2} + \tfrac{7}{4}x_2^2 + x_3^2$$

$$= \left(x_1 + \tfrac{1}{2}x_2\right)^2 + \tfrac{7}{4}x_2^2 + x_3^2$$

positive sum of squares

$\downarrow$

Convex function

$$\cancel{\left(x_1 + 3x_2\right)^2 \ldots 9x_2^2} \; -7x_2^2$$

$$\min_x f(x) \quad g(x) \le 0$$

f, g convex $\rightarrow$ problem is convex

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convex

$$\min_x \sum w_i |x_i| \qquad w_i \geq 0$$

LP

$$\min_{\alpha\, x} \sum w_i \alpha_i$$

$\alpha_i \geq |x_i|$ $\rightarrow$ $\alpha_i^* = |x_i^*|$ optimal value

$|x_i| = \max(x_i, -x_i)$

$\alpha_i \geq x_i$

$\alpha_i \geq -x_i$

$t \geq \max |x_i|$ $\rightarrow$ $t \geq |x_i|$

$\rightarrow$ $t \geq x_i$

$t \geq -x_i$

LP

slide 6

(2)

$$\min_{\substack{\alpha_1,\alpha_2,\alpha_3\\ x_1,x_2,x_3}} \alpha_1 + \alpha_2 + \alpha_3$$

$\alpha_1 \geq x_1$ $\quad$ $\alpha_2 \geq x_2$ $\quad$ $\alpha_3 \geq x_3$

$\alpha_1 \geq -x_1$ $\quad$ $\alpha_2 \geq -x_2$ $\quad$ $\alpha_3 \geq -x_3$

$x_1 + x_2 + x_3 \geq 3$

$x_1 - 2x_2 + 4x_3 \geq 1$

LP

(1)

$$\min_{x} \underbrace{\max(x_1, x_2, x_3, x_4)}_{t}$$

$\geq \max(x_1, x_2, x_3, x_4)$

$$\min_{x,t} t$$

$t \geq x_1$

$t \geq x_2$

$t \geq x_3$

$t \geq x_4$

$-2 \leq x_1 + x_2 - 2x_3 + x_4 + 5 \leq 2$

LP

max $\min(x_1, x_2)$

not convex

....

split into 2 problems

$x_1 \leq x_2$

$\min x_1$

$x_2 \leq x_1$

$\min x_2$

$\min(x_1 + x_2 - 4,\ x_1 - 2x_2 + 3) \leq 5$

not convex.

Split into 2 problems!

$x_1 + x_2 - 4 \leq 5$

$x_1 + x_2 - 4 \leq x_1 - 2x_2 + 3$

$x_1 - 2x_2 + 3 \leq 5$

$x_1 - 2x_3 + 3 \leq x_1 + x_2 - 4$

✓

$$\max_{x_1 x_2 x_3} \; e^{x_1^{x_2^3}} + \tanh \frac{x_1^{x_2}}{x_1^2 + x_2 + x_3^8} + e^{\sinh(1-\cosh(x_1 x_2 x_3))}$$

→ H, Df are easy to compute.

✓

$\theta \to \boxed{\text{ctrl}} \to J$ performance by many experiment

$\cancel{H, Df}$ → numerical approximation

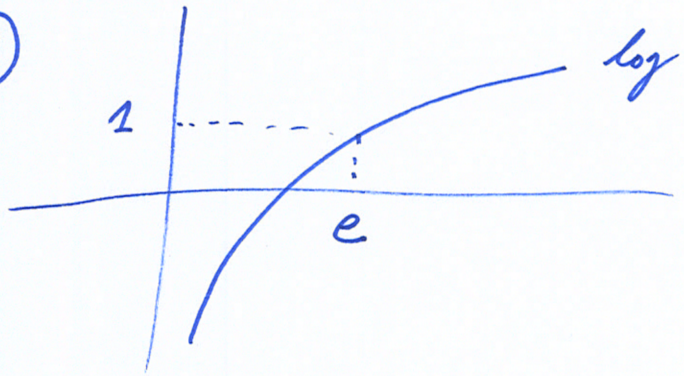
$\sim \frac{n^2}{2}$ $\quad$ n function evaluation

$$\left.\frac{\partial f}{\partial x_i}\right|_{x_0} \approx \frac{f\left(x_0 + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \leftarrow i \\ \vdots \\ 0 \end{bmatrix} \delta\right) - f(x_0)}{\delta}$$

✓

if evaluation f is time consuming, if n large $\Big\}$ → no numerical ∇f, H can be advisable slide 11

(1)



$\log |\ldots| \le 1 \rightarrow |\ldots| \le e$

$\log = \log_e$ $(= \ln)$

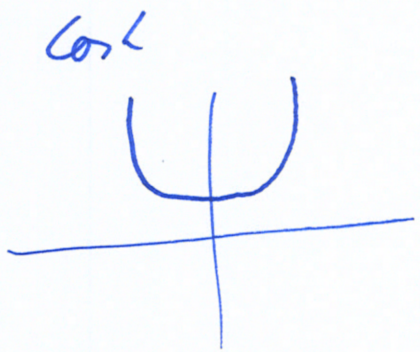
$(\neq \log_{10})$

$\min_x \; -4x_1 - 5x_2 + 6x_3$ linear

$-e \le 2x_1 + 7x_2 + 5x_3 \le e$
$x_1, x_2, x_3 \ge 0$ linear

→ LP!

(2)



→ $\cosh(x_1 + x_2 + x_3)$ is convex

x^2 is convex → $(5x_1 - 6x_2 + 7x_3 + 6)^2$ is convex

max

$\max(\cosh(\ldots), (\ldots)^2)$ is convex

→ convex optimization problem

$\underbrace{\|x\|_2}_{\text{convex}} - 10 \le 0$

$$\|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_i x_i^2}$$

$$\|x\|_p = \sqrt[p]{\sum_i |x_i|^p}$$

$p \geq 1 \to$ norm function

↓

convex in x

$$\|x\|_1 = \sum_i |x_i|$$

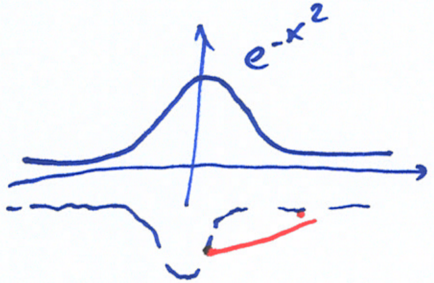
$$\|x\|_\infty = \max_i |x_i|$$

$$\sqrt[100]{\cancel{2^{100}} + 3^{100}}$$

$$\frac{1}{2} x^T H x$$

3 $\quad \min_x \; -e^{-x_1^2 - x_2^2}\left(x_1^2 + x_1 x_2 + 6x_1\right)$

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{array}{l} \to |H_{11}| = 2 > 0 \\ \to |H| = -1 < 0 \end{array} \quad \cancel{H \succ 0}$$

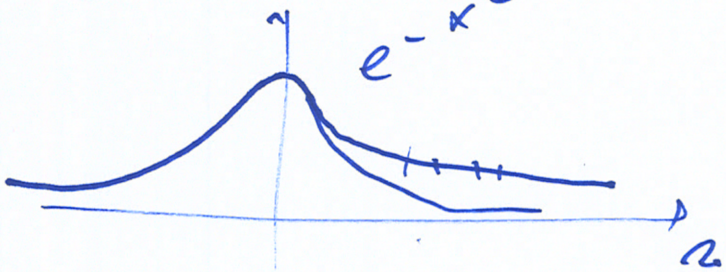


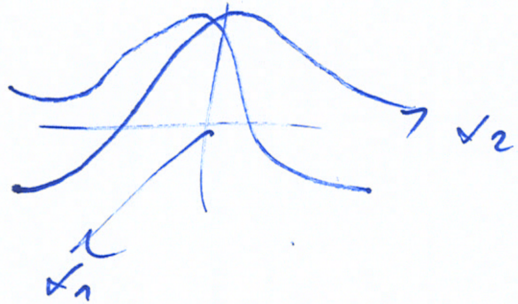
not convex

$(x_1 + \dots x_2)^2 \to$ Cannot be written as positive sum of squares

nonlinear non-convex

$$e^{-(x_1^2+x_2^2)} =$$





is convex if $f''>0$

$$f' = e^{-x^2}\cdot(-2x)$$

$$f'' = e^{-x^2}(-2x)^2 - 2e^{-x^2}$$

or

$$e^x \approx 1+x+\cdots$$

$$e^{-x^2} \approx 1-x^2+\cdots$$

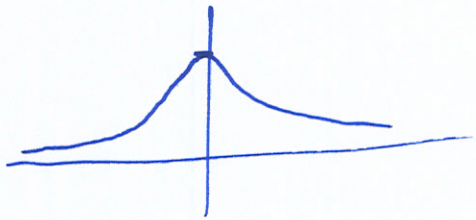
4

$$x_1 = 1 - x_2 - x_3$$

$$\min_{x_2 x_3} \frac{(1 - x_2 - x_3)\, x_2 x_3}{1 + (1 - x_2 - x_3)^6 + x_2^4 + x_3^2}$$

$x_2 x_3 \to$ not convex

$\dfrac{1}{1+x^2} \to$ not convex



fix $x_2 = 1 \to (2 - x_3)\, x_3$

$\to 2x_3 - x_3^2$

not convex

no convex