Optimal Semistable Control for Continuous-Time Coupled Systems
Qing Hui

Abstract—In this paper, we develop distributed $H_2$ semistability theory for linear dynamical systems. Using this theory, we design distributed $H_2$ optimal semistable controllers for linear dynamical systems. Unlike the standard $H_2$ optimal control problem, a complicating feature of the distributed $H_2$ optimal semistable stabilization problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. Necessary and sufficient conditions for existence of solutions to the semistable Lyapunov equation are derived and a design framework for distributed optimal controllers is presented.

I. INTRODUCTION

Dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles, autonomous underwater vehicles, distributed sensor networks, air and ground transportation systems, swarms of air and space vehicle formations, and congestion control in communication networks, to cite but a few examples. A unique feature of the closed-loop dynamics under any control algorithm in dynamical networks is the existence of a continuum of equilibria representing a desired state of convergence. Under such dynamics, the desired limiting state is not determined completely by the system dynamics, but depends on the initial system state as well [1], [2].

The dependence of the limiting state on the initial state is not limited to dynamical network systems, it is also seen in the dynamics of compartmental systems [3] which arise in chemical kinetics [4], and biomedical [5], environmental [6], economic [7], power [8], and thermodynamic systems [9]. In all such systems possessing a continuum of equilibria, semistability, and not asymptotic stability, is the relevant notion of stability. Semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. Semistability then implies Lyapunov stability, and is implied by asymptotic stability.

Semistability was first introduced in [10] for linear systems, and applied to matrix second-order systems in [11]. Nonlinear extensions were considered in [12] and [13], which give several stability results for systems having a continuum of equilibria based on nontangency and arc length of trajectories, respectively. References [1], [2] build on the results of [12], [13] and give semistable stabilization results for nonlinear network dynamical systems. Distributed optimal semistable stabilization, however, has never been considered in the literature.

In this paper, we develop a distributed $H_2$ optimal semistable control framework for linear dynamical systems. Unlike the standard $H_2$ optimal control problem, a complicating feature of the distributed $H_2$ optimal semistable stabilization problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. We present necessary and sufficient conditions for existence of solutions to the semistable Lyapunov equation. Based on these conditions, we convert the distributed $H_2$ optimal control problem into a constrained convex optimization problem. Then this convex optimization problem can be used to solve the original distributed $H_2$ optimal control problem.

II. PROBLEM FORMULATION

The notion we use in this paper is fairly standard. Specifically, $\mathbb{R}$ (resp., $\mathbb{C}$) denotes the set of real (resp., complex) numbers, $\mathbb{R}^n$ (resp., $\mathbb{C}^n$) denotes the set of $n \times 1$ real (resp., complex) column vectors, $\mathbb{R}^{n \times m}$ (resp., $\mathbb{C}^{n \times m}$) denotes the set of $n \times m$ real (resp., complex) matrices, $(\cdot)^T$ denotes transpose, $(\cdot)^*$ denotes complex conjugate transpose, $(\cdot)^\dagger$ denotes the group generalized inverse, and $I_n$ or $I$ denotes the $n \times n$ identity matrix. Furthermore, we write $\| \cdot \|$ for the Euclidean vector norm, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix $A$, $\text{spec}(A)$ for the spectrum of the square matrix $A$, $\text{tr}(\cdot)$ for the trace operator, $\mathbb{E}$ for the expectation operator, and $A \geq 0$ (resp., $A > 0$) to denote the fact that the Hermitian matrix $A$ is nonnegative (resp., positive) definite. Finally, we write $B_e(x)$, $x \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball with radius $\varepsilon$ and center $x$, $\otimes$ for the Kronecker product, $\oplus$ for the Kronecker sum, and $\text{vec}(\cdot)$ for the column stacking operator.

In this paper, we consider $q$ continuous-time linear systems $\dot{G}_i$, $i = 1, 2, \ldots, q$, given by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0,$$

where $x_i(t) \in \mathbb{R}^n$ denotes states of the $i$th system, $u_i(t) \in \mathbb{R}^m$ denotes inputs of the $i$th system, $A_i \in \mathbb{R}^{n \times n}$, and $B_i \in \mathbb{R}^{n \times m}$. Note that this model includes both homogeneous systems [14] and heterogeneous systems [15]. We can rewrite the overall system as the compact form

$$\dot{x} = Ax + Bu,$$

where $x \triangleq [x_1^T, \ldots, x_q^T]^T \in \mathbb{R}^{nq}$, $u = [u_1^T, \ldots, u_q^T]^T \in \mathbb{R}^{mq}$, $A \triangleq \text{block-diag} [A_1, \ldots, A_q]$, and $B \triangleq \text{block-diag} [B_1, \ldots, B_q]$.

This work was supported in part by a research grant from Texas Tech University.

Q. Hui is with the Department of Mechanical Engineering, Texas Tech University, Lubbock, TX 79409-1021, USA (qing.hui@ttu.edu).
In this section, we consider the distributed controller design $u_i = K_i x$ so that the closed-loop system is semistable, that is, 
$$
\lim_{t \to \infty} x_i(t) = \alpha_i, \quad i = 1, 2, \ldots, n, 
$$
where $\alpha_i$ is a constant and determined by the initial condition, the cost function

$$
J(u, x_0) = \int_0^\infty \left( (x(s) - \alpha)^T Q (x(s) - \alpha) + (u(s) - \beta)^T R (u(s) - \beta) \right) ds
$$

is minimized, where $Q = Q^T \geq 0$, $R = R^T > 0$, $\alpha = \lim_{t \to \infty} x(t)$, and $\beta = \lim_{t \to \infty} u(t)$. Here, we assume that the communication topology is not determined yet, that is, the controller gain $K_i$ does not have a fixed structure. We call this problem P1. The first question on this optimal control problem is about well-posedness. Is this optimal control problem a well-defined problem? Under what conditions is the optimal control problem well defined? These questions are important since some optimal control problems for dynamically coupled systems are indeed ill-posed under certain circumstances as the following example shows.

**Example 2.1:** Consider the linear system given by

$$
\begin{align*}
\dot{x}_1 &= u_1, \\
\dot{x}_2 &= u_2.
\end{align*}
$$

Assume that the distributed controllers are given by

$$
\begin{align*}
u_1 &= k_{11} x_1 + k_{12} x_2, \\
u_2 &= k_{21} x_1 + k_{22} x_2,
\end{align*}
$$

where $k_{ij} \in \mathbb{R}$, $i, j = 1, 2$. The closed-loop system is given by

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}
= \begin{bmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}.
$$

For simplicity, we assume that $\lim_{t \to \infty} x_1(t) = \lim_{t \to \infty} x_2(t)$. Then we must have

$$
\begin{align*}
k_{11} + k_{12} &= 0, \\
k_{21} + k_{22} &= 0,
\text{and} \\
k_{12} + k_{21} &> 0.
\end{align*}
$$

Hence, 

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}
= \begin{bmatrix}
-k_{12} & k_{12} \\
k_{21} & -k_{21}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
$$

and

$$k_{12} + k_{21} > 0.
$$

Then we have

$$
x(t) = \begin{bmatrix}
\frac{k_{21} + k_{12} e^{-tk_{21}}}{k_{21} + k_{12} + k_{21}} & \frac{k_{12} - k_{12} e^{-tk_{12}}}{k_{21} + k_{12} + k_{21}} \\
\frac{k_{12} - k_{21} e^{-tk_{12}}}{k_{21} + k_{12} + k_{21}} & \frac{k_{12} + k_{12} e^{-tk_{12}}}{k_{21} + k_{12} + k_{21}}
\end{bmatrix} x(0)
$$

and

$$
\lim_{t \to \infty} x(t) = \begin{bmatrix}
\frac{k_{21}}{k_{21} + k_{12} + k_{21}} & \frac{k_{12}}{k_{21} + k_{12} + k_{21}} \\
\frac{k_{12}}{k_{21} + k_{12} + k_{21}} & \frac{k_{21}}{k_{21} + k_{12} + k_{21}}
\end{bmatrix} x(0).
$$

Next, the cost function is chosen to be

$$
J(u, x_0) = \int_0^\infty [(u_1(s) - u_{e1})^2 + (u_2(s) - u_{e2})^2] ds,
$$

where $u_{ei} = \lim_{t \to \infty} u_i(t)$, $i = 1, 2$. Substituting (15) and (16) into (17) yields

$$
J(u, x_0) = x^T(0) \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} x(0),
$$

where

$$a_{11} = \frac{k_{12}^2 + k_{21}^2}{2(k_{12} + k_{21})}, \quad a_{12} = -\frac{k_{21}^2}{k_{12} + k_{21}}, \quad a_{21} = a_{12}, \quad a_{22} = a_{11}.
$$

Thus,

$$J(u, x_0) = a_{11} (x_1(0) - x_2(0))^2.
$$

Note that $a_{11} > 0$ and $J(u, x_0)$ does not have a minimum value for any $k_{12}, k_{21} \in \mathbb{R}$ satisfying $k_{12} + k_{21} > 0$. Hence, in this case, the optimal control problem P1 is not well-defined.

Next, if we change the cost function (17), then the optimal control problem P1 may be well-defined.

**Example 2.2:** Consider the linear system given by (5) and (6) with distributed control input (7) and (8). For simplicity, we assume that $\lim_{t \to \infty} x_1(t) = \lim_{t \to \infty} x_2(t)$. If we choose the cost function given by

$$
J(u, x_0) = \int_0^\infty [(x_1(s) - x_{e1})^2 + (x_2(s) - x_{e2})^2 + (u_1(s) - u_{e1})^2 + (u_2(s) - u_{e2})^2] ds,
$$

where $x_{ei} = \lim_{t \to \infty} x_i(t)$, $i = 1, 2$, then $J(u, x_0)$ is given by (20) with

$$a_{11} = \frac{k_{12}^2 + k_{21}^2}{2(k_{12} + k_{21})^3} + \frac{k_{21}^2}{2(k_{12} + k_{21})}. $$

Note that $a_{11}$ can be rewritten as

$$a_{11} = \frac{k_{12}^2 + k_{21}^2}{2(k_{12} + k_{21})^2} \left[ \frac{1}{k_{12} + k_{21}} + (k_{12} + k_{21}) \right].
$$

Since $(k_{12}^2 + k_{21}^2)/(2(k_{12} + k_{21})^2) \geq 1/4$ and $1/(k_{12} + k_{21}) + (k_{12} + k_{21}) \geq 2$, it follows that $a_{11} \geq 1/2$, where the equality holds if and only if $k_{12} = k_{21} = 1/2$. Hence, $\min_u J(u, x_0) = (1/2)(x_1(0) - x_2(0))^2$ and the optimal controller gain $K$ is unique and given by

$$K = \begin{bmatrix}
\frac{1}{2} & -1 & 1 \cr -1 & 1 & -1
\end{bmatrix}.
$$

Unlike the standard $H_2$ problem, the optimal control problem P1 does not necessarily have a unique solution. We have the following example to show this point.
Example 2.3: Consider the linear system given by
\[
\begin{align*}
\dot{x}_1 &= -x_1 + u_1, \\
\dot{x}_2 &= -x_2 + u_2.
\end{align*}
\] (25) (26)
Assume that the distributed controllers are given by (7) and (8). The closed-loop system is given by
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
k_{11} - 1 & k_{12} \\
k_{21} & k_{22} - 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}.
\] (27)
For simplicity, we assume that \( \lim_{t \to \infty} x_1(t) = \lim_{t \to \infty} x_2(t) \). Then we must have
\[
\begin{align*}
k_{11} + k_{12} &= 1, \\
k_{21} + k_{22} &= 1, \\
k_{11} + k_{22} &< 2.
\end{align*}
\] (28) (29) (30)
Hence, (13) and (14) hold. Furthermore, we have (15) and (16). Next, let the cost function be
\[
J(u, x_0) = \int_0^\infty [(u_1(s) - u_{e_1})^2 + (u_2(s) - u_{e_2})^2] ds.
\] (31)
Substituting (15) and (16) into (31) yields (18) and (20), and
\[
a_{11} = \frac{k_{12}^2(k_{11}^2 + k_{21}^2)}{2(k_{12} + k_{21})^3} - \frac{k_{12}k_{11}k_{21} + k_{22}k_{21}}{(k_{11} + k_{21})^3} + \frac{k_{21}^2(k_{21}^2 + k_{22}^2)}{2(k_{12} + k_{21})^3}. 
\] (32)
Note that \( a_{11} \) can be rewritten as
\[
a_{11} = \frac{(k_{12}k_{11} - k_{12}k_{21})^2 + (k_{21}k_{12} - k_{21}k_{22})^2}{2(k_{12} + k_{21})^3}.
\] (33)
Hence, \( a_{11} \geq 0 \) and \( a_{11} = 0 \) if and only if \( k_{12}k_{11} = k_{12}k_{21} \) and \( k_{21}k_{12} = k_{21}k_{22} \), which implies that \( \min_u J(u, x_0) = 0 \).

Example 2.4: Consider Example 2.3. Let \( k_{11} = k_{12} = k_{21} = k_{22} = 1/2 \). Then
\[
K = \frac{1}{2} \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}, \quad A + BK = \frac{1}{2} \begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix},
\] (37)
Clearly the pair \((A + BK, Q + K^T R K)\) is not semiobservable in the sense of [19]. It is easy to verify that there does not exist \( P \geq 0 \) such that \((A + BK)^T P + P(A + BK) + Q + K^T R K = 0\), and hence, we cannot use the method developed in [19] to find the optimal controller \( u = K x \).

Hence, it is worthwhile developing a technique to solve this issue. In this paper, we will investigate this problem by developing distributed \( H_2 \) semistable control theory.

Now we go back to our general optimal control problem P1. The closed-loop system can be written as
\[
\dot{x}(t) = \tilde{A} x(t), \quad x(0) = x_0, \quad t \geq 0,
\] (38)
where \( \tilde{A} \triangleq A + BK \) and \( K = [K_1^T, \ldots, K_q^T]^T \in \mathbb{R}^{m_q \times n_q} \).

III. DISTRIBUTED \( H_2 \) SEMISTABILITY THEORY

Suppose the closed-loop system (38) is semistable. Then it follows from [21, p. 438] that \( \lim_{t \to \infty} x(t) = x_e \), where \( x_e = (I_{n_q} - \tilde{A} \tilde{A}^\#) x(0) \).

Suppose the closed-loop system (38) is semistable. Then it follows from [21, p. 438] that \( \lim_{t \to \infty} x(t) = x_e \), where \( x_e = (I_{n_q} - \tilde{A} \tilde{A}^\#) x(0) \).

Suppose the closed-loop system (38) is semistable. Then it follows from [21, p. 438] that \( \lim_{t \to \infty} x(t) = x_e \), where \( x_e = (I_{n_q} - \tilde{A} \tilde{A}^\#) x(0) \).

Suppose the closed-loop system (38) is semistable. Then it follows from [21, p. 438] that \( \lim_{t \to \infty} x(t) = x_e \), where \( x_e = (I_{n_q} - \tilde{A} \tilde{A}^\#) x(0) \).
Lemma 3.3: If (38) is semistable, then $P$ given by (40) satisfies
\[ \tilde{A}^T (\tilde{A}^TP + P\tilde{A} + \tilde{R}) \tilde{A} = 0. \] (42)
Equation (42) is a Lyapunov equation for semistability of (38). We call (42) the semistable Lyapunov equation. Note that if (38) is asymptotically stable, then the semistable Lyapunov equation (42) reduces to the standard Lyapunov equation $\tilde{A}^TP + P\tilde{A} + \tilde{R} = 0$. The following result is immediate.

Lemma 3.4: If (38) is semistable, then for every nonnegative-definite matrix $\tilde{R} = \tilde{R}^T \in \mathbb{R}^{nq \times nq}$, there exists a nonnegative-definite matrix $P = P^T \in \mathbb{R}^{nq \times nq}$ such that (42) holds.

Definition 3.1: Let $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times n}$. The pair $(A, C)$ is weakly semiobservable if
\[ \bigcap_{k=1}^{n} \mathcal{N}(CA^k) = \mathcal{N}(A). \] (43)
Recall from Definition 2.3 of [19] that the pair $(A, C)$ is semiobservable if and only if
\[ \bigcap_{k=1}^{n} \mathcal{N}(CA^k^{-1}) = \mathcal{N}(A). \] (44)
It is easy to see from the definitions of semiobservability and weak semiobservability that $(A, C)$ is weakly semiobservable if and only if $(A, CA)$ is semiobservable. Motivated by Lemma 4.1 of [22], we have the following lemma.

Lemma 3.5: If there exist matrices $P \geq 0$ and $\tilde{R} \geq 0$ in $\mathbb{R}^{nq \times nq}$ such that (42) holds and the pair $(A, \tilde{R})$ is weakly semiobservable, then i) $\mathcal{N}(PA) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}(\tilde{R}A)$ and ii) $\mathcal{N}(\tilde{A}) \cap \mathcal{R}(\tilde{A}) = \{0\}$.

Proof: Note that $(\tilde{A}, \tilde{R})$ is weakly semiobservable if and only if
\[ \bigcap_{k=1}^{nq} \mathcal{N}(\tilde{R}A^k) = \mathcal{N}(\tilde{A}). \] (45)
Since $\tilde{A}^T \tilde{R}\tilde{A}A^{-k} = \tilde{A}^T \tilde{R}\tilde{A}^k$ and $\mathcal{N}(\tilde{A}^T \tilde{R}\tilde{A}) = \mathcal{N}(\tilde{R}\tilde{A}A^{-k}) = \mathcal{N}(\tilde{R}\tilde{A}^k)$, it follows that
\[ \bigcap_{k=1}^{nq} \mathcal{N}(\tilde{A}^T \tilde{R}A^k) = \bigcap_{k=1}^{nq} \mathcal{N}(\tilde{R}A^k) = \mathcal{N}(\tilde{A}). \] (46)
Let $\hat{P} \triangleq \tilde{A}^T P \tilde{A}$ and $\hat{Q} \triangleq \tilde{A}^T \tilde{R} \tilde{A}$. Then (42) becomes
\[ \tilde{A}^T \hat{P} + \hat{P} \tilde{A} + \hat{Q} = 0 \] (47)
and (46) becomes
\[ \bigcap_{k=1}^{nq} \mathcal{N}(\hat{Q} \tilde{A}^k) = \mathcal{N}(\tilde{A}). \] (48)
Now the result directly follows from Lemma 4.1 of [22] by noting that $\mathcal{N}(\hat{P}) = \mathcal{N}(PA)$ and $\mathcal{N}(\hat{Q}) = \mathcal{N}(\tilde{R}A)$.

The part of the converse result for Lemma 3.4 can be stated as follows.

Lemma 3.6: Assume that for every nonnegative-definite matrix $\tilde{R} = \tilde{R}^T \in \mathbb{R}^{nq \times nq}$ satisfying $(\tilde{A}, \tilde{R})$ is weakly semiobservable, there exists a nonnegative-definite matrix $P = P^T \in \mathbb{R}^{nq \times nq}$ such that (42) holds. Then (38) is semistable.

The following lemmas are needed for the main result of this section.

Lemma 3.7: Consider the linear dynamical system (38). Then (38) is semistable if and only if for every weakly semiobservable pair $(\tilde{A}, \tilde{R})$ with nonnegative-definite $\tilde{R}$, there exists an $nq \times nq$ matrix $P \geq 0$ such that (42) holds.

Lemma 3.8: Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times n}$. If $A$ and $B$ are semistable, then $A \oplus B$ is semistable.

Lemma 3.9: Let $x \in \mathbb{R}^{n}$ and $A \in \mathbb{R}^{n \times n}$, and assume $A$ is semistable. Then $\int_{0}^{\infty} e^{\tilde{A}x} dt$ exists if and only if $x \in \mathcal{R}(A)$. In this case, $\int_{0}^{\infty} e^{\tilde{A}x} dt = -A^#x$.

The next result gives a necessary and sufficient condition for guaranteeing semistability of (38) by means of the semistable Lyapunov equation.

Theorem 3.1: Consider the linear dynamical system (38). Then (38) is semistable if and only if for every weakly semiobservable pair $(\tilde{A}, \tilde{R})$ with nonnegative-definite $\tilde{R}$, there exists an $nq \times nq$ matrix $\hat{P} > 0$ such that
\[ \tilde{A}^T \hat{P} + \hat{P} \tilde{A} + \tilde{A}^T \tilde{R} \tilde{A} = 0. \] (49)
Such a $\hat{P}$ is not unique. Furthermore, if $(\tilde{A}, \tilde{R})$ is weakly semiobservable and $\tilde{P}$ satisfies (49), then
\[ \hat{P} = \int_{0}^{\infty} e^{\tilde{A}^T t} \tilde{R} e^{\tilde{A} t} dt + \alpha x x^T, \] (50)
Proof: Suppose $(\tilde{A}, \tilde{R})$ is weakly semiobservable. Then it follows from Lemma 3.7 that there exists an $nq \times nq$ matrix $P \geq 0$ such that (42) holds. Since, by Lemma 3.5, $\mathcal{N}(\tilde{A}) \cap \mathcal{R}(\tilde{A}) = \{0\}$, it follows from [7, p. 119] that $A$ is group invertible. Thus, let $L \triangleq I_{nq} - \tilde{A} \tilde{A}^#$ and note that $L^2 = L$. Hence, $L$ is the unique $nq \times nq$ matrix satisfying $\mathcal{N}(L) = \mathcal{R}(\tilde{A})$, $\mathcal{R}(L) = \mathcal{N}(\tilde{A})$, and $Lx = x$ for all $x \in \mathcal{N}(\tilde{A})$. Now, define
\[ \hat{P} \triangleq \tilde{A}^T P A + L^T L. \] (51)
Next, we show that $\hat{P}$ is positive definite. Consider the function $V(x) = x^T P x$, $x \in \mathbb{R}^{nq}$. If $V(x) = 0$ for some $x \in \mathbb{R}^{nq}$, then $P \tilde{A} x = 0$ and $L x = 0$. It follows from i) of Lemma 3.5 that $x \in \mathcal{N}(\tilde{A})$, and $L x = 0$ implies that $x \in \mathcal{R}(\tilde{A})$. Now, it follows from ii) of Lemma 3.5 that $x = 0$. Hence, $\hat{P}$ is positive definite. Next, since $L \tilde{A} = \tilde{A} - \tilde{A} \tilde{A}^# \tilde{A} = 0$, it follows that
\[ \tilde{A}^T \hat{P} + \hat{P} \tilde{A} + \tilde{A}^T \tilde{R} \tilde{A} = \tilde{A}^T \tilde{A}^T P A + \tilde{A}^T P \tilde{A} + \tilde{A}^T \tilde{R} \tilde{A}
+ \tilde{A}^T \tilde{R} \tilde{A} + \tilde{A}^T L^T L + L^T L \tilde{A}
= (L \tilde{A})^T L + L^T L \tilde{A} = 0. \] (52)
Conversely, if there exists $\hat{P} > 0$ such that (49) holds, consider the function $U(x) = x^T \hat{P} x$, $x \in \mathbb{R}^{nq}$. Then $U(x) = -x^T \tilde{A}^T \tilde{R} \tilde{A} x \leq 0$ and $U^{-1}(0) = \mathcal{N}(\tilde{R} \tilde{A})$. To obtain the largest invariant set $\mathcal{M}$ contained in $\mathcal{N}(\tilde{R} \tilde{A})$, consider a solution $x(t)$ of (38) such that $\tilde{R} \tilde{A} x(t) = 0$ for
all $t \geq 0$. On $M$, it follows that $\tilde{R}\tilde{A}^{k-1}\frac{dx}{dt} = 0$ for all $t \geq 0$ and $k = 1, \ldots, n$, and hence, $\tilde{R}\tilde{A}^{k-1}x(t) = 0$ for all $t \geq 0$ and $k = 1, \ldots, n$. Now, it follows from (45) that $x(t) \in \mathcal{N}(\tilde{A})$ for all $t \geq 0$. Thus, $M \subseteq \mathcal{N}(\tilde{A})$. Since $\mathcal{N}(\tilde{A})$ consists of equilibrium points, it follows that $M = \mathcal{N}(\tilde{A})$. For $x_e \in \mathcal{N}(\tilde{A})$, Lyapunov stability of $x_e$ now follows by considering the Lyapunov function $U(x - x_e)$.

For any $\tilde{P} > 0$ satisfying (49) and $M \geq 0$, let $\tilde{T} = \tilde{P} + L^TML$. Clearly, $\tilde{T} \geq \tilde{P} > 0$. It is easy to verify that $\tilde{T}$ is a solution to (49), and hence, such a $\tilde{P}$ is not unique.

Finally, since $\tilde{A}$ is semistable, it follows from the above result that there exists an $n \times n$ positive-definite matrix $\tilde{P}$ such that (49) holds or, equivalently, $(\tilde{A} \oplus \tilde{A})^T \tilde{P} = -\text{vec} \tilde{A}^T \tilde{R} \tilde{A}$. Hence, $\text{vec} \tilde{A}^T \tilde{R} \tilde{A} \in \mathcal{R}((\tilde{A} \oplus \tilde{A})^T)$. Next, it follows from Lemma 3.8 that $\tilde{A} \oplus \tilde{A}$ is semistable, and hence, by Lemma 3.9,

\[
\text{vec}^{-1} \left((\tilde{A} \oplus \tilde{A})^T \tilde{A} \right) + \text{vec}^{-1} \left((\tilde{A} \oplus \tilde{A})^T \tilde{R} \tilde{A} \right) = -\int_0^\infty \text{vec}^{-1} \left((\tilde{A} \oplus \tilde{A})^T \tilde{R} \tilde{A} \right) dt
= -\int_0^\infty \text{vec}^{-1} \left((\tilde{A}^T)^T \tilde{R} \tilde{A} \right) dt
= -\int_0^\infty \text{vec}^{-1} \tilde{R} \tilde{A} dt,
\]

where in (53) we used the facts that $(X \otimes X)^T = X^T \otimes X$, $X^T \otimes Y = X \otimes Y$, and $\text{vec}(X) = (Z \otimes X) \text{vec} Y$ [21, Chapter 7]. Hence,

\[
\tilde{P} = \int_0^\infty \text{vec}^{-1} \tilde{R} \tilde{A} dt = \text{vec}^{-1}(z),
\]

where $z$ satisfies $z \in \mathcal{N}((\tilde{A} \oplus \tilde{A})^T)$ and $\text{vec}^{-1}(z) = (\text{vec}^{-1}(z))^T \geq 0$ (The nonnegativity of $\text{vec}^{-1}(z)$ is guaranteed by Theorem 4.2a of [23]). Since $(\tilde{A} \oplus \tilde{A})^T$ is semistable, it follows that the general solution to the equation $(\tilde{A} \oplus \tilde{A})^T z = 0$ is given by $z = \lambda x \otimes x$, where $x \in \mathcal{N}(\tilde{A})$ and $\lambda \in \mathbb{R}$. Hence, $\text{vec}^{-1}(z) = \text{vec}^{-1}(\lambda x \otimes x) = \lambda x x^T$, $\alpha > 0$, where we used the fact that $xy^T = \text{vec}^{-1}(y \otimes x)$ (Proposition 7.1.8 of [21]).

Next, we convert the optimal control problem P1 into a constrained quadratic programming problem based on Theorem 3.1.

**Theorem 3.2:** Consider the linear dynamical system (38). Assume $(\tilde{A}, \tilde{R})$ is weakly semisobservable for nonnegative definite $\tilde{R}$. Let $S_{\text{min}}$ be a solution to the minimization problem

\[
\min \left\{ \text{tr}(\tilde{A}^\#)^T S \tilde{A}^\# V : S > 0 \text{ and } \tilde{A}^T S + S \tilde{A} + \tilde{A}^T \tilde{R} \tilde{A} = 0 \right\},
\]

where $V \in \mathbb{R}^{nq \times nq}$, $V \geq 0$. Then for $P$ given by (40),

\[
\text{tr}(\tilde{A}^\#)^T S_{\text{min}} \tilde{A}^\# V = \text{tr} PV.
\]

Furthermore, $S_{\text{min}}$ is not unique. Finally, all the solutions to the minimization problem (55) can be parametrized as

\[
P_{\text{min}} = S_{\text{min}} + \alpha xx^T, \quad x \in \mathcal{N}(\tilde{A}) \cap \mathcal{N}(\tilde{A}^\#), \quad \alpha > 0.
\]

Note that it follows from Theorem 3.2 that the optimal control problem P1 can be recast as a linear matrix inequality (LMI) using the techniques developed in [19]. Due to space limitation, we do not discuss it here.

**IV. DISTRIBUTED $H_2$ SEMISTABILITY CONTROL DESIGN FOR COUPLED SYSTEMS: FREE COMMUNICATION TOPOLOGY CASE**

In this section, we use distributed $H_2$ semistability theory developed in Section III to present semistable optimal controller design for the optimal control problem P1. Specifically, using Theorem 4.1, we show that the optimal control problem P1 can be recast as a new minimization problem.

**Theorem 4.1:** Consider the linear dynamical system (38). Assume $(\tilde{A}, \tilde{R})$ is weakly semisobservable for nonnegative definite $\tilde{R}$. Let $S_{\text{min}}$ be a solution to the minimization problem (55) and $Z_{\text{min}}$ be a solution to the minimization problem

\[
\min \left\{ \text{tr} Z V : Z > 0 \text{ and } \tilde{A}^T Z + Z \tilde{A} + \tilde{A}^T \tilde{R} \tilde{A} = 0 \right\}. \quad (58)
\]

Then

\[
\text{tr}(\tilde{A}^\#)^T S_{\text{min}} \tilde{A}^\# V = \text{tr}(\tilde{A}^\#)^T Z_{\text{min}} \tilde{A}^\# V. \quad (59)
\]

The new optimization problem (58) can be solved numerically using either the interior-point method or the subgradient method [24], [25]. Furthermore, when $K$ has a special structure, Theorem 4.1 can be significantly simplified. To elucidate this, we need the following lemma.

**Lemma 4.1:** Let $A \in \mathbb{R}^{n \times n}$. If $-A$ is idempotent, that is, $A^2 = -A$, then $A$ is semistable.

**Example 4.1:** Consider the optimal control problem P1. Assume $m = n = 1$, $A = 0$, and $B = Q = R = I_n$. Now, it follows from Theorem 4.1 that the optimal control problem P1 is equivalent to the minimization problem (58). In this case, $K^T Z + Z K + K^T K + K^T K K = 0$. To simplify our discussion, we assume that $K^\#$ is idempotent and symmetric. Then it follows from Lemma 4.1 that $K$ is semistable. Furthermore, $K^T (Z - I_n) + (Z - I_n) K = 0$. Since $K$ is semistable, it follows from Theorem 3.1 that $Z - I_n = xx^T$, $x \in \mathcal{N}(K)$. Hence, $Z_{\text{min}} = I_n$. In this case, it follows from iii) of Proposition 6.2.2 of [21, p. 229] that $K^\# = K$, and hence, the cost function is given by $-\text{tr}(K^T V)$. For $g = 2$, we solve $K^2 = -K = -K^T$ to obtain two nonzero matrices

\[
K_1 = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}, \quad K_2 = \frac{1}{2} \begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix}.
\]

Note that $-\text{tr}(K_1 V) \geq -\text{tr}(K_2 V)$. The second matrix $K_2$ corresponds to distributed semistable control design, which is in accordance with the result in Example 2.2. It is clear that the proposed method is much easier to capture the optimal control gain $K$ than the direct method in Example 2.2.

Motivated by Example 4.1, we have the following distributed $H_2$ semistable control design for homogeneous coupled systems.

**Corollary 4.1:** Consider the optimal control problem P1. Assume $m = n$, $A = 0$, and $B = Q = R = I_{nq}$. Then
the optimal control problem $P_1$ has a nonzero solution $K_{\text{min}}$ given by the minimization problem

$$\min\{-\operatorname{tr}(KV) : K^2 + K = 0, \ K = K^T, \ K \neq 0\}. \quad (61)$$

V. CONCLUSION

In this paper, we extended $H_2$ theory to include distributed semistable systems. Using this framework along with constrained convex optimization, we developed a distributed $H_2$ optimal semistable stabilization framework for linear dynamical systems. In the future, we will consider the case where the communication topology between subsystems is fixed [26]–[29] as well as switching [30].

REFERENCES


