A Switched State Feedback Law for the Stabilization of LTI Systems

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Abstract—Inspired by prior work in the design of switched feedback controllers for second order systems, we develop a switched state feedback control law for the stabilization of LTI systems of arbitrary dimension. The control law operates by switching between two static gain vectors in such a way that the state trajectory is driven onto a stable \(n-1\) dimensional hyperplane (where \(n\) represents the system dimension). We begin by briefly examining relevant geometric properties of the phase portraits in the case of two-dimensional systems and show how these geometric properties can be expressed as algebraic constraints on the switched vector fields that are applicable to LTI systems of arbitrary dimension. We then describe an explicit procedure for designing stabilizing controllers and illustrate the closed-loop transient performance via two examples.

I. INTRODUCTION

The study of switched linear systems is a problem that has pervaded the systems and control literature for over five decades (see, e.g., [1]—[37]). With roots in relay feedback systems [30] and certain branches of optimal control [5], the primary perspective for the study of switched linear systems has evolved into the following basic question: can we artificially introduce switching into systems design so as to increase performance? While this question is simply stated, the answer is not. Indeed, the idea of introducing switching into systems design has led to a tremendous amount of research over the past decade-and-a-half which attempts to address this issue from a variety of different technical perspectives. A brief survey of the existing literature on the topic leads to two immediate conclusions, one in that a number of difficult problems have been already formulated and solved, and another in that there are still quite a number of open questions which need to be addressed in order to make switched system design a mature engineering field.

While there are many varied technical approaches to designing switched linear systems, a basic theme which is followed by most is encapsulated in the following problem, first described by Liberzon and Morse in [14]. We consider a switching system of the form

\[
\dot{x} = A_{\sigma(t)}x(t)
\]  

(1)

where \(x(t) \in \mathbb{R}^n\) is the continuous state, \(\sigma(t) \in \{1, 2, \ldots, k\}\) is a piecewise constant function of time (referred to as the switching signal), and \(A_i, 1 \leq i \leq k\) are given \(n \times n\) linear transformations. A generic design problem that can be posed for such a setup is the following: construct a switching signal \(\sigma(t)\) which makes the switching system of Eqn. 1 asymptotically stable.

The above problem, and certain generalizations of it, have led to a number of problems/techniques that have been studied in the literature: quadratic stabilizability techniques attempt to find a (piecewise) quadratic Lyapunov function which can be used to produce a switching law that minimizes a piecewise quadratic cost function at every time instant ([13], [14], [28], [29]); techniques have been developed for low order systems (via phase portraits and/or algebraic techniques) which can effectively utilize unstable behavior of linear subsystems to create stable switched interconnections ([1], [11], [15], [22]—[26], [32]—[35]); [10] utilizes the Youla parameterization to devise a method of switching between stabilizing controllers for arbitrary switching signals; extensions from asymptotic stability to L2 gain stability have been considered in [9], [21], [25], [26], [31], [36], [37]; some recent work considers switched system design over polyhedra/polyhedral Lyapunov functions ([8], [17]—[19]).

A. Tradeoffs: “General” Methods vs. Low Order Methods

A qualitative examination of the literature indicates that methods for designing switching controllers typically fall into one of two categories: methods that focus on low order systems (typically no higher than two to three states) which exploit algebraic and geometric properties of the corresponding state-space descriptions to cleverly achieve stability through switching, and methods which apply to general (arbitrary order) state-space descriptions which are typically less reliant on system structure. The latter of these two families of problems appears to have a larger following for a good reason: methods which do not depend on order can be applied to a larger class of problems. Moreover, while low order methods often involve nonlinear/nonconvex constraints on the corresponding decision parameters (see, e.g., [11], [34]), general order methods are often formulated in a manner such that the resulting constraints have a linear structure (e.g., the Youla parameterization-based method of [10] or linear matrix inequalities that result from quadratic stabilizability methods). Hence, the resulting constraints can be solved efficiently in high dimension.

Nevertheless, while general order methods provide obvious benefits, they are not without their detriments. First, general order methods tend to focus on asymptotic behavior without paying explicit attention to transient characteristics. While methods which focus on asymptotic behavior in linear systems design often produce good transient behavior, the same cannot typically be said in switching systems design. For instance, methods which rely upon quadratic stabilization techniques many times produce closed-loop controllers which switch very frequently and which often
produce “jagged” state trajectories (see, e.g., [28], [29] and the examples presented therein). A more concerning issue, however, lies in that general order methods typically depend upon a restricted set of matrices $A_i$ in order to operate properly. In some of the simplest methods, which are restricted to switch between stabilizing controllers, each matrix $A_i$, $1 \leq i \leq k$ is assumed to be Hurwitz (correspondingly Schur for discrete-time problems) [9], [10], [28], [36]. In less restrictive methods, a common assumption (typically used in quadratic stabilizability methods) is that some convex combination of the $A_i$’s is Hurwitz (resp. Schur) [28], i.e., that there exist $\tau_i \geq 0$, $1 \leq i \leq k$ with $\sum \tau_i = 1$ such that $\sum \tau_i A_i$ is a Hurwitz (resp. Schur) matrix. While this second condition is clearly less restrictive than the first, it does exclude certain “good” choices of switching laws, as is demonstrated by the following example.

**Example 1.1:** This example is based on the author’s prior work in [22]—[26]. Consider a double integrator in the controllability canonical form, i.e., a plant of the form

$$
\begin{align*}
\dot{x}_1 &= 0\ x_1 + 0\ x_2 + 1\ u \\
y &= x_1
\end{align*}
$$

under the feedback law $u = v(x)y$, where $v(x)$ is given by

$$
v(x) = \begin{cases} 
-1 & x_1(x_1 + x_2) > 0 \\
1 & x_1(x_1 + x_2) \leq 0 
\end{cases}.
$$

The above control law corresponds to a switched state feedback law that switches between the matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
$$

A sample phase portrait of the resulting closed-loop interconnection is illustrated in Fig. 1.1, where the shaded region denotes where $v(x) = 1$ and the non-shaded region denotes where $v(x) = -1$. We shall not describe the operation of the above control law in detail here ([22]—[26] provide very detailed descriptions of the performance and robustness of a class of control laws which include this example), but we simply point out that the control law operates by switching between two unstable systems ($A_2$ being marginally unstable, $A_1$ being exponentially unstable) in a stable way: all state trajectories are driven onto the stable eigenvector $[1 \ -1]^T$ of $A_2$ in finite time where they evolve forever more. Moreover, we point out that such a control law can not be found using standard quadratic stabilizability methods. For the particular example of the plant of Eqn. 2 and 3 with $v(x)$ given by Eqn. 4, not only are the matrices $A_1$ and $A_2$ both individually unstable ($A_1$ has eigenvalues of $\pm 1$, while $A_2$ has eigenvalues of $\pm j$), but no convex combination of these matrices is Hurwitz stable either. Indeed, any convex combination of $A_1$ and $A_2$ takes the form

$$
\begin{bmatrix} 0 & 1 \\ w & 0 \end{bmatrix}
$$

with $|w| \leq 1$. For $w \geq 0$, the above matrix has eigenvalues $\pm \sqrt{w}$, while for $w < 0$, it has eigenvalues of $\pm j\sqrt{|w|}$.

The previous example serves to illustrate a simple point: while general order methods may cover an overall broader class of systems to which they can be applied, they can “miss” certain forms of control that have good behavior because the corresponding conditions on the matrices $A_i$ are too strict. On the other hand, the major criticism of a control law such as the one depicted in the example is that it is derived only for low order systems, and no immediate extensions to arbitrary systems of general dimension have been apparent—until now.

The goal of this paper is to describe an extension of the control laws of the previous example that can be generalized to LTI systems of arbitrary finite dimension. Deferring exact details of the problem description to the next section, we consider an extension where we switch between two static state feedback controllers to drive the state of the plant onto a stable manifold of dimension $n - 1$ (for a system of dimension $n$).

Due to very limited space, this paper presents results only; a detailed version of this manuscript, including all relevant proofs and additional related work, can be found in [27].

**II. GEOMETRIC CONSIDERATIONS**

The basic problem that we consider in this document is the following: given a reachable continuous-time LTI system $\dot{x} = Ax + Bu$, with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n}$, find $K_1$, $K_2$, $F_1$, $F_2 \in \mathbb{R}^{1 \times n}$ such that the switched system

$$
\dot{x} = \begin{cases} 
(A + BK_1)x & x^T F_1 F_2 x \leq 0 \\
(A + BK_2)x & x^T F_1 F_2 x > 0
\end{cases}
$$

is globally exponentially stable. For notational simplicity, we shall frequently refer to the matrices $A_1 \triangleq A + BK_1$ and $A_2 \triangleq A + BK_2$.

While there are many ways that the vectors $K_1$, $K_2$, $F_1$, and $F_2$ can be selected so as to achieve stability, here, we focus our efforts on designing control laws that, in a sense, mimic the geometric behavior of the second order control laws studied in [22]—[26], i.e., controllers that drive the state $x$ of Eqn. 7 onto a stable hyperplane of dimension $n - 1$.

For second order systems, it is easy to design control laws with particular geometric properties by examining phase portraits. If, however, one desires to adapt such results to higher dimension, these geometric properties must somehow
be translated into (relatively simple) algebraic constraints. In this section, we examine two second order examples to demonstrate the relevant geometric features of the control laws that we wish to design, and we show how to translate these geometric features into algebraic constraints that can be used to develop design algorithms for LTI systems of arbitrary dimension.

To begin, we shall start by examining the example of the last section in more detail. The matrices $A_1$ and $A_2$ are as in Eqn. 5, and corresponding values of $K_i$ and $F_i$, $i = 1, 2$ can be determined by inspection: $K_1 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$, $K_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, and $F_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$. As we shall describe in a moment, for a given selection of $K_1$ and $K_2$, the choice of $F_1$ is unique to within a scaling factor, whereas the choice of $F_2$ is not unique. A more detailed diagram depicting the switching law of Example 1 (for an “arbitrary” choice of $F_2$) is provided in Fig. II. Here $v_2$ represents the stable eigenvector of $A_1$ with eigenvalue $\lambda_2 = -1$, and $N$ represents a vector normal to $v_2$ of clockwise orientation. The vector $v_1$ represents the unstable eigenvector of $A_1$ with eigenvalue $\lambda_1 = 1$, and $w_1$ represents a vector normal to $v_1$ of clockwise orientation. $F_2$ represents a normal vector of clockwise orientation to one side of the switching boundary between $A_1$ and $A_2$.

The first relevant geometric feature of Fig. II that we point out is that, since the objective of the switched feedback law is to drive the state $x$ onto the stable manifold of $A_1$, the stable eigenvector $v_2$ is always a switching boundary. That is, we can always choose $F_1 = N$, where $N$ is some normal vector of appropriate orientation to the stable manifold. The second relevant feature we point out is that the region in the state space where $A_1$ is used cannot contain the unstable eigenvector $v_1$ of $A_1$. If such a situation were to occur, then any initial condition that were to lie along $v_1$ would grow exponentially for all time, and the resulting system would be unstable. Since $A_1$ is used in the region where $x'N'F_2x \leq 0$, the prior constraint can be represented algebraically via

$$v_1'N'F_2v_1 > 0.$$  

A third important feature of the switching law depicted in Fig. II relates to the “unidirectional” motion of the phase portraits. As can be seen from the sample phase portrait in the figure, the state trajectory always rotates in a clockwise direction so that the angle $\theta(t)$ of the state trajectory is always non-decreasing. Such a condition guarantees lack of Zeno behavior along the switching boundaries and, hence, guarantees existence of solutions. Geometrically, this means that the vector fields $A_1x$ and $A_2x$ have to “point” in the same direction across the boundaries $Nx = 0$ and $F_2x = 0$.

To see the algebraic consequences of this along the boundary defined by $F_2$, consider the set of $x$ such that $F_2x = 0$ and $Nz \geq 0$. Geometrically, this set of $x$ corresponds to all points lying along the ray in the second quadrant that are perpendicular to $F_2$. For this set of $x$, if the phase portrait is to rotate clockwise, then $F_2x \geq 0$ for both $x = A_1x$ and $x = A_2x$. Note that, for the set of $x$ given by $F_2x = 0$ and $Nz \leq 0$, the condition is reversed: $F_2x \leq 0$ for both $x = A_1x$ and $x = A_2x$. Both of these sets of conditions can be combined to form a pair of quadratic constraints:

$$x'N'F_2A_1x \geq 0 \quad \forall x : \quad F_2x = 0 \tag{9}$$

$$x'N'F_2A_2x \geq 0 \quad \forall x : \quad F_2x = 0 \tag{10}$$

While perhaps not immediately obvious, satisfaction of the above quadratic constraints also guarantees that the phase portrait will reach the stable eigenvector $v_2$ in finite time, hence automatically ensuring that the switched system is well-behaved along the boundary defined by $Nz = 0$.

While simple, the conditions of Eqn. 8—10 represent the essential geometric properties for switching laws of the form Eqn. 7, and we shall exploit these properties to determine algorithms for selecting vectors $K_1$, $K_2$, and $F_2$ which guarantee exponential stability of the system of Eqn. 7 in the next section. There is, however, one small additional caveat related to the matrix $A_2$ that needs to be explored. In this example, $A_2$ was designed to have complex eigenvalues so as to induce rotation in the corresponding phase portraits. In general, it is not necessary for $A_2$ to have complex eigenvalues in order to induce rotation (and, hence, proper operation) of the switching law of Eqn. 7, as we now illustrate. Consider the problem of switching between matrices $A_1$ and $A_2$ where $A_1$ is as in Eqn. 5 but where $A_2$ is now given by

$$A_2 = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}. \tag{11}$$

The matrix $A_2$ has real eigenvalues $\lambda_2 = 2$ and $\lambda_1 = 3$ as is depicted in the diagram of Fig. 3 (the dotted lines represent the corresponding eigenvectors of $A_2$). A heuristic description of how the state evolves under the switching law depicted in the figure for the depicted initial condition $x(0)$ is as follows. First, since $x(0)$ initially evolves according to the matrix $A_1$, the state begins to move “closer” (in an angular sense) to the unstable eigenvector $v_1$ of $A_1$. In doing so, the state trajectory “passes over” the unstable eigenvectors of the matrix $A_2$. When the state trajectory crosses the boundary $F_2x = 0$, the state begins to evolve according to $x = A_2x$. Since, in the absence of switching, the state trajectory should tend toward the eigenvector with maximal
Two other major necessary conditions which arise are the following: the vector $F_1$ in Eqn. 7 is of the form $F_1' = w_1$, where $w_1$ is the left eigenvector of $A + BK_1$ corresponding to the dominant eigenvalue $\lambda_1$, and $F_2' = w_1 + \mu w_2$ where $w_2$ is a left eigenvector of $A + BK_1$ corresponding to eigenvalue $\lambda_2$ with $\mu \in \mathbb{R}$. It should be noted that not every value of $\mu \in \mathbb{R}$ yields a stabilizing controller, but, rather, a range of stabilizing values of $\mu$ exists that is a function of the eigenvalues $\lambda_1, \lambda_2, \tilde{\lambda}_1$, and $\tilde{\lambda}_2$ as well as the corresponding eigenvectors. This range of values of $\mu$, along with constraints on the corresponding eigenvalues $\lambda_1$ and $\lambda_2$ of $A+BK_2$, are described compactly in the following design algorithm.

A. Basic Design Algorithm

We now go about describing a basic process which can be used to design stabilizing control laws of the form in Eqn. 7. We assume that $K_1$ has been chosen so that the matrix $A + BK_1$ has a real dominant eigenvalue $\lambda_1$, and $n-1$ stable eigenvalues, at least one of which ($\lambda_2$) is real. Design of a stabilizing control law is equivalent to finding vectors $F_1$, $F_2$, and $K_2$. This can be achieved in multiple ways by carrying out the following steps:

Step 1: Pick one real negative eigenvalue of $A + BK_1$ and call this $\lambda_2$. Compute left eigenvectors $w_1$ and $w_2$ of $A + BK_1$ corresponding to eigenvalues $\lambda_1$ and $\lambda_2$. Choose $F_1' = w_1$.

Step 2, option 1: Select a gain vector $K_2$ such that

1) $A + BK_2$ has the same eigenvalues as $A + BK_1$, with the exception of the eigenvalues $\lambda_1$ and $\lambda_2$.

2) The remaining two eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of $A + BK_2$ are not real and are not eigenvalues of $A + BK_1$.

Compute a right eigenvector $\tilde{v}_1$ corresponding to eigenvalue $\lambda_1$, and find some value of $\mu$ such that

$$\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} \leq \frac{w_1' \tilde{v}_1}{w_2' \tilde{v}_2} \mu < 0.$$ 

For such a value of $\mu$, set $F_2' = w_1 + \mu w_2$.

Step 2, option 2: Select a gain vector $K_2$ such that

1) $A + BK_2$ has the same eigenvalues as $A + BK_1$, with the exception of the eigenvalues $\lambda_1$ and $\lambda_2$.

2) The remaining two eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of $A + BK_2$ are not real and are not eigenvalues of $A + BK_1$, and satisfy the condition $(\tilde{\lambda}_1, \tilde{\lambda}_2) \in \Lambda$, where $\Lambda$ is given by

$$\left\{ (\tilde{\lambda}_1, \tilde{\lambda}_2) \in \mathbb{R}^2 : \min\{\tilde{\lambda}_1, \tilde{\lambda}_2\} > \lambda_1 \right\} \cup \left\{ (\tilde{\lambda}_1, \tilde{\lambda}_2) \in \mathbb{R}^2 : \max\{\tilde{\lambda}_1, \tilde{\lambda}_2\} < \lambda_2 \right\}.$$ 

Compute a right eigenvector $\tilde{v}_1$ corresponding to eigenvalue $\lambda_1$, and find some value of $\mu$ such that

$$\frac{\lambda_1 - \lambda_1}{\lambda_1 - \lambda_2} \leq \frac{w_1' \tilde{v}_1}{w_2' \tilde{v}_2} \mu < \min \left\{ -\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_1}, -\frac{\lambda_2 - \lambda_2}{\lambda_2 - \lambda_1} \right\}.$$ 

III. NECESSARY CONDITIONS AND DESIGN ALGORITHM

The sufficient conditions presented in the last section imply a number of necessary conditions for the sufficient conditions, themselves, to hold. We briefly summarize some of the major necessary conditions that arise, and then present a design algorithm that results from these necessary conditions which can be used to design stabilizing switching controllers of the form Eqn. 7.

The first major necessary condition which arises from the conditions of the last section is that $A + BK_1$ and $A + BK_2$ must have $n-2$ eigenvalues in common. While this is not at all obvious (see [27] for a proof), such a result allows us to characterize simple stability conditions in terms of the eigenvalues that are different between these two matrices. In particular, we denote $\lambda_1$ and $\lambda_2$ as the eigenvalues of $A + BK_1$ that are not eigenvalues of $A + BK_2$. The eigenvalue $\lambda_1$ is assumed to be the real, dominant eigenvalue of $A + BK_1$, and $\lambda_2$ is a negative real eigenvalue (i.e., a stable eigenvalue whose corresponding eigenspace lies on the stable hyperplane associated with $A + BK_1$). We denote by $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ the eigenvalues of $A + BK_2$ which are not eigenvalues of $A + BK_1$.
For such a value of \( \mu \), set \( F'_2 = w_1 + \mu w_2 \).

Step 2, option 3: Select a gain vector \( K_2 \) such that

1) \( A + BK_2 \) has the same eigenvalues as \( A + BK_1 \), with the exception of the eigenvalues \( \lambda_1 \) and \( \lambda_2 \).

2) The remaining two eigenvalues \( \tilde{\lambda}_1 \) and \( \tilde{\lambda}_2 \) of \( A + BK_2 \) are real and equal, are not eigenvalues of \( A + BK_1 \), and satisfy the condition

\[
\tilde{\lambda}_1 \in \Lambda, \quad \Lambda = \{ \lambda : \lambda > \lambda_1 \cup \lambda < \lambda_2 \}.
\]

Compute a generalized right eigenvector \( \tilde{v}_2 \) corresponding to eigenvalue \( \lambda_1 \), and find some value of \( \mu \) such that

\[
\frac{\lambda_1 - \lambda_1 w'_1 \tilde{v}_2}{\lambda_1 - \lambda_2 w'_2 \tilde{v}_2} < \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_1}.
\]

For such a value of \( \mu \), set \( F'_2 = w_1 + \mu w_2 \).

IV. EXAMPLES

In this section, we provide an example to illustrate the design techniques described in the previous sections. Here, we consider a sixth-order integrator in the controllability canonical form (i.e., where the matrix \( A \) is a companion matrix with companion entries on the bottom row, and where the \( B \) vector is of the form \( B = [0 \ 0 \cdots 0 \ 1] \)) where the gain vector \( K_1 \) is selected such that the matrix \( A + BK_1 \) is given by

\[
A + BK_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
54 & 81 & -9 & -70 & -44 & -11
\end{bmatrix}.
\] (14)

The characteristic polynomial of \( A + BK_1 \) is given by

\[
(s + 3)^3(s + 2)(s + 1)(s - 1).
\] (15)

Our objective is to find a switched feedback controller of the form Eqn. 7 such that the state trajectory is driven onto the stable invariant subspace of the matrix \( A + BK_1 \), spanned by the (generalized) eigenvectors corresponding to the eigenvalues \(-3, -2\) and \(-1\). To do so requires three objects: selection of a gain vector \( K_2 \), and computation of the switching boundary vectors \( F_1 \) and \( F_2 \). We shall actually design two separate switching controllers—one where the matrix \( A + BK_2 \) has complex eigenvalues, and one where it has purely real eigenvalues—to illustrate the design procedure for the first two options listed in Step 6.

1) Controller 1: Complex Eigenvalues: Recall that the matrices \( A + BK_1 \) and \( A + BK_2 \) must have \( n - 2 \) eigenvalues in common. Therefore, only two eigenvalues are allowed to differ between the characteristic polynomials of \( A + BK_1 \) and \( A + BK_2 \). For this example, we (arbitrarily) will “move” the eigenvalues located at \(-1\) and \(-1\) of the matrix \( A + BK_1 \) to eigenvalues of \( \pm j \) for the matrix \( A + BK_2 \), so that the matrix \( A + BK_2 \) has characteristic polynomial

\[
(s + 3)^3(s + 2)(s^2 + 1).
\] (16)

The gain vector \( K_2 \) corresponding to the above characteristic polynomial is given by

\[
K_2 = \begin{bmatrix}
-54 & -81 & -99 & -92 & -46 & -11
\end{bmatrix}.
\] (17)

The switching boundary vector \( F_1 \) is a normal vector to the stable invariant subspace of the matrix \( A + BK_1 \):

\[
F_1 = \begin{bmatrix}
54 & 135 & 126 & 56 & 12 & 1
\end{bmatrix}.
\] (18)

Recall, now, that \( F_2 = w'_1 + \mu w'_2 \), where \( w'_1 = F_1 \) is a left eigenvector of \( A + BK_1 \) corresponding to the eigenvalue 1, and \( w_2 \) is some other left eigenvector of \( A + BK_1 \). Specifically, \( w_2 \) corresponds to the left eigenvector with the eigenvalue that is “removed” from \( A + BK_1 \) to form the characteristic polynomial of \( A + BK_2 \) which, in this case, is a left eigenvector corresponding to the eigenvalue \(-1\):

\[
w_2 = \begin{bmatrix}
-54 & -27 & 36 & 34 & 10 & 1
\end{bmatrix}.
\] (19)

Setting \( F'_2 = w_1 + \mu w_2 \) and carrying out the computation for step 2, option 1 of the design procedure for computing the stabilizing range of \( \mu \), we find that any value of \( \mu < 0 \) achieves a stable closed-loop interconnection. If we choose \( \mu = -1 \), the switching boundary vector \( F_2 \) is given via

\[
F_2 = \begin{bmatrix}
108 & 162 & 90 & 22 & 2 & 0
\end{bmatrix}.
\] (20)

Fig. IV.1 shows a sample phase portrait for the resulting closed-loop system for the initial condition \( x(0) = [1 \ 0 \ 0 \ 0 \ 0 \ 0] \). The trajectory was computed in MATLAB by approximating the continuous-time system via
the sampled-data system
\[ x((k + 1)T) = \begin{cases} \tilde{A}_1 x(kT) & x'F_1' F_2 x \leq 0 \\ \tilde{A}_2 x(kT) & x'F_1' F_2 x > 0 \end{cases} \] (21)
where \(\tilde{A}_i = \exp((A + BK_i)T)\), with \(T = 0.001\). The waveforms \(x_1(t)\) through \(x_6(t)\) represent the individual components of the state vector \(x(t)\), and the waveform denoted “Switching signal” is given by
\[ \sigma(t) = \text{sgn}(x(t)' F_2 F_1 x(t)). \] (22)

The switching signal \(\sigma(t)\) indicates which of the matrices \(A_1\) or \(A_2\) is being used at any given time; when \(\sigma(t) = 1\), the state evolves according to \(\dot{x} = (A + BK_2)x\), while when \(\sigma(t) = -1\), the state evolves according to \(\dot{x} = (A + BK_1)x\).

Several comments are in order. First, note that \(\sigma(t)\) switches between the values +1 and -1 multiple times. This contradicts the fact that the exact solution to the nonlinear differential equation should have \(\sigma(t) = -1\) for all \(t \geq t_0\) for some \(t_0 > 0\) (corresponding to the fact that, once the state is driven onto the stable hyperplane, it never leaves).

Note in general, however, for the sampled-data system of Eqn. 21, the gain vector will not switch from \(K_2\) to \(K_1\) at the exact point in time that the state trajectory crosses the stable hyperplane but will, rather, switch some small amount of time after this has happened. Therefore, as a general rule, \(\sigma(t)\) will vary between +1 and -1.

Nevertheless, as the plots in the figure seem to suggest, the state trajectory remains well-behaved despite this issue. In fact, one can formally prove that the switching control laws derived here are globally exponentially stable even in the presence of sufficiently small time delays. As heuristic verification of stability for this particular example, we uniformly grid the unit box \(|x|_\infty = 1\) with grid size \(\Delta = 0.2\) and simulate the closed-loop differential equation for every initial condition \(x(0)\) on this grid. For each resulting state trajectory, we compute the value \(|x(20)|_2/|x(0)|_2\) and find that the maximum value of this quantity over all initial conditions on the grid is 6.88 x 10^-4 (corresponding to the initial condition \(x(0) = [-0.4 -1 -1 -0.4 0 0.0 0]\).

2) Controller 2: Real, Non-repeated Eigenvalues: For this example, we choose to move the eigenvalue 1 and one of the eigenvalues located at -3 from the matrix \(A + BK_1\) to eigenvalues 2 and 3 for the matrix \(A + BK_2\), corresponding to a characteristic polynomial
\[ (s + 3)^2(s + 2)(s + 1)(s - 2)(s - 3). \] (23)

The gain vector \(K_2\) which achieves this characteristic polynomial is
\[ K_2 = [ -108 -144 3 52 10 -4 ]. \] (24)

Since the value of \(F_1\) depends only upon the matrix \(A + BK_1\), \(F_1\) is the same as for the previous controller and is given by Eqn. 18. To compute a choice of \(F_2\), we first compute the left eigenvector \(w_2\) corresponding to the eigenvalue -3:
\[ w_2 = [ -18 -21 10 20 8 1 ]. \] (25)

We can choose \(F_2 = w_1' + \mu w_2'\) for any value of \(\mu\) which satisfies the condition in step 2, option 2 of the last section. Using the values \(\lambda_1 = 1, \lambda_2 = -3, \lambda_1 = 2\) and \(\lambda_2 = 3\), and carrying out the computation for some appropriate choice of \(\hat{v}_1\), we find that this condition reduces to \(\mu < -5\). Hence, by choosing \(\mu = -6\), we arrive at
\[ F_2 = [ 162 261 66 -64 -36 -5 ]. \] (26)

A sample phase portrait for the initial condition \(x(0) = [ 1 0 0 0 0 0 0 ]\) is shown in Fig. IV-2.

V. CONCLUSION

We have described a new switched state feedback control architecture based off of our previous work in designing switched output feedback control laws for second order systems. After describing geometric properties that lead to a set of algebraic conditions that are sufficient for stabilizability, we presented a set of necessary conditions, along with a basic design algorithm. Finally, we presented an example where we designed two different switching controllers of the form Eqn. 7.

The work presented here is hardly an end. First, the problem of examining switched output feedback controllers is an important problem which must be investigated since the full state vector is not always immediately available and must be estimated. Also, we hope to extend the results related to finite L2 gain stability for the second order counterpart of this problem [25] to problems of general dimension, so as to formally quantify the effects of disturbances on performance for the class of switched feedback systems considered here. The
investigation of applications where these switched feedback control laws can have significant impact in performance over linear control design is an important area of future research, as well.

As a related aside, in addition to proving all of the result presented here in a formal manner, [27] also examines an application of the control laws presented here for minimizing Lyapunov exponents. The application not only demonstrates practical use for the switching controllers presented here, but also provides a class of problems where the selection of the matrix $A + BK_1$ has a “natural” choice (as opposed to the arbitrary selections that we made in the examples here).

As a final remark, it has been stated by some that the necessary conditions on the eigenvalues of the matrices $A + BK_1$ and $A + BK_2$ appear to be very restrictive. In the opinion of the author, however, these conditions should be viewed in a very positive light, for several reasons. First, without some sort of simply-characterized set of necessary conditions, the design problem as posed at the begin of Section II would likely be intractable. Second, existence of gain vectors $K_1$ and $K_2$ which satisfy the assumptions on eigenvalues placement is trivially guaranteed by the assumed reachability of the pair $(A, B)$; by virtue of the standard pole placement problem of linear systems theory which guarantees that the spectrum of a reachable linear system can be placed arbitrarily, vectors $K_1$ and $K_2$ which satisfy the necessary conditions always exist, and selection of these vectors for given spectra is, again, standard. Finally, from a qualitative perspective, the fact that global exponential stability for a nonlinear system can be characterized via a condition on (only two) eigenvalues is, itself, a rare result which is interesting both theoretically and from a design perspective (since the characterization of stabilizing controllers in this framework has a simply parameterized form as indicated in the paper).

REFERENCES