2023-10-26

$x \to \boxed{?} \to f(x)$

for nonlinear, non convex.

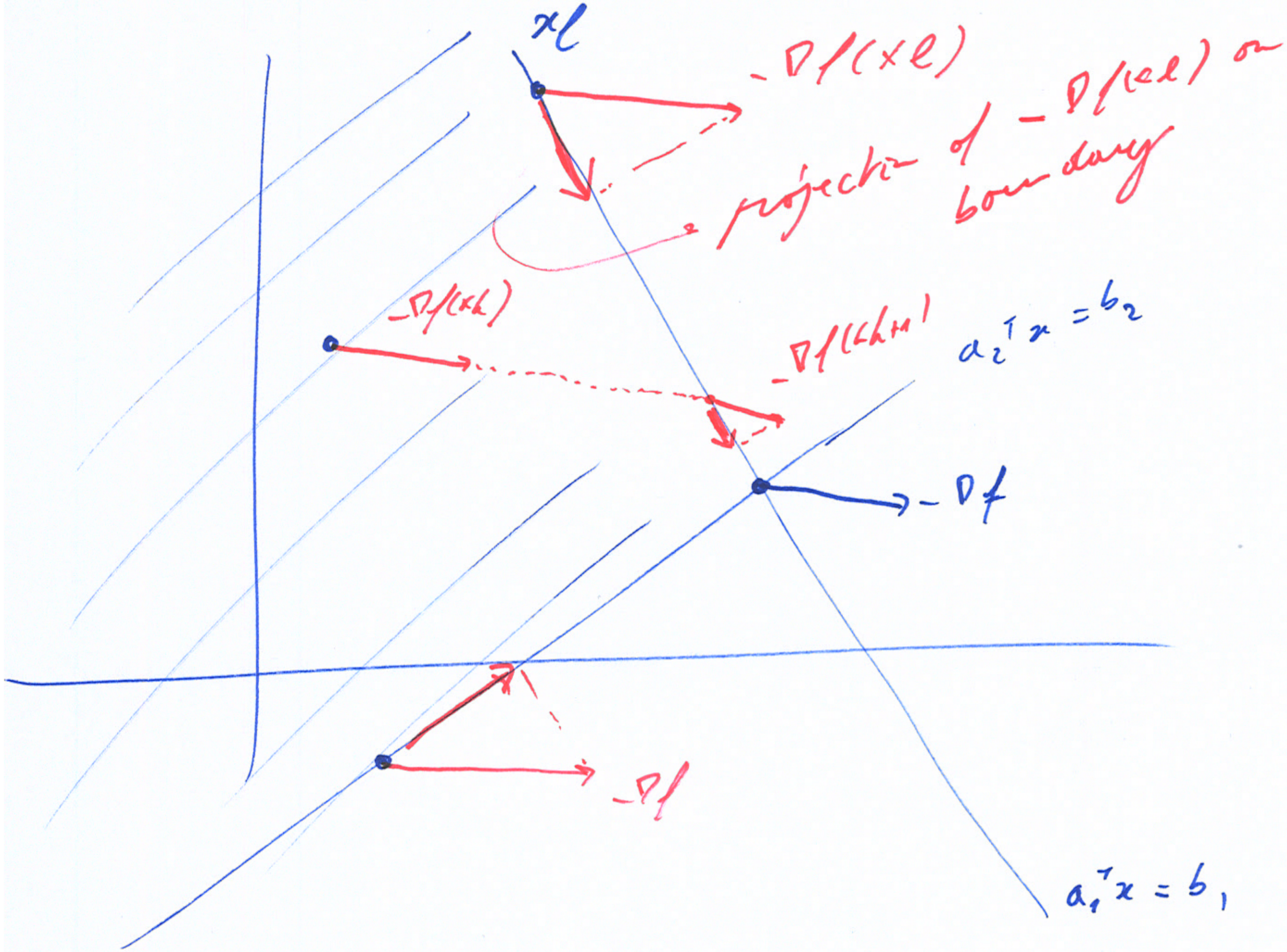
if f takes long to evaluate /
if # variables is very large
→ numerical computation of ∇f, H becomes intractable

→ gradient free

multi-start {
- Powell
- perpendicular
- Nelder-Mead ($n \leq 3$)
- penalty / barrier with

multi-min {
- genetic
- simulated annealing

Convex → interior-point (as cutting-plane & ellipsoid use gradient)



$$Ax \le b$$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} x \le \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

1) if x_k is inside feasible set

$$A\left(x_k - s_k \nabla f(x_k)\right) \le b \quad \rightarrow \quad \text{bound on } s_k$$

2) if x_l is on boundary of feasible set

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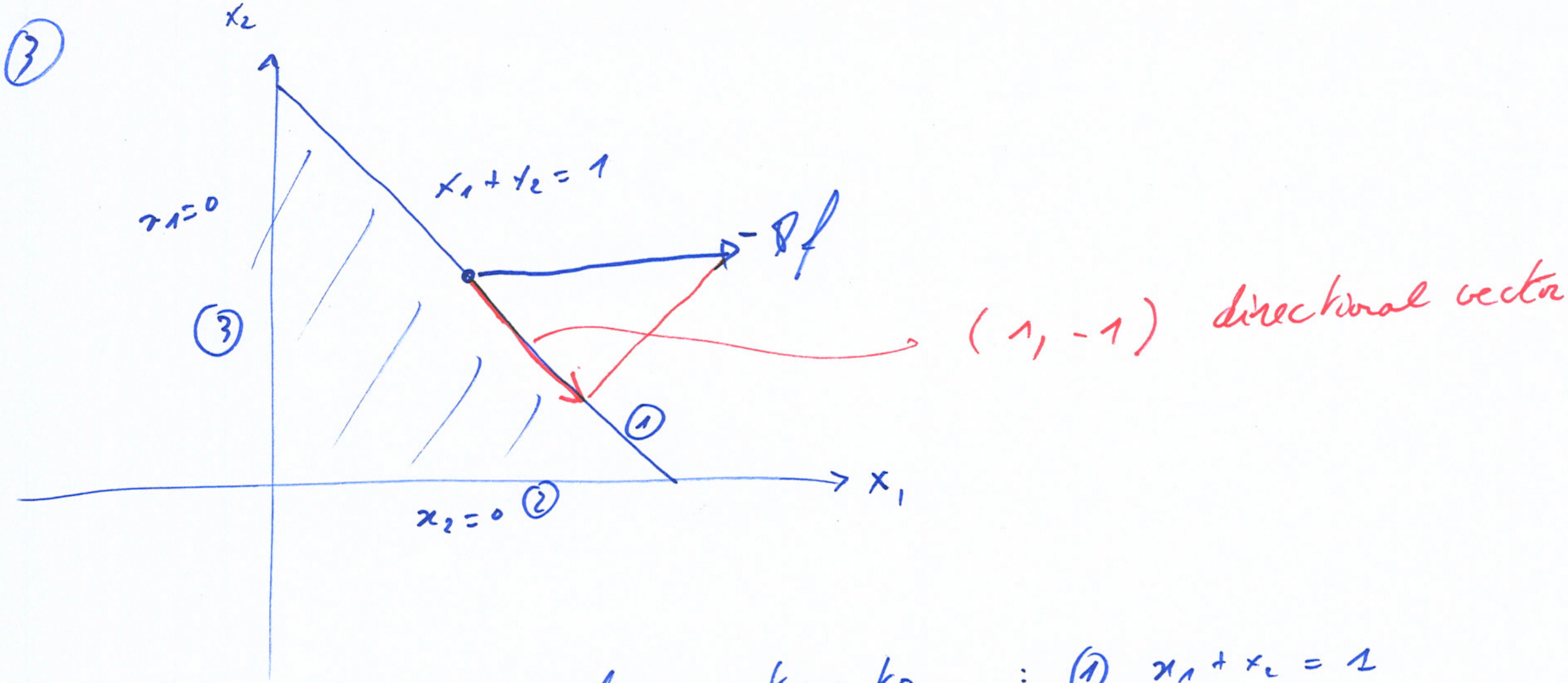


$$x = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

→ active constraints : (1) $x_1 + x_2 = 1$

inactive constraints : (2) $x_2 > 0$

(3) $x_1 > 0$

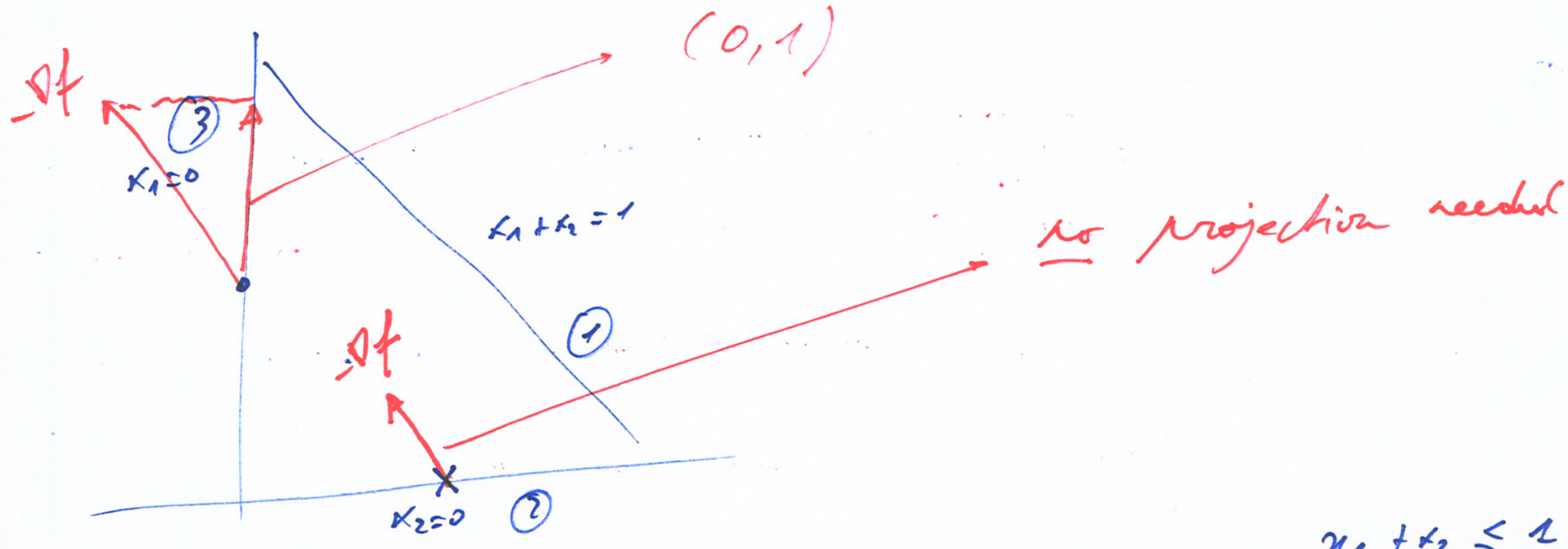
→ project on active constraint

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$x_2 \geq 0$ $\quad 0.5 - \alpha \geq 0$

$\alpha \leq 0.5$

(4)



$x_1+x_2\le 1$

$0 + 0.5 + \lambda \le 1$

$\lambda \le 0.5$

$$\begin{bmatrix}0\\0.5\end{bmatrix} \rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0.5\end{bmatrix} + \lambda\begin{bmatrix}0\\1\end{bmatrix}$$

↳ active constraint: (3)
inactive constraint: (1) & (2)

$\begin{bmatrix}0.5\\0\end{bmatrix}$ → $-\nabla f$ points into feasible set
→ no projection

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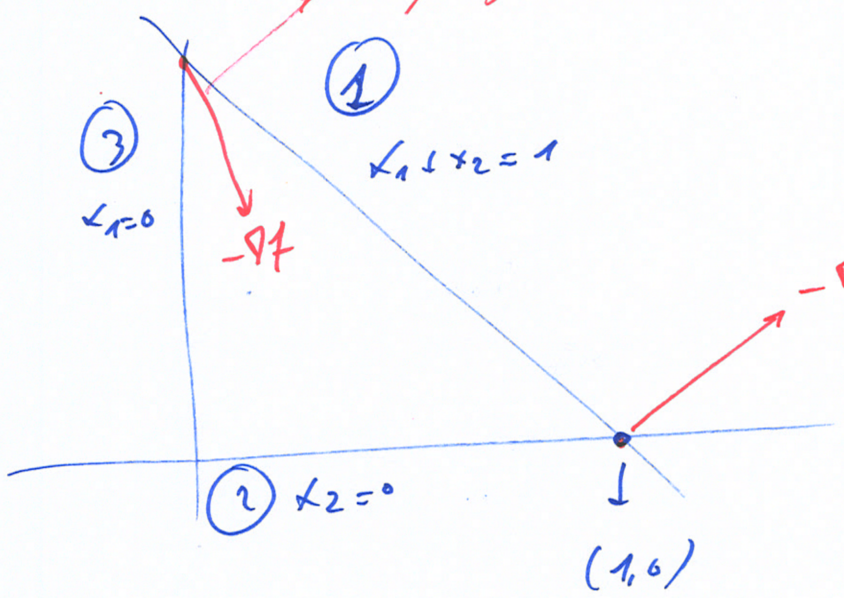
no projection needed



$-\nabla f$ → project on 1 & 2

" " $\begin{bmatrix}1\\0\end{bmatrix}$

→ " "

→ 0 vector

→ (1,0) is optimum.

→ at (1,0) → 2 active constraints : 1 $x_1+x_2=1$

2 $x_2=0$

1 inactive constraint 3 $x_1=0$

⑥ a) $\min\limits_{x_1 x_2} \max(x_1+x_2,\ 2x_1-x_2+7)$ s.t.

b) $\min\limits_{x_1 x_2} \min(x_1+x_2,\ 2x_1-x_2+7)$ s.t.

c) $\max\limits_{x_1 x_2} \max(x_1+x_2,\ 2x_1-x_2+7)$ s.t.

d) $\max\limits_{x_1 x_2} \min(x_1+x_2,\ 2x_1-x_2+7)$ s.t.

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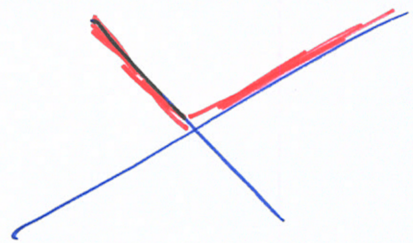
a) $t = \max(x_1+x_2,\ 2x_1-x_2+7)$



$$\min_t t \quad \text{s.t.} \quad t = \max(x_1+x_2,\ 2x_1-x_2+7)$$

$$t \geq \max(x_1+x_2,\ 2x_1-x_2+7)$$

$$\min_t t$$

$$\hookrightarrow t^* = \max(x_1^*+x_2^*,\ 2x_1^*-x_2^*+7)$$

LP

$$\min_t t \qquad t \geq x_1+x_2$$
$$t \geq 2x_1-x_2+7$$

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b) $\min_{x_1 x_2} \min(x_1 + x_2, 2x_1 - x_2 + 7)$

$\min_{t,x} t$ $\quad t = \min(x_1 + x_2, 2x_1 - x_2 + 7)$

$t \leq \min(x_1 + x_2, 2x_1 - x_2 + 7)$

$\cancel{t^* = \min(x_1^* + x_2^*, 2x_1^* - x_2^* + 7)}$

split

1. $\min \; x_1 + x_2$ $\quad [x_1 + x_2 \leq 2x_1 - x_2 + 7]$
   → solve

2. $\min \; 2x_1 - x_2 + 7$ $\quad [2x_1 - x_2 + 7 \leq x_1 + x_2]$
   → solve

take one with smallest obj. function

c) $\max_x \max(\varepsilon_1, \varepsilon_2) \Rightarrow -\min_x -\max(\varepsilon_1, \varepsilon_2) \Rightarrow -\min_x \min(-\varepsilon_1, \varepsilon_2)$

d) $\max_x \min(\varepsilon_1, \varepsilon_2) \Rightarrow -\min_x -\min(\varepsilon_1, \varepsilon_2) \Rightarrow -\min_x \max(-\varepsilon_1, -\varepsilon_2)$

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$\max_{x,t} t \quad t = \min(\varepsilon_1, \varepsilon_2)$

$\max_{\alpha,t} t \qquad t \leq \min(\varepsilon_1, \varepsilon_2) \quad - \quad t^* = \min(\varepsilon_1^*, \varepsilon_2^*)$

$\max_{\alpha,t} t \quad t \leq \varepsilon_1$

$t \leq \varepsilon_2$

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LMI

$$F_0 = \begin{bmatrix} 1 & 4 \\ 4 & 8 \end{bmatrix} \qquad F_1 = \begin{bmatrix} 0 & 5 \\ 5 & 3 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 1 & -9 \\ -9 & -3 \end{bmatrix} \qquad F_i = F_i^T$$

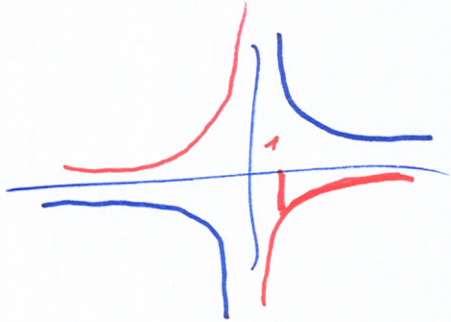
$$F_0 + F_1 \cdot x_1 + F_2 \cdot x_2 \underbrace{> 0}_{\text{positive definite}}$$

$$\begin{bmatrix} 1 & 4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ 5 & 3 \end{bmatrix} x_1 + \begin{bmatrix} 1 & -9 \\ -9 & -3 \end{bmatrix} \cdot x_2 > 0$$

$$x \in \mathbb{R}^2 : \underbrace{\begin{bmatrix} 1 + x_2 & 4 + 5x_1 - 9x_2 \\ 4 + 5x_1 - 9x_2 & 8 + 3x_1 - 3x_2 \end{bmatrix}}_{A(x)} \underbrace{> 0}_{\text{positive definite}}$$

⑩ Oct. 2020 P2

- $$-\min_{x\in\mathbb{Z}^4} - \frac{1}{(5x_1+4x_2-2x_2-8x_4-7)^5}$$



$$\min_{x\in\mathbb{Z}^4} 5x_1+4x_2-2x_2-8x_4-7$$

linear!

- Constraints

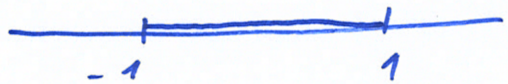
(1) $\rightarrow$ linear.

(2) $\|x\|_1 \le |2|$

convex

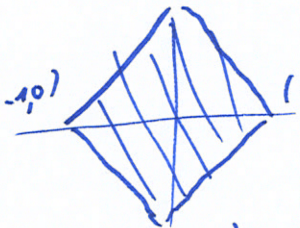
$\|x\|_1 \le 1$ $x\in\mathbb{R}$ $-1 \le x \le 1$ linear

$|x| \le 1$



$|x_1|+|x_2| \le 1$

$x_1+x_2 \le 1$
$x_1-x_2 \le 1$
$-x_1-x_2 \le 1$
$-x_1+x_2 \le 1$



linear

(11)

$$|x_1| + |x_2| \leq 1$$

$$\max(x_1, -x_1) + \max(x_2, -x_2) \leq 1$$

$$\max(\underline{x_1} + x_2, \underline{x_1 -} x_2, -\underline{x_1} + x_2, -\underline{x_1 -} x_2) \leq 1$$

$$\left.\begin{aligned} x_1 + x_2 &\leq 1 \\ x_1 - x_2 &\leq 1 \\ -x_1 + x_2 &\leq 1 \\ -x_1 - x_2 &\leq 1 \end{aligned}\right\} \text{ linear.}$$

(3) → $-196 \leq \dots \leq 196$ → linear.

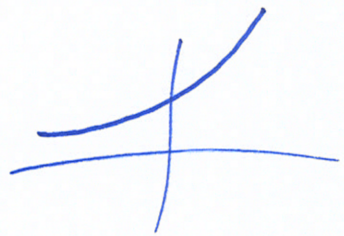
(4) → linear

MILP → branch-and-bound method for MILP

(12)

P5 $\beta = 1$, $\alpha = 2$

Convex?



- $3 \exp\left((\text{linear})^4 - 1\right)$

$\exp(x^4 - 1)$ $\quad x \in \mathbb{R}.$

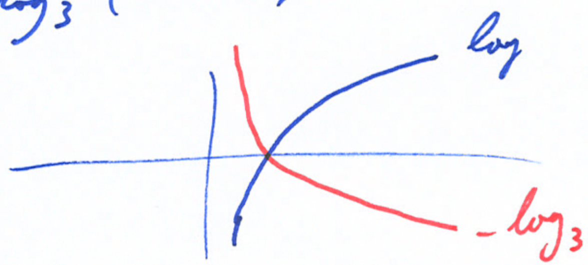
$h = \exp \to$ convex, $\nearrow$

$g = x^4 - 1 \to$ convex

$\downarrow$

$\exp(x^4 - 1)$ is convex

- $-\log_3(\text{linear})$



$\to$ convex $\xrightarrow{\text{linear arg.}}$ convex.

- sum of convex $\to$ result convex $\to$ obj. function is <u>convex</u>

(13)

(1)

$7x_1 + 9x_2 \geq 1$ AND

$8x_3 + x_4 \geq 1$ AND

$x_2 + 4x_3 + x_4 \geq 1$

linear → convex)

(2)

$9 - x_1^2 - x_2^4 - x_3^6 - x_4^8 \leq 36$

$-27 - x_1^2 - x_2^4 - x_3^6 - x_4^8 \leq 0$

is always true!

↳ $x \in \mathbb{R}^4$

↳ convex

$100 - x^2 \leq 0$

not convex

$100 - x^2$

is not convex

→ Convex problem.

exception

1) no cutting plane, no ellipsoid, no interior point

→ which algo?

↓

SQP

GA

multi-start?

not needed as for convex; ~~[illegible]~~ local = global!

needed as genetic does always yield local optimum!

(14)

$\max(7x_1 + 9x_2,\ 8x_3 + x_4,\ x_2 + 4x_3 + x_4) \geq 1$

$7x_1 + 9x_2 \geq 1$ **OR** $8x_3 + x_4 \geq 1$ **OR** $x_2 + 4x_3 + x_4 \geq 1$

→ split into 3 problems

P1	P2	P3
$7x_1 + 9x_2 \geq 1$ linear & convex	$8x_3 + x_4 \geq 1$ linear & convex	$x_2 + 4x_3 + x_4 \geq 1$ convex.
(2)	(2)	(2)

select overall minimum. due to $\min_x f(x)$

original problem : nonconvex.
→ split it into 3 convex problems