Safe Adaptive Switching Control with No SCLI Assumption

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Abstract—A new approach is proposed to dismiss the SCLI assumption in the safe adaptive control [1]. Fictitious signals and cost functions that are newly developed in this paper achieve the cost-detectability without imposing the SCLI assumption unlike [1] and [2] where the SCLI assumption is imposed on candidate controllers and on matrix fraction description forms of candidate controllers, respectively. Since this approach has virtually no assumption on candidate controllers, the safe adaptive control with this approach is able to contain virtually any controller in a candidate controller set.

I. INTRODUCTION

The safe adaptive control [3] has been developed to achieve stability of a system under a large class of uncertainty in a plant and disturbance signals. The adaptive switching control scheme exploits the collected data in a real-time experiment rather than employs any assumption on a plant and disturbance signals. Based on the concept of the unfalsified control [4], the safe adaptive control stabilizes a system with an unknown plant and unknown disturbance signals using ε-hysteresis algorithm [5] whenever there exists a stabilizing controller, which is called a feasible controller, in a candidate controller set, provided that a plant-independent cost function of the switching algorithm is cost-detectable.

Due to the lack of knowledge on a plant and disturbance signals, it may be the first idea to place in a candidate controller set as many controllers that have possibility to stabilize the plant with the disturbance signals as possible. However, there is a restriction on the candidate controllers, which is called the SCLI assumption [1], i.e. each candidate controller has to be causally left invertible and the causal left inverse has to be incrementally stable. Fictitious signals for SCLI candidate controllers are well-defined and make $\mathcal{L}_2$-gain-related cost functions cost-detectable.

In order to expand the range of controllers that can be placed in a candidate controller set, the matrix fraction description method is employed in [2] and [6]. By factorizing a non-SCLI linear controller in a matrix fraction description form, the controller and a reference signal are reorganized into linear stable factors and a new reference signal. This new controller, composed of the linear stable factors, satisfies the SCLI condition. Then, fictitious reference signals for the new controllers with respect to the new reference signal, together with $\mathcal{L}_2$-gain-related cost functions, ensure the cost-detectability. In [2], the matrix fraction description method is also applied to one-degree-of-freedom nonlinear controllers that can be factorized into incrementally stable nonlinear matrix factors. Consequently, the SCLI assumption still plays a key role in the matrix fraction description method.

In this paper, another method to generate fictitious signals is proposed. Fictitious signals obtained in this method and redefined $\mathcal{L}_2$-gain-related cost functions are proved to be sufficient to directly achieve the cost-detectability without imposing the SCLI assumption on any form of candidate controllers. Two main ingredients in this approach are an auxiliary signal and subcontrollers. The auxiliary signal is the external input signal that is added to a controller-output signal in an adaptive control system and is exactly known. In many cases, the auxiliary signal is a zero signal but it can be any signal. Each candidate controller has two degrees of freedom and is assumed to have a corresponding controller, called a subcontroller, that stabilizes the candidate controller. If the candidate controller is stable itself, the subcontroller can simply be a zero controller that produces only a zero signal. By considering the auxiliary signal, the fictitious signals, newly defined in this paper, are generated using observed signals in the adaptive control system without inverting candidate controllers. When a candidate controller is connected in the loop of the adaptive control system, the fictitious signal for the candidate controller is the same as a vector of a zero, the reference signal in the adaptive control system, and the auxiliary signal. When the candidate controller is not connected in the loop of the adaptive control system, the subcontroller stabilizes the candidate controller by making a closed-loop system and the fictitious signal for the candidate controller is obtained from subtraction between input and output signals of the candidate controller and input and output signals of the plant. With the help of the subcontroller, the mapping from the observed signals in the adaptive control system to the fictitious signal for the candidate controller is guaranteed to be stable. Since the cost-detectability is achieved with the new fictitious signals, redefined $\mathcal{L}_2$-gain-related cost functions, and virtually no assumption on candidate controllers, the safe adaptive control with this approach can contain virtually any controller in a candidate controller set. Another notable advantage of this approach is that the fictitious signals are generated by mere subtraction between observed signals.

The paper is organized as follows. In Section II-A, an adaptive switching control system with an auxiliary signal is carefully described. The main results of the safe adaptive control are summarized in Section II-B. In Section III, new fictitious signals are introduced and $\mathcal{L}_2$-gain-related cost functions are redefined. Together they achieve the cost-detectability and, hence, the safe adaptive control is achieved. Conclusion follows in Section IV.

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II. ADAPTIVE SWITCHING CONTROL

A. Adaptive Control System Formulation

The norm $\| \cdot \|$ is the $L_2$-norm and denote by $L_2^m$ the $L_2$ space of $m$-dimensional functions of time, i.e. $L_2^m = \{ x : [0, \infty) \mapsto \mathbb{R}^m \ | \ |x| < \infty \}$. Define a truncated version of the $L_2$-norm

$$\| x \|_t \triangleq \sqrt{\int_0^t x^T(\tau) x(\tau) d\tau}$$

for any function of time $x$ and denote the extended space of $L_2^m$ by $L_{\infty}^m = \{ x : [0, \infty) \mapsto \mathbb{R}^m \ | \ |x| < \infty , \forall t \in [0, \infty) \}$. Define a truncated version of the $L_\infty$-norm

$$\| y \|_{\infty} \triangleq \sup_{t \in [0, \infty)} \| y(t) \|$$

An adaptive control system in Fig. 1 is considered as a mapping (or a system) $H : L_{2c} \mapsto L_{2c}$, with input $x \in L_{2c}$ is said to be stable if there exist constants $\alpha_1, \beta_1 \geq 0$ such that for any given $x \in L_{2c}$

$$\| Hx \|_t \leq \alpha_1 \| x \|_t + \beta_1 \quad \text{for all } t \geq 0.$$

Otherwise, $H$ is said to be unstable.

An adaptive control system in Fig. 1 is considered as a mapping from three system-input signals, i.e. a reference signal $r \in L_{2r}$, an auxiliary signal $s \in L_{2s}$, and a disturbance signal $d \in L_{2d}$, to an observed system-output signal $z = [u \ y]^T$ that consists of a plant-input signal $u \in L_{2u}$ and a measured plant-output signal $y \in L_{2y}$. The plant-input signal $u$ is the sum of the controller-output signal and the auxiliary signal $s$. The name ‘auxiliary’ comes from the fact that in most cases it is a zero signal. The reference signal $r$ and the auxiliary signal $s$ are known and the disturbance signal $d$ is unknown and given as a deterministic signal. The plant $P : L_{2u} \times L_{2d} \mapsto L_{2y}$ is an unknown mapping from $u$ and $d$ to $y$. When the plant has a state, the plant $P$, in this paper, represents not only its unknown dynamics but also its initial condition, which is also unknown.

A candidate controller is paired with another controller, which is called a subcontroller, i.e. for any given $C \in \mathbb{C}$ the candidate controller $C$ has a subcontroller $K$ paired with $C$. The subcontroller $K$ is designed to stabilize the candidate controller $C$ in the sense that for any given time $\bar{t} \in [0, \infty)$ if two switches in Fig. 2 are closed for $\forall t \in [\bar{t}, \infty)$, then a mapping from $[r \ y]^T$ to a signal that is produced anywhere in the system in Fig. 2 is stable as in Definition 1. Although the subcontroller $K$ in Fig. 2 is depicted to use only the controller-output signal of $C$, actually the subcontroller $K$ is allowed to use not only the controller-output signal of $C$ but also every information on $C$ with perfect knowledge of the controller $C$. If $C$ is stable itself, $K$ is given as a zero subcontroller whose output signal is $0$ for $\forall t \geq 0$.

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A switching algorithm, at each selecting time, selects a candidate controller from the candidate controller set $\mathbb{C}$ and keeps its controller-output signal delivered to the loop of the adaptive control system until the next selecting time. Denote by $\hat{C}$ the sequence of controllers that are chosen and connected in the loop of the adaptive control system by the switching algorithm and let $\hat{C}_t$ denote the candidate controller that is connected in the loop of the adaptive control system at time $t \geq 0$.

For any given $C \in \mathbb{C}$ and its corresponding subcontroller $K$, when the candidate controller $C$ is selected by the switching algorithm and is connected in the loop of the adaptive control system, the subcontroller $K$ is disconnected as shown in Fig. 3 a). When the candidate controller $C$ is not connected in the loop of the adaptive control system, the subcontroller $K$ is connected as shown in Fig. 3 b).
controller is not in the loop of the adaptive control system. Fig. 3 shows that, for any given $C \in \mathbb{C}$ and its corresponding subcontroller $K$, we have

$$
\begin{align*}
  r_C(t) &= r(t) \\
  y_C(t) &= y(t) \\
  u_C(t) &= u(t) - s(t)
\end{align*}
$$

for any time $t \geq 0$ when $C$ is connected in the loop of the adaptive control system. When $C$ is not connected in the loop of the adaptive control system, the mapping from $[r \ y]^T$ to $z_C = [r_C \ u_C \ y_C]^T$ is stable as in Definition 1. Note that if $C$ is stable itself and, hence, $K$ is a zero subcontroller, then $r_C(t) = r(t)$ and $y_C(t) = y(t)$ for all $t \geq 0$.

For the purpose of switching, candidate controllers, at each time $t \geq 0$, are prioritized on the basis of a mapping $V : \mathbb{C} \times \mathcal{L}_{2e}(m_e + m_a + m_p) \times \mathcal{L}_{2e}(m_u + m_v) \rightarrow \mathcal{L}_{2e}$ that is called a cost mapping and maps from a candidate controller in $\mathbb{C}$ and two signals in $\mathcal{L}_{2e}(m_e + m_u + m_v)$ and $\mathcal{L}_{2e}(m_u + m_v)$ to a function of time. For any given $C \in \mathbb{C}$, $x_{zc} \in \mathcal{L}_{2e}(m_e + m_a + m_p)$, and $x_z \in \mathcal{L}_{2e}(m_u + m_v)$ the image of the cost mapping, $V(C, x_{zc}, x_z)$, is a function of time and is called a cost function for $C$ whose evaluated value at time $t \geq 0$ is denoted by $(V(C, x_{zc}, x_z))(t)$. At each time $t \geq 0$, every candidate controller is assessed by the evaluated value of its cost function at time $t$.

The cost mapping $V$ is assumed to be causal, which means that $(V(C, x_{zc}, x_z))(t)$ depends only on $C$, $x_{zc}(\tau)$, and $x_z(\tau)$ for $\forall \tau \in [0, t]$. In an experiment, to calculate the cost function for a given $C$, the input and the output signal of $C$, $z_C$, and the observed system-output signal $z$ are utilized as $x_{zc}$ and $x_z$, respectively. Thus, the causality of $V$ is inevitable since, at each time $t \geq 0$, the observed data $z_C$ and $z$ is available between time 0 and time $t$ in the experiment and cost functions $V(C, z_C, z)$ for $\forall C \in \mathbb{C}$ need to be evaluated with the available data.

### B. Safe Adaptive Control

The main results of the safe adaptive switching control in [1] is summarized in the following.

**Definition 2** (Cost-detectability) Given the reference signal $r$, the auxiliary signal $s$, the candidate controller set $\mathbb{C}$, and the cost mapping $V$ in the adaptive control system in Section II-A, the pair $(V, \mathbb{C})$ is said to be cost-detectable if, for every sequence of switched controllers $\hat{C}$ with finitely many switches and the accordingly observed system-output signal $z = [u \ y]^T$ and controller-input-output signal for $C_f$, $z_{C_f} = [r_{C_f} \ u_{C_f} \ y_{C_f}]^T$, where $C_f$ denotes the final controller in the controller sequence $\hat{C}$, the following statements are equivalent:

1) $(V(C_f, z_{C_f}, z))(t)$ is bounded as $t$ increases to infinity.
2) For any given finite constant $\rho_1 > 0$, a function of time

$$
\dot{V}(t) \leq \max_{0 \leq \tau \leq t} \left\| z \right\|_{\tau}^2 + \rho_1 $$

is bounded as $t$ increases to infinity where $w = [r \ s]^T$.

The cost-detectability in Definition 2 is the same as the cost-detectability in [1].

**Definition 3** (Feasibility) Given the unknown disturbance signal $d$, the input-output description of the plant $\mathbb{Z}(P, d)$, the cost mapping $V$, and the input-output descriptions of the candidate controllers $\mathbb{Z}_C(C)$ for $\forall C \in \mathbb{C}$ in the adaptive control system in Section II-A, a controller $C$ is said to be a feasible controller if there exists a finite constant $\alpha_2 \geq 0$ such that for any given $x_{zc} \in \mathbb{Z}_C(C)$ and $x_z \in \mathbb{Z}(P, d)$,

$$(V(C, x_{zc}, x_z))(t) \leq \alpha_2 \quad \text{for } \forall t \geq 0.$$ 

The adaptive control problem is said to be feasible if the candidate controller set $\mathbb{C}$ contains at least one feasible controller.

In Definition 3, whether a controller is a feasible controller or not depends on the plant including its initial condition and the disturbance signal in the experiment conducted from time 0 to $\infty$.

Given the input and the output signals of $C$, $z_C$, for $\forall C \in \mathbb{C}$, the observed system-output signal $z$, and the cost mapping $V$ in the adaptive control system in Section II-A, the $\epsilon$-Hysteresis Switching Algorithm [5] is employed.

**Algorithm 1** (\epsilon-Hysteresis Switching Algorithm)

$$
\hat{C}_t = \arg\min_{C \in \mathbb{C}} \left\{ (V(C, zC, z))(t) - \epsilon_\delta_{CC'} \right\}
$$

where $\epsilon > 0$ is a constant, $\delta_{ij}$ is the Kronecker’s $\delta$, and $C' = \lim_{t \rightarrow 0} \hat{C}_t$.

Convergence of the switching algorithm in a finite number of switches is stated in the following lemma.

**Lemma 1** (Convergence)[7] Consider the adaptive control system in Section II-A, together with Algorithm 1. Suppose that 1) $V(C, x_{zc}, z)$ is nondecreasing in time for any given $C \in \mathbb{C}$, $x_{zc} \in \mathcal{L}_{2e}(m_e + m_a + m_p)$, and $x_z \in \mathcal{L}_{2e}(m_u + m_v)$, and 2) for any given $C \in \mathbb{C}$, $x_{zc} \in \mathcal{L}_{2e}(m_e + m_a + m_p)$, and $x_z \in \mathcal{L}_{2e}(m_u + m_v)$, the cost-detectability in Definition 2 is satisfied.

For the above assumptions and Lemma 1, Algorithm 1 is known to be convergent.
and $2_{e}^{(m_{u}+m_{y})}$ and 2) the adaptive control problem is feasible as in Definition 3. Then, the number of switches is finite and $(V(C)_{t}, z_{C_{t}}, z_{t}) (t)$ remains bounded as $t$ increases to infinity.

In [3] is the convergence result for infinite number of candidate controllers.

**Lemma 2:** [1] Consider the adaptive control system in Section II-A, together with Algorithm 1. Suppose that 1) $V(C, x_{C}, x_{z})$ is nondecreasing in time for any given $C \in \mathbb{C}$, $x_{z} \in \mathbb{L}_{2e}^{(m_{u}+m_{u}+m_{y})}$, and $x_{z} \in \mathbb{L}_{2e}^{(m_{u}+m_{y})}$, 2) the adaptive control problem is feasible as in Definition 3, and 3) the pair $(V, C)$ is cost-detectable. Then, for any given finite constant $\rho_1 > 0$, the function of time

$$\hat{V}(t) = \max_{0 \leq \tau \leq t} \|z\|_{\tau} + \rho_1$$

is bounded for $\forall t \in [0, \infty)$ where $w = [r \ s]^{T}$ and $z = [u \ y]^{T}$.

**Proof.** By Lemma 1, the number of switches is finite and $(V(C_{t}, z_{C_{t}}, z_{t})) (t)$ remains bounded as $t$ increases to infinity. The cost-detectability in Definition 2 completes the proof. \hfill $\square$

### III. COST DETECTABLE COST FUNCTION

The first and the third assumptions of Lemma 2, i.e. a nondecreasing property in time and a cost-detectability property, are imposed only on the cost mapping. In this section, those assumptions are proven to be satisfied by an $\mathbb{L}_{2e}$-gain-related cost mapping defined in Definition 5 without any assumption on candidate controllers.

**Definition 4:** (Fictitious signal) Given the candidate controller set $C$ in the adaptive control system in Section II-A, for any given $C \in \mathbb{C}$, $x_{C} \in \mathbb{L}_{2e}^{m_{u}}$, $x_{y} \in \mathbb{L}_{2e}^{m_{y}}$, and $x_{z}, x_{y_{C}} \in \mathbb{L}_{2e}^{y}$ a fictitious signal for $C$ is defined by

$$\hat{w}(x_{z}, x_{z}) = \begin{bmatrix} \hat{q}(z_{C}, z_{C}) \\ \hat{r}(z_{C}, z_{C}) \\ \hat{s}(z_{C}, z_{C}) \\ x_{u_{C}} - x_{y_{C}} \\ x_{y_{C}} \\ x_{u} - x_{u_{C}} \end{bmatrix}$$

where $z_{C} = [x_{u} \ x_{y}]^{T}$, $x_{u_{C}} = C(x_{y_{C}}, y_{C})$, and $x_{z} = [x_{r_{C}} \ x_{u_{C}} \ x_{y_{C}}]^{T}$.

The fictitious signal $\hat{w}(x_{z}, x_{z})$ is a hypothetical signal that would have exactly reproduced $x_{z}$ had the controller $C$ been in a fictitious closed-loop system in Fig. 4 with a plant $X_{P}$: $\mathbb{L}_{2e}^{m_{u}} \times \mathbb{L}_{2e}^{m_{y}} \rightarrow \mathbb{L}_{2e}^{m_{y}}$ and a disturbance signal $x_{d} \in \mathbb{L}_{2e}^{m_{y}}$ that satisfy $x_{z} \in \mathbb{Z}$. The fictitious signal is uniquely determined and is a function of time whose evaluated value at time $t \geq 0$ is denoted by $(\hat{w}(x_{z}, x_{z})) (t) = \begin{bmatrix} \hat{q}(z_{C}, z_{C}) (t) \\ \hat{r}(z_{C}, z_{C}) (t) \\ \hat{s}(z_{C}, z_{C}) (t) \\ x_{u_{C}} - x_{y_{C}} (t) \\ x_{y_{C}} (t) \\ x_{u} - x_{u_{C}} (t) \end{bmatrix}$.

**Remark 1:** (Fictitious signal in the adaptive control system in Section II-A) Consider the adaptive control system in Section II-A with the reference signal $r$, an auxiliary signal $s, w = [r \ s]^{T}$, the candidate controller set $C$, the input and the output signals of $C$, $z_{C} = [r_{C} \ u_{C} \ y_{C}]^{T}$, for $\forall C \in \mathbb{C}$, the plant-input signal $u$, the measured plant-output signal $y$, and $z = [u \ y]^{T}$. For any given $C \in \mathbb{C}$ and its corresponding $K$, a fictitious signal for $C$ is given by

$$\hat{w}(z_{C}, z) = \begin{bmatrix} \hat{q}(z_{C}, z) \\ \hat{r}(z_{C}, z) \\ \hat{s}(z_{C}, z) \\ y_{C} - y \\ r_{C} - r_{C_{t}} \\ u - u_{C} \end{bmatrix}$$

Note that all signals that are needed to generate the fictitious signal $\hat{w}(z_{C}, z)$ are observed in the adaptive control system and the fictitious signal is obtained from mere subtraction between the observed signals. For any given time $t \geq 0$ if $C$ is connected on the loop of the adaptive control system at time $t$, then $z_{C}(t) = \begin{bmatrix} r(t) \\ u(t) - s(t) \\ y(t) \end{bmatrix}$ and, hence, $(\hat{w}(z_{C}, z))(t) = \begin{bmatrix} 0 \\ r(t) \\ s(t) \end{bmatrix}$. If $C$ is not connected on the loop of the adaptive control system, then its corresponding subcontroller $K$ makes a closed-loop system with $C$ and stabilizes the candidate controller $C$ so that a mapping from $[r \ y]^{T}$ to $z_{C}$ is stable and, hence, a mapping from $[r \ z]^{T}$ to $\hat{w}(z_{C}, z)$ is stable. Therefore, a mapping from $[w \ z]^{T}$ to $\hat{w}(z_{C}, z)$ is always stable whether or not the candidate controller is connected in the loop of the adaptive control system. Later, Theorem 2 guarantees stability of a mapping from $w$ to $z$, then stability of a mapping from $w$ to $z_{C}$ and stability of a mapping from $w$ to $\hat{w}(z_{C}, z)$ follow.

**Definition 5:** (Cost-detectable cost function) Given the candidate controller set $C$, the input-output description of the candidate controllers $Z_{C}(C)$ for $\forall C \in \mathbb{C}$, and the cost mapping $V$ in the adaptive control system in Section II-A, the cost mapping $V$ is said to be $\mathbb{L}_{2e}$-gain-related if

1) $(V(C, x_{z}, x_{z})) (t)$ is nondecreasing in time $t$ for any given $C \in \mathbb{C}$, $x_{z} \in \mathbb{L}_{2e}^{(m_{u}+m_{u}+m_{y})}$, and $x_{z} \in \mathbb{L}_{2e}^{(m_{u}+m_{y})}$ and

2) for any given $C \in \mathbb{C}$ and a set $A$ satisfying

$$A \subset \left\{ [x_{1} \ x_{2}]^{T} | x_{1} \in Z_{C}(C), x_{2} \in \mathbb{L}_{2e}^{(m_{u}+m_{y})} \right\}$$

the following statements are equivalent:

- There is a constant $\alpha_{3} \geq 0$ such that for any given $[x_{z} \ x_{z}]^{T} \in A$

  $$(V(C, x_{z}, x_{z})) (t) < \alpha_{3} \quad \text{for} \quad \forall t \geq 0$$

- For any given finite constant $\rho_{2} > 0$, there is a constant $\alpha_{4} \geq 0$ such that for any given $[x_{z} \ x_{z}]^{T} \in A$

  $$\max_{0 \leq \tau \leq t} \|\hat{w}(x_{z}, x_{z})\|_{\tau} < \alpha_{4} \quad \text{for} \quad \forall t \geq 0$$

- There is a constant $\alpha_{3} \geq 0$ such that for any given $[x_{z} \ x_{z}]^{T} \in A$

  $$(V(C, x_{z}, x_{z})) (t) < \alpha_{3} \quad \text{for} \quad \forall t \geq 0$$

- For any given finite constant $\rho_{2} > 0$, there is a constant $\alpha_{4} \geq 0$ such that for any given $[x_{z} \ x_{z}]^{T} \in A$

  $$\max_{0 \leq \tau \leq t} \|\hat{w}(x_{z}, x_{z})\|_{\tau} < \alpha_{4} \quad \text{for} \quad \forall t \geq 0$$
where \( \tilde{w}(x_C, x_z) \) is a fictitious signal for \( C \) as in Definition 4.

The definition of an \( \mathcal{L}_{2e} \)-gain-related cost mapping in Definition 5 is the same as the definition of an \( \mathcal{L}_{2e} \)-gain-related cost functional in [1] in the sense that \( \max_{0 \leq \tau \leq t} \|w(x_C, x_z)\|_{\tau} \) is bounded as \( t \) increases to infinity if, and only if, stability of a fictitious system \((X_P, x_d, C)\) in Fig. 4, where a plant \( X_P : \mathcal{L}^{m_u}_{2e} \times \mathcal{L}^{m_w}_{2e} \rightarrow \mathcal{L}^{m_y}_{2e} \) and a disturbance signal \( x_d \in \mathcal{L}^{m_d}_{2e} \) satisfies \( x_z \in Z(X_P, x_d) \) and is unsatisfied by the input-output pair \((\hat{w}(x_C, x_z), x_z)\). Note that Definition 5 does not have an existence requirement of the fictitious signal \( \tilde{w}(x_C, x_z) \) since the fictitious signal \( \tilde{w}(x_C, x_z) \) always exists and is unique.

An obvious example of an \( \mathcal{L}_{2e} \)-gain-related cost mapping is a mapping \( V \) such that for any given \( C \in \mathbb{C} \), \( x_C \in Z_C(C) \), and \( x_z \in \mathcal{L}^{m_z}_{2e} \),

\[
(\tilde{V}(C, x_C, x_z, t)) (t) = \max_{0 \leq \tau \leq t} \left\| \tilde{w}(x_C, x_z) \right\|_{\tau} + 1 \quad \text{for all } t \geq 0.
\]

**Theorem 1:** Given the reference signal \( r \), the auxiliary signal \( s \), the candidate controller set \( \mathbb{C} \), the cost mapping \( V \), the input and the output signals of \( C \),\( z_C = [r_C \ u_C \ y_C]^T \), for all \( C \in \mathbb{C} \), the plant-input signal \( u \), the measured plant-output signal \( y \), and \( z = [u \ y]^T \) in the adaptive control system in Section II-A, the cost mapping \( V \) is \( \mathcal{L}_{2e} \)-gain-related as in Definition 5. Then, the pair \((V, C)\) is cost-detectable.

**Proof.** Suppose that there are finite number of switches and denote the final controller and the final switching time by \( C_f \) and \( t_f < \infty \), respectively.

After the final switch, it can be obtained from (1) that

\[
z_{C_f}(t) = \begin{bmatrix} r_{C_f}(t) \\ y_{C_f}(t) \\ u_{C_f}(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ y(t) \\ u(t) - s(t) \end{bmatrix} =: \begin{bmatrix} r(t) \\ y(t) \\ u(t) - u_{C_f}(t) \end{bmatrix} \quad \text{for all } t \geq t_f.
\]

Remark 1 provides the fictitious signal for the final controller

\[
(\tilde{w}(z_{C_f}, z)) (t) = \begin{bmatrix} \tilde{q}(z_{C_f}, z) \\ \tilde{r}(z_{C_f}, z) \\ \tilde{s}(z_{C_f}, z) \end{bmatrix} (t) = \begin{bmatrix} y_{C_f}(t) - y(t) \\ r_{C_f}(t) \\ u(t) - u_{C_f}(t) \end{bmatrix} \quad \text{for all } t \geq 0,
\]

from which, with (2), it follows that

\[
(\hat{w}(z_{C_f}, z)) (t) = \begin{bmatrix} \hat{q}(z_{C_f}, z) \\ \hat{r}(z_{C_f}, z) \\ \hat{s}(z_{C_f}, z) \end{bmatrix} (t) = \begin{bmatrix} 0_{m_y} \\ r(t) \\ s(t) \end{bmatrix} \quad \text{for all } t \geq t_f
\]

where \( 0_{m_y} \) is a zero vector in \( \mathbb{R}^{m_y} \) and, hence, it can be obtained that

\[
\left\| \hat{w}(z_{C_f}, z) - 0_{m_y} \right\|_{t_f} \leq \left\| \tilde{q}(z_{C_f}, z) - \hat{q}(z_{C_f}, z) \right\|_{t_f} + \left\| \tilde{r}(z_{C_f}, z) - r(t) \right\|_{t_f} + \left\| \tilde{s}(z_{C_f}, z) - s(t) \right\|_{t_f} \leq \rho_3 < \infty \quad \text{for all } t \geq t_f
\]

where \( w = [r \ s]^T \). The third inequality in (3) comes from the fact that the signals are in \( \mathcal{L}_{2e} \). From the triangle inequality and (3), it can be obtained that

\[
\begin{align*}
\left\| \tilde{w}(z_{C_f}, z) \right\|_{t} &\leq \left\| \hat{w}(z_{C_f}, z) - 0_{m_y} \right\|_{t} + \left\| w(t) \right\|_{t} \\
&\leq \left\| w \right\|_{t} + \rho_3 \quad \text{for all } t \geq 0
\end{align*}
\]

and

\[
\begin{align*}
\left\| w \right\|_{t} &\leq \left\| 0_{m_y} \right\|_{t} + \left\| \tilde{w}(z_{C_f}, z) \right\|_{t} \\
&\leq \left\| \tilde{w}(z_{C_f}, z) \right\|_{t} + \rho_3 \quad \text{for all } t \geq 0.
\end{align*}
\]

The two statements in the definition of the cost-detectability in Definition 2 is proved to be equivalent by showing 1) the sufficiency part and 2) the necessity part.

1) If the cost function \((V(C_f, z_{C_f}, z)) (t)\) is bounded as \( t \) increases to infinity, for a given finite constant \( \rho_2 > 0 \) there exists, by Definition 5 with \( C = C_f \) and \( A = \{ z_{C_f}, z \}^T \), a finite constant \( \alpha_4 \geq 0 \) such that

\[
\max_{0 \leq \tau \leq t} \left\| \tilde{w}(z_{C_f}, z) \right\|_{\tau} \leq \alpha_4 \quad \text{for all } t \geq 0,
\]

which implies that

\[
\left\| z \right\|_{t} \leq \alpha_4 \left( \left\| \tilde{w}(z_{C_f}, z) \right\|_{t} + \rho_2 \right) \leq \alpha_4 \left( \left\| w \right\|_{t} + \rho_3 + \rho_2 \right) \quad \text{for all } t \geq 0.
\]

The second inequality in (6) comes from (4). From (6), it follows that for any given finite constant \( \rho_1 > 0 \)

\[
\hat{V}(t) = \max_{0 \leq \tau \leq t} \left\| \tilde{w}(z_{C_f}, z) \right\|_{\tau} \leq \alpha_4 \max_{0 \leq \tau \leq t} \left\| w \right\|_{\tau} + \rho_3 + \rho_2 \leq \alpha_4 \left( 1 + \frac{\rho_3 + \rho_2 - \rho_1}{\rho_1} \right) < \infty \quad \text{for all } t \geq 0.
\]

2) If for a given finite constant \( \rho_1 > 0 \) the function of time

\[
\hat{V}(t) = \max_{0 \leq \tau \leq t} \left\| \tilde{w}(z_{C_f}, z) \right\|_{\tau} + \rho_2
\]

is bounded as \( t \) increases to infinity, there exists a finite constant \( \alpha_5 \geq 0 \) such that

\[
\left\| \tilde{w}(z_{C_f}, z) \right\|_{t} \leq \alpha_5 \left( \left\| w \right\|_{t} + \rho_1 \right) \leq \alpha_5 \left( \left\| \tilde{w}(z_{C_f}, z) \right\|_{t} + \rho_3 + \rho_1 \right) \quad \text{for all } t \geq 0.
\]

The second inequality in (7) comes from (5). From (7), it follows that for any given finite constant \( \rho_2 > 0 \)

\[
\max_{0 \leq \tau \leq t} \left\| \tilde{w}(z_{C_f}, z) \right\|_{\tau} \leq \alpha_5 \max_{0 \leq \tau \leq t} \left\| \tilde{w}(z_{C_f}, z) \right\|_{\tau} + \rho_3 + \rho_1 \leq \alpha_5 \left( 1 + \frac{\rho_3 + \rho_1 - \rho_2}{\rho_2} \right) < \infty \quad \text{for all } t \geq 0,
\]

which means by Definition 5 that \((V(C_f, z_{C_f}, z))(t)\) is bounded as \( t \) increases to infinity.

Therefore, from 1) and 2), the proof is completed. \( \square \)
The following theorem presents a safe adaptive switching control with the new fictitious signals and the redefined $\mathcal{L}_{\infty}$-gain-related costs.

**Theorem 2:** Consider the adaptive control system in Section II-A, together with Algorithm 1. Suppose that 1) the cost mapping $V$ is $\mathcal{L}_{\infty}$-gain-related as in Definition 5 and 2) the adaptive control problem is feasible as in Definition 3. Then, for any given finite constant $\rho_1 > 0$ the function of time

$$
\dot{V}(t) = \max_{0 \leq \tau \leq t} \frac{\|z\|}{\|u\| + \rho_1}
$$

is bounded for $\forall t \in [0, \infty)$ where $w = [r \ s]^T$ and $z = [u \ y]^T$.

**Proof.** The pair $(V, C)$ is cost-detectable by Theorem 1. By Definition 5, $V(C, x_{zc}, x_z)$ for any given $C \in \mathbb{C}$, $x_{zc} \in \mathcal{L}_{\infty}^{(m_r + m_q + m_u)}$, and $x_z \in \mathcal{L}_{\infty}^{(m_u + m_u)}$ is nondecreasing in time. Since all the conditions of Lemma 2 are satisfied, Lemma 2 completes the proof. \hfill $\square$

Theorem 2 guarantees that, for the adaptive control system in Section II-A with Algorithm 1 and an $\mathcal{L}_{\infty}$-gain-related cost mapping, there exists, for a given finite constant $\rho_1 > 0$, a finite constant $\alpha_6 \geq 0$ such that

$$
\|z\| \leq \alpha_6 (\|w\|_{\tau} + \rho_1) \quad \text{for} \quad \forall t \geq 0 \quad (8)
$$

provided that the adaptive control problem is feasible. When the auxiliary signal $s$ is a zero signal, i.e. $s(t) = 0$ for $\forall t \geq 0$, the inequality (8) reduces to

$$
\|z\| \leq \alpha_6 (\|r\|_{\tau} + \rho_1) \quad \text{for} \quad \forall t \geq 0,
$$

which is the same result that the safe adaptive control in [1] attains with the SCLI assumption.

In Theorem 2, since there is no assumption on the candidate controllers and the cost mapping can be deliberately derived to be $\mathcal{L}_{\infty}$-gain-related, the only major assumption is the feasibility assumption. Consider an $\mathcal{L}_{\infty}$-gain-related cost mapping $V$ such that for any given $C \in \mathbb{C}$, $x_{zc} \in \mathcal{Z}_C(C)$, and $x_z \in \mathcal{L}_{\infty}^{(m_u + m_u)}$

$$
\left(V(C, x_{zc}, x_z) \right) (t) = \max_{0 \leq \tau \leq t} \frac{\|z\|_{\tau}}{\|\bar{w}(x_{zc}, x_z)\|_{\tau} + 1} \quad (9)
$$

where $\bar{w}(x_{zc}, x_z)$ is the fictitious signal for $C$ as in Definition 4. From Definition 3 and 5, a controller $C \in \mathbb{C}$ is a feasible controller in Theorem 2 with (9) if, and only if, there exists a finite constant $\alpha_2 \geq 0$ such that for any given $x_{zc} \in \mathcal{Z}_C(C)$ and $x_z \in \mathcal{Z}(P, d)$

$$
\left(V(C, x_{zc}, x_z) \right) (t) \leq \alpha_2 \quad \text{for} \quad \forall t \geq 0,
$$

which is equivalent to the condition that there exists a finite constant $\alpha_4 \geq 0$ such that for any given $x_{zc} \in \mathcal{Z}_C(C)$ and $x_z \in \mathcal{Z}(P, d)$

$$
\|z\|_{\tau} \leq \alpha_4 (\|\bar{w}(x_{zc}, x_z)\|_{\tau} + 1) \quad \text{for} \quad \forall t \geq 0. \quad (10)
$$

The signal $x_z = [x_q \ x_r]^T$ in the condition (10) can be interpreted as an observed system-output signal of a fictitious system in Fig. 4 with $X_P = P$ and $x_d = d$. If we apply to the fictitious system arbitrary signals $x_q \in \mathcal{N}_{2e}^{m_q}$, $x_r \in \mathcal{L}_{2e}^{m_r}$, and $x_z \in \mathcal{L}_{2e}^{m_z}$, instead of the fictitious signals $\tilde{q}(x_{zc}, x_z)$, $\tilde{r}(x_{zc}, x_z)$, and $\tilde{s}(x_{zc}, x_z)$, respectively, and the controller $C$ has an ability to attain finite constants $\alpha_7, \beta_7 \geq 0$ such that for any given $x_q \in \mathcal{N}_{2e}^{m_q}$, $x_r \in \mathcal{L}_{2e}^{m_r}$, and $x_z \in \mathcal{L}_{2e}^{m_z}$

$$
\|x_z\|_{\tau} \leq \alpha_7 \left\| \begin{bmatrix} x_q & x_r & x_z \end{bmatrix}^T \right\|_{\tau} + \beta_7 \quad \text{for} \quad \forall t \geq 0,
$$

then the controller $C$ is a feasible controller in the adaptive control system. This ability of $C$ totally depends on the plant and the disturbance signal. Thus, when the plant and the disturbance signal are entirely unknown before the experiment, it is not possible to determine if a controller is feasible or not so that placing as many controllers in the candidate controller set as possible would be the best.

**IV. CONCLUSION**

The SCLI assumption, the only assumption on candidate controllers in the safe adaptive control [1], can be removed when we use the fictitious signals and the cost functions introduced in this paper. By injecting an auxiliary signal into the adaptive control system, needs for inverses of candidate controllers disappear. Candidate controllers are stabilized by their corresponding subcontrollers while they are not connected in the adaptive switching control system. This guarantees that the fictitious signal generator is stable. The new fictitious signals and the new cost functions achieve cost-detectability with no restriction on candidate controllers.

With the fictitious signals and the cost functions, the only assumption left in the safe adaptive control is the feasibility assumption. Conditions for a controller to be a feasible controller remains to be specified and loosened based on knowledge of a plant and disturbance signals.

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**REFERENCES**