The disturbance rejection by measurement feedback problem revisited

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Abstract—In this paper, we revisit the disturbance rejection by measurement feedback (DRMF) problem in a structural framework. We prove that the study of this problem can be restricted to a subset of the state space and that the DRMF problem reduces to an unknown input observer problem on this subset. This result allows to explain why we need to estimate a sufficient number of state variables and early enough in order to estimate the disturbance effect and compensate for it via the control input. We give the minimal number of sensors to be implemented for solving the DRMF problem and prove that measuring in some parts of the state space is useless. Our analysis is performed in the context of structured systems which represent a large class of parameter dependent linear systems.

Index Terms: Linear structured systems, Disturbance rejection, Sensor failure, Sensor classification

I. INTRODUCTION

This paper is concerned with linear systems which are affected by unmeasurable disturbances and we look for exact disturbance rejection (i.e., a zero disturbance-controlled output transfer matrix) by measurement feedback. Other approaches allow to stabilize and minimize some norm of this transfer matrix, see for example [1]. The problem of disturbance rejection by state feedback is a very well known problem [2], [3]. In the case where the state is not available for measurement, the problem is more complex. The problem of disturbance rejection by measurement feedback has been solved in an elegant way in geometric terms, see [4], [5].

We consider here linear structured systems which represent a large class of parameter dependent linear systems. Generic properties for such systems can be obtained easily from a graph naturally associated with the systems. This approach was pioneered by Lin [6]. In this framework, the Disturbance Rejection by Measurement Feedback (DRMF) problem has been solved via a graph approach in [7], [8], [9]. For single input single output discrete time systems, we have the intuitive idea that the DRMF problem is potentially solvable when the time for the disturbances to affect controlled outputs is longer than the time to get from output measurements information on the disturbances plus the time to annihilate the effect of the disturbances on the outputs with the help of the control inputs. The proposed graph approach illustrates and generalizes the above idea for multivariable systems. It is clear that the solvability of this problem highly relies on the sensor network. Sensor location has already been studied in a structural framework for two other problems, the observability in [10], [11], [12] and the Fault Detection and Isolation problem in [13], [14].

In this paper we revisit the DRMF problem. We give a new necessary and sufficient condition for the structural solvability of the DRMF problem. As a consequence, we prove that the problem reduces to an unknown input observer problem on a subset of the state space. This subset consists of the states for which a disturbance affecting directly these states can be rejected by state feedback. The observation problem amounts to estimate the disturbance effect before it leaves this subset. This allows to explain why we need to measure a sufficient number of state variables early enough to be able to estimate the disturbance effect and compensate for it via the control input. Consequently, we give the minimal number of sensors to be implemented for solving the DRMF problem. We prove also that the sensors measuring only states out of a given subset are useless for solving the DRMF problem. Our analysis comes within the context of structured systems which represent a large class of linear systems. The generic results are obtained directly from the system associated graph.

The outline of this paper is as follows. We formulate the problem of disturbance rejection by measurement feedback in section II. The linear structured systems are presented in section III as well as the known structural results on the DRMF problem. In section IV we revisit the DRMF problem and give our main Theorem. The sensor location problem is considered in section V: we give the minimal number of sensors for solving the DRMF problem and characterize an important set of useless sensors. Some concluding remarks end the paper.

II. DISTURBANCE REJECTION BY MEASUREMENT FEEDBACK (DRMF)

We consider the linear system $\Sigma$ given by:

$$\begin{align*}
\Sigma \quad \begin{cases}
\dot{x}(t) &= Ax(t) + Bu(t) + Ed(t) \\
y(t) &= Cx(t) \\
z(t) &= Hx(t)
\end{cases},
\end{align*}$$

(1)

where $u(t) \in \mathbb{R}^m$ is the control input, $d(t) \in \mathbb{R}^q$ is the unmeasurable disturbance, $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the regulated output and $z(t) \in \mathbb{R}^r$ the measured output provided by a sensor network. For such a system, we have the transfer matrix:

$$\begin{bmatrix}
y(s) \\
z(s)
\end{bmatrix} = \begin{bmatrix}
G(s) & K(s) \\
M(s) & N(s)
\end{bmatrix} \begin{bmatrix}
u(s) \\
d(s)
\end{bmatrix}$$

(2)

The problem of disturbance rejection amounts to find a
dynamic measured output feedback compensator
\[ \Sigma_{zu} \begin{cases} \dot{w}(t) = Lw(t) + Rz(t) \\ u(t) = Sw(t) + Pz(t) \end{cases} , \]
(3)
such that in closed loop the disturbances will have no effect on the regulated output. In transfer matrix terms, we look for a dynamic compensator (see Figure 1) \( u(s) = F(s)z(s) \), where \( F(s) \) is a proper rational matrix, such that the closed loop system transfer matrix from disturbance \( d \) to controlled output \( y \) is identically zero:
\[ G(s)F(s)(I - M(s)F(s))^{-1}N(s) + K(s) = 0 \]
(4)
This problem received a very elegant solution in geometric terms, see [15], [4]. A geometric necessary and sufficient condition for the solvability of the DRMF problem is:
\[ N^* \subset \mathcal{V}^* \]
(5)
where \( N^* \) is the minimal \((H, A)\)-invariant subspace containing \( Im E \) and \( \mathcal{V}^* \) is the maximal \((A, B)\)-invariant subspace contained in \( Ker C \).

In fact, the subspace \( \mathcal{V}^* \) appeared first in solving the disturbance rejection by state feedback problem [16], [2], while \( N^* \) was introduced for solving the dual problem which is the partial state estimation independently of a disturbance [3], [17]. Since the DRMF combines these two problems, the very nice condition (5) appears as very natural. It is quite clear that the solvability of the DRMF problem is highly dependent on the sensor network. In particular, in condition (5), the subspace \( N^* \) is computed from the relations between the disturbances and the measured outputs and involves only matrices \( E, A \) and \( H \).

In this paper we will revisit the DRMF problem in a structural way. We give new results and show in particular that this problem turns out to be equivalent to an observation problem on a subsystem of the original one. This will provide us with some useful information on the minimal number of sensors to be implemented and on their possible location.

III. LINEAR STRUCTURED SYSTEMS

A. Definitions

In this subsection we recall some definitions and results on linear structured systems. More details can be found in [18]. We consider linear systems of type (1) with parameterized entries and denoted by \( \Sigma_{\Lambda} \) as follows:
\[ \Sigma_{\Lambda} \begin{cases} \dot{x}(t) = A_{\Lambda}x(t) + B_{\Lambda}u(t) + E_{\Lambda}d(t) \\ y(t) = C_{\Lambda}x(t) \\ z(t) = H_{\Lambda}x(t) \end{cases} \]
(6)
This system is called a linear structured system if the entries of the composite matrix \( J_{\Lambda} = \begin{bmatrix} A_{\Lambda} & B_{\Lambda} & E_{\Lambda} \\ C_{\Lambda} & 0 & 0 \\ H_{\Lambda} & 0 & 0 \end{bmatrix} \) are either fixed zeros or independent parameters (not related by algebraic equations). \( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \) denotes the set of independent parameters of the composite matrix \( J_{\Lambda} \).

For such systems, one can study generic properties, i.e. properties which are true for almost all values of the parameters collected in \( \Lambda \) [19]. More precisely, a property is said to be generic (or structural) if it is true for all values of the parameters outside a proper algebraic variety of the parameter space. A directed graph \( G(\Sigma_{\Lambda}) = (V, W) \) can be associated with the structured system \( \Sigma_{\Lambda} \) of type (6):
- the vertex set is \( V = U \cup D \cup X \cup Y \cup Z \) where \( U, D, X, Y \) and \( Z \) are the input, disturbance, state, regulated output and measured output sets given by \( \{u_1, u_2, \ldots, u_m\}, \{d_1, d_2, \ldots, d_q\}, \{x_1, x_2, \ldots, x_n\}, \{y_1, y_2, \ldots, y_p\} \) and \( \{z_1, z_2, \ldots, z_r\} \), respectively,
- the arc set is \( W = \{\{u_i, x_j\}|B_{\Lambda, ji} \neq 0\} \cup \{\{d_i, x_j\}|A_{\Lambda, ji} \neq 0\} \cup \{\{x_i, y_j\}|C_{\Lambda, ji} \neq 0\} \cup \{\{x_i, z_j\}|H_{\Lambda, ji} \neq 0\} \), where \( A_{\Lambda, ji} \) (resp. \( B_{\Lambda, ji} \), \( C_{\Lambda, ji} \), \( H_{\Lambda, ji} \)) denotes the entry \((j, i)\) of the matrix \( A_{\Lambda} \) (resp. \( B_{\Lambda} \), \( C_{\Lambda} \), \( H_{\Lambda} \)).

Let \( V_1, V_2 \) be two nonempty subsets of the vertex set \( V \) of the graph \( G(\Sigma_{\Lambda}) \). We say that there exists a path from \( V_1 \) to \( V_2 \) if there are vertices \( i_0, i_1, \ldots, i_r \) such that \( i_0 \in V_1 \), \( i_r \in V_2 \), \( i_t \in V \) for \( t = 0, 1, \ldots, r \) and \( (i_{t-1}, i_t) \notin W \) for \( t = 1, 2, \ldots, r \). We call the path simple if every vertex on the path occurs only once. Two paths from \( V_1 \) to \( V_2 \) are said to be disjoint if they consist of disjoint sets of vertices, \( r \) paths from \( V_1 \) to \( V_2 \) are said to be disjoint if they are mutually disjoint, i.e. any two of them are disjoint. A set of \( r \) disjoint and simple paths from \( V_1 \) to \( V_2 \) is called a linking from \( V_1 \) to \( V_2 \) of size \( r \).

Example 1: Consider the following example of a structured system whose matrices of Equation (6) are the following:
\[ A_{\Lambda} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ \lambda_2 & 0 & \lambda_3 & 0 \end{bmatrix} , \quad B_{\Lambda} = \begin{bmatrix} 0 \\ 0 \\ \lambda_4 \\ 0 \end{bmatrix} , \quad E_{\Lambda} = \begin{bmatrix} \lambda_5 \\ \lambda_6 \\ 0 \\ 0 \end{bmatrix} \]
\[ C_{\Lambda} = \begin{bmatrix} 0 & 0 & 0 & \lambda_7 \end{bmatrix} , \quad H_{\Lambda} = \begin{bmatrix} 0 & \lambda_8 & 0 & 0 \end{bmatrix} . \]
The associated graph is depicted in Figure 2.
In this example, \( \{d, x_1, x_4, y\} \) and \( \{d, x_2, x_3, x_4, y\} \) are \( D - Y \) paths but the maximal size of a \( D - Y \) linking is one.

**B. Disturbance Rejection by Measurement Feedback for structured systems**

In order to solve the DRMF problem in the context of structured systems we will define two important sets of vertices in the graph \( G(\Sigma_A) \).

**Definition 1:** Consider \( \Sigma_A \) a structured system of type (6) with associated graph \( G(\Sigma_A) \). Let us define the vertex set \( \mathcal{I}^* \) as follows:

\[ \mathcal{I}^* = \{ x_i \in X \mid \text{the maximal size of a linking in } G(\Sigma_A) \text{ from } U \cup \{x_i\} \text{ to } Y \text{ is the same as the maximal size of a linking in } G(\Sigma_A) \text{ from } U \text{ to } Y, \text{ and the minimal number of vertices in } X \cup U \text{ is the same for both such maximal linkings} \} \]

The set \( \mathcal{I}^* \) corresponds to the states for which an un-accessible disturbance affecting directly these states can be rejected by state feedback [20]. Notice that \( \mathcal{I}^* \) can be computed independently of the sensor network since its computation involves uniquely the matrices \( A, B \) and \( C \) in (6).

In a dual way, we can define the set of vertices \( \mathcal{J}^* \):

**Definition 2:** Consider \( \Sigma_A \) a structured system of type (6) with associated graph \( G(\Sigma_A) \). Let us define the vertex set \( \mathcal{J}^* \) as follows:

\[ \mathcal{J}^* = \{ x_j \in X \mid \text{the maximal size of a linking in } G(\Sigma_A) \text{ from } D \text{ to } Z \cup \{x_j\} \text{ is the same as the maximal size of a linking in } G(\Sigma_A) \text{ from } D \text{ to } Z, \text{ and the minimal number of vertices in } X \cup Z \text{ is the same for both such maximal linkings} \} \]

The set \( \mathcal{J}^* \) corresponds to the states which can be estimated from the measured outputs through an observer independently from the disturbance. Notice that \( \mathcal{J}^* \) depends on the sensor network. Its computation involves uniquely the matrices \( A, E \) and \( H \) in (6).

From the definitions of \( \mathcal{I}^* \) and \( \mathcal{J}^* \), the geometric condition for disturbance rejection by measurement feedback can be translated for linear structured systems as follows: [8], [9]

**Theorem 1:** Consider \( \Sigma_A \) a structured system of type (6) with associated graph \( G(\Sigma_A) \). The problem of disturbance rejection by measurement feedback is generically solvable if and only if:

\[ \mathcal{I}^* \cup \mathcal{J}^* = X \]  \hspace{1cm} (7)

It can be shown that for a structured system, \( \mathcal{I}^* \) coincides generically with the fixed part of the maximal \((A, B)\)-invariant subspace contained in \( \ker C \) when the parameters vary. Similarly the complement in \( X \) of \( \mathcal{J}^* \) is the envelope of the minimal \((H, A)\)-invariant subspace containing \( \text{Im } E \) when the parameters vary.

Consider again Example 1. We have \( \mathcal{I}^* = \{x_1, x_2, x_3\} \), \( \mathcal{J}^* = \{x_3, x_4\} \). Since \( \mathcal{I}^* \cup \mathcal{J}^* = X \), the DRMF problem is generically solvable for this example.

From the definitions of \( \mathcal{I}^* \) and \( \mathcal{J}^* \), checking condition (7) amounts to compute in \( G(\Sigma_A) \) some maximal linkings with minimal number of vertices. This can be done using standard algorithms of combinatorial optimization as max-flow min-cost techniques [21], [22]. This means that for a given sensor network, the solvability of the DRMF problem can be checked in polynomial time.

It is of interest to derive general results on the solvability of the DRMF problem with respect to the possible location of sensors without having to check (7) for any possibility. This will be the purpose of the next section.

**IV. THE DRMF PROBLEM REVISITED**

In this section we state the main contribution of this paper, i.e., a new necessary and sufficient condition for the DRMF problem to be solvable. This theorem will give a new insight into the problem and provide with useful information on the number and the location of the sensors to be implemented. Let us give first a preliminary definition:

**Definition 3:** Consider \( \Sigma_A \) a structured system of type (6) with associated graph \( G(\Sigma_A) \). Define \( F_{Z^*} \), the frontier of \( \mathcal{I}^* \) as the set of vertices:

\[ F_{Z^*} = \{ x_i \in \mathcal{I}^* \mid \exists (x_i, x_j) \in W, x_j \notin \mathcal{I}^* \}. \]

**Lemma 1:** Consider \( \Sigma_A \) a structured system of type (6) with associated graph \( G(\Sigma_A) \). Let \( L_{D, Z} \) be a maximal linking in \( G(\Sigma_A) \) from \( D \) to \( Z \) having a minimal number of vertices in \( X \cup Z \) denoted \( N_{D, Z} \). Let \( \mathcal{J}^* \) be as in Definition 2. Then for any \( x_i \in L_{D, Z}, x_i \notin \mathcal{J}^* \).

**Proof:** From the calculation of \( \mathcal{J}^* \) it follows directly, by considering \( x_i \) as a new measurement, that the size \( \mu \) of the maximal linking from \( D \) to \( Z \) can not be reduced. So we have two cases:

1. If \( \mu \) increases, \( x_i \notin \mathcal{J}^* \) by Definition 2.
2. If \( \mu \) remains the same, the path from \( D \) to \( Z \) through \( x_i \) which is needed for the construction of the linking \( L_{D, Z} \) is now ended at \( x_i \), so \( N_{D, Z} \) reduces at least by 1. Then \( x_i \notin \mathcal{J}^* \).

State now the main result of this paper:

**Theorem 2:** Consider \( \Sigma_A \) a structured system of type (6) with associated graph \( G(\Sigma_A) \) and affected by the disturbances \( d_1, \ldots, d_q \). The DRMF problem is generically solvable if and only if:

1. \( d_1, \ldots, d_q \) affect only state vertices of \( \mathcal{I}^* \), i.e. for any \( (d_i, x_j) \in W, x_j \notin \mathcal{I}^* \).
2. The maximal size of a linking in \( G(\Sigma_A) \) from \( D \) to \( Z \) is the same as the maximal size of a linking in \( G(\Sigma_A) \) from \( D \) to \( Z \cup F_{Z^*} \), and the minimal number of vertices in \( X \) is the same for both such maximal linkings.

**Proof:**

**Necessity:** Let \( L_{D, Z} \) be a maximal linking in \( G(\Sigma_A) \) from \( D \) to \( Z \) having a minimal number of vertices in \( X \cup Z \) denoted \( N_{D, Z} \). \( L_{D, Z} \) is also a maximal linking which has a minimal number of state vertices \( X_{D, Z} = N_{D, Z} - r_{D, Z} \), where \( r_{D, Z} \) is the size of the linking \( L_{D, Z} \).

1. Prove first that Condition 1 is necessary. Assume that there exists an arc \( (d_i, x_j) \in W \), such that \( x_j \notin \mathcal{I}^* \). We will prove that \( x_j \notin \mathcal{J}^* \), therefore \( \mathcal{I}^* \cup \mathcal{J}^* \neq X \) and the DRMF problem is not solvable. If \( x_j \) belongs to \( L_{D, Z} \) then \( x_j \notin \mathcal{J}^* \) by Lemma 1. If \( x_j \) does not belong
to any \(L_{D,Z}\), we have two following cases depending on whether \(d_i\) belongs to \(L_{D,Z}\) or not:

- If \(d_i\) does not belong to any \(L_{D,Z}\), by adding the arc \((d_i, x_j) \in W\) to \(L_{D,Z}\), we have a maximal linking from \(D\) to \(Z \cup x_j\). The size of this new linking is greater than the size of \(L_{D,Z}\), therefore \(x_j \notin J^*\).

- If \(d_i\) belongs to some \(L_{D,Z}\) by a path from \(d_i\) to a measurement \(z_k\), since \((d_i, x_j) \in W\), we can construct a new maximal linking from \(D\) to \(Z \cup x_j\) by exchanging the path from \(d_i\) to \(z_k\) by the arc \((d_i, x_j)\) of length 1 i.e. which is shorter. Then the number of vertices in \(X \cup Z\) in this new linking is smaller than \(N_{D,Z}\) and \(x_j \notin J^*\).

2) Prove now that Condition 2 is necessary.

\(\mathcal{I}^*\) corresponds to the states for which a disturbance affecting these states can be generically rejected by state feedback. Conversely (see [9]), a disturbance (i.e. an unknown signal) affecting states out of \(\mathcal{I}^*\) cannot be generically rejected by state feedback. Assume that the states variables of \(F_{T^*}\) cannot be estimated from the available measurements. Since every vertex of \(F_{T^*}\) has a successor out of \(\mathcal{I}^*\), the unknown effect of a disturbance on \(F_{T^*}\) will propagate out of \(\mathcal{I}^*\), therefore it cannot be rejected by state feedback and consequently by measurement feedback. Thus, without prior knowledge on the disturbances, it is necessary to estimate exactly from the measurements the disturbance effect on the frontier \(F_{T^*}\) of \(\mathcal{I}^*\). This corresponds to the standard model following problem: find a causal dynamical system \(O\) (input \(Z\), output \(\hat{F}_{T^*}\)) such that \(\hat{F}_{T^*} = F_{T^*}\) for zero initial conditions (see Figure 3), where \(\Sigma_1\), \(\Sigma_2\) are corresponding subsystems of \(\Sigma\).

![Fig. 3. Model following problem](image)

This model following problem has a solution if and only if the infinite structure (number of infinite zeros and multiplicities) of \(\Sigma_2\) is equal to that of \([\Sigma_1, \Sigma_2]\). This is a well known result proved in geometric terms [23] or using a transfer matrix approach via the Smith form of matrices over the ring of proper rational functions [24]. The above infinite structure equality is then necessary for solving the DRMF problem. To prove this equality, it is sufficient to prove the equality of the number of infinite zeros and of the sum of the infinite zero orders [25].

In the framework of structured systems, the number of infinite zeros, which is equal to the transfer matrix rank of a system, is equal to the maximal size of an input-output linking in the graph of the system and the sum of the infinite zeros orders is equal to the minimal number of state vertices for such a maximal linking [20]. The equality of the infinite structures for \(\Sigma_2\) and \([\Sigma_1, \Sigma_2]\), which is a necessary condition for the DRMF problem, coincides then with condition 2.

**Sufficiency:** Assume now that the two conditions hold. Consider \(x_j \notin \mathcal{I}^*\). Let \(L_{D,Z,\mathcal{U}_{x_j}}\) be a maximal linking in \(G(\Sigma)\) from \(D\) to \(Z \cup x_j\) having a minimal number of vertices in \(X \cup Z\) denoted \(N_{D,Z,\mathcal{U}_{x_j}}\). We denote \(|L_{D,Z}|, |L_{D,Z,\mathcal{U}_{x_j}}|\) the size of \(L_{D,Z}\) and \(L_{D,Z,\mathcal{U}_{x_j}}\) respectively. Suppose that \(x_j \notin J^*\), we have two cases:

- \(|L_{D,Z}| < |L_{D,Z,\mathcal{U}_{x_j}}|\), then there exists a path from a disturbance \(d_i\) to \(x_j\) in \(L_{D,Z,\mathcal{U}_{x_j}}\). By the first condition, this path must intersect \(F_{T^*}\), i.e. there exists \(x_k \in F_{T^*}\) such that \(|L_{D,Z}| < |L_{D,Z,\mathcal{U}_{x_k}}|\) and this is impossible from the second condition.

- \(|L_{D,Z}| = |L_{D,Z,\mathcal{U}_{x_j}}|\), since \(x_j \notin J^*\) then \(N_{D,Z} > N_{D,Z,\mathcal{U}_{x_j}}\). Therefore \(X_{D,Z} \geq X_{D,Z,\mathcal{U}_{x_j}}\), where \(X_{D,Z}\), \(X_{D,Z,\mathcal{U}_{x_j}}\) are the number of vertices in \(X\) of \(L_{D,Z}\), \(L_{D,Z,\mathcal{U}_{x_j}}\), and there exists a path from a disturbance \(d_i\) to \(x_j\) in \(L_{D,Z,\mathcal{U}_{x_j}}\). This path must contain \(x_k \in F_{T^*}\) from Condition 1 and we have \(|L_{D,Z,\mathcal{U}_{x_k}}| = |L_{D,Z,\mathcal{U}_{x_j}}| = |L_{D,Z}|\) but \(X_{D,Z,\mathcal{U}_{x_k}} < X_{D,Z,\mathcal{U}_{x_j}} \leq X_{D,Z}\). This is impossible from the second condition.

It follows that if \(x_j \notin \mathcal{I}^*\), then \(x_i \in J^*\) so \(\mathcal{I}^* \cup J^* = X\) and by Theorem 1 the DRMF problem is generically solvable.

**Interpretation:**

Important consequences of Theorem 2 are the following:

- It is sufficient to study the DRMF problem on a part of the state space corresponding to \(\mathcal{I}^*\). For the problem to be solvable, disturbances must affect directly only states of \(\mathcal{I}^*\) since this set corresponds to the states for which a disturbance affecting directly these states can be rejected by state feedback and because a dynamic output feedback cannot do better than a full state feedback.

- The second condition as seen previously corresponds with a condition for solving a model matching problem. This model matching problem is in fact equivalent to an Unknown Input Observer problem [26], [17], [27]. The condition expresses the fact that the measurements must allow to reconstruct the effects of the disturbances before they leave \(\mathcal{I}^*\).

**Example 2:** Consider the following example of a structured system whose associated graph is depicted in Figure 4. In this example, \(\mathcal{I}^* = \{x_1\}\). The disturbance affects a state vertex which is not in \(\mathcal{I}^*\), the DRMF problem is not solvable.

**Example 3:** Consider now the example of a structured system whose associated graph is depicted in Figure 5. In this example, \(\mathcal{I}^* = \{x_1, x_2, x_3, x_4\}\). The disturbances affect directly state vertices which are in \(\mathcal{I}^*\). The frontier is \(F_{T^*} = \{x_3, x_4\}\). The maximal size of a linking in \(G(\Sigma)\) from \(D\) to \(Z\) is two and is the same as the maximal size
of a linking in $G(\Sigma \Lambda)$ from $D$ to $Z \cup F_I^\ast$. A linking from $D$ to $Z$ of maximal size with minimal number of vertices in $X$ is $\{(d_1, x_1, x_3, z_1), (d_2, x_2, x_4, z_2)\}$. A linking from $D$ to $Z \cup F_I^\ast$ of maximal size with minimal number of vertices in $X$ is $\{(d_1, x_1, z_2), (d_2, x_2, x_4)\}$. Since these two linkings have not the same number of vertices in $X$, the DRMF problem is then not solvable. The reader can check to that $\mathcal{J}^\ast = \{\emptyset\}$ so condition (7) of Theorem 1 is not satisfied.

V. SENSOR LOCATION FOR DRMF

In this section we will examine the consequences of Theorem 2 on the possible sensor location for solving the DRMF problem. The first result concerns the minimal number of sensors to be implemented.

Proposition 1: Consider $\Sigma \Lambda$ a structured system of type (6) with associated graph $G(\Sigma \Lambda)$. The problem of disturbance rejection by measurement feedback is generically solvable only if the number of sensors is greater than or equal to the maximal size of a linking in $G(\Sigma \Lambda)$ from $D$ to $F_I^\ast$.

Proof:
From condition 2, the maximal size of a linking in $G(\Sigma \Lambda)$ from $D$ to $Z$ is the same as the maximal size of a linking in $G(\Sigma \Lambda)$ from $D$ to $Z \cup F_I^\ast$. It follows that the maximal size of a linking in $G(\Sigma \Lambda)$ from $D$ to $Z$ must be greater than or equal to the maximal size of a linking in $G(\Sigma \Lambda)$ from $D$ to $F_I^\ast$. This implies that the number of sensors must be greater than or equal to the maximal size of a linking in $G(\Sigma \Lambda)$ from $D$ to $F_I^\ast$.

Example 4: In the example of Figure 6, $\mathcal{I}^\ast = \{x_1, x_2, x_3, x_4\}$. The frontier is $F_I^\ast = \{x_3, x_4\}$. The maximal size of a linking in $G(\Sigma \Lambda)$ from $D$ to $F_I^\ast$ is two. From Proposition 1 at least two sensors are required, then the DRMF problem is not solvable.

In fact, we can see that the effect of disturbance $d_1$ can propagate independently with the effect of disturbance $d_2$ to the regulated output $y$ by the path $\{d_1, x_1, x_3, x_7, y\}$. But with only one measurement $z$ as in here, we have only the information on the combination effect of $d_1$ and $d_2$. So that, we cannot compensate the effect of $d_1$ via the control input and therefore the DRMF is not solvable.

We will show now that it is useless to measure variables outside $\mathcal{I}^\ast$, this result has been proved in a different way by the authors in [28]. For this purpose let us give first a definition:

Definition 4: For the DRMF problem, we call admissible sensor set a set of sensors $V_Z \subset Z = \{z_1, z_2, \ldots, z_n\}$ such that the DRMF problem remains generically solvable with the output set $V_Z$.

Theorem 3: Consider $\Sigma \Lambda$ a structured system of type (6) with associated graph $G(\Sigma \Lambda)$. Assume that the Disturbance Rejection by Measurement Feedback problem is generically solvable. Let $z_j \in Z$ be such that for any $(x_i, z_j) \in W$, $x_i \in \mathcal{I}^\ast = X \setminus \mathcal{I}^\ast$. Then an admissible sensor set $V_z = \{z_j\}$ with $V_z \setminus \{z_j\}$ is still an admissible sensor set for the DRMF problem.

In other words, a sensor measuring only states out of $\mathcal{I}^\ast$ is of no use for solving the DRMF problem.

Proof:
Let $z_j \in Z$ be such that for any $(x_i, z_j) \in W$, $x_i \in \mathcal{I}^\ast = X \setminus \mathcal{I}^\ast$. Then, if there exists a path from $D$ to $z_j$, this path will intersect $F_I^\ast$, in some $x_k$.

1. If $z_j$ belongs to a maximal size linking from $D$ to $Z$ having a minimal number of vertices in $X$, we can replace the path from $D$ to $z_j$ by the path from $D$ to $x_k$ for having a maximal size linking from $D$ to $Z \cup F_I^\ast$, with a smaller number of vertices in $X$. Then by Theorem 2, the DRMF is not solvable.

2. For the same reason, $z_j$ cannot be contained in a maximal size linking from $D$ to $Z \cup F_I^\ast$, having a minimal number of vertices in $X$.

Then $z_j$ does not play any role in Condition 2 of Theorem 2. When Condition 2 is satisfied, removing $z_j$ will leave this condition satisfied. Consequently, $z_j$ is clearly useless for the DRMF problem.

The interest of Theorem 2 for sensor location is as follows. Compute first $\mathcal{I}^\ast$ (notice that $\mathcal{I}^\ast$ can be computed without the knowledge of the sensor network). Then check
if disturbances satisfy condition 1 of Theorem 2. If it is the case, compute via Proposition 1 the minimal number of required sensors.

From Theorem 3, it turns out that the sensors must measure states inside $\mathcal{T}$. The problem reduces to find a sensor network satisfying the above requirements and such that condition 2 of Theorem 2 is fulfilled.

VI. Concluding Remarks

In this paper we gave a new necessary and sufficient condition for the DRMF problem to be solvable. We proved that the disturbance rejection by measurement feedback problem reduces to an unknown input observer problem on a subset of the state space. This structural result allowed us to study the DRMF problem irrespective of the sensors network and then to determine the minimal number of sensors to be implemented and to show that it is useless for the problem to measure states in some region of the state space. We have shown that the proposed structural analysis is well adapted to explain the phenomena involved in the DRMF problem and is useful for tackling sensor location problems.

References