Abstract—This paper presents a Diesel engine selective catalytic reduction (SCR) ammonia surface coverage ratio profile control design for a two-cell SCR system, which enables a staircase ammonia coverage ratio profile for simultaneous high NOx conversion efficiency and low tailpipe ammonia slip. The novel two-cell SCR architecture greatly increases the system complexity, and controlling ammonia surface coverage ratios of two cells at different levels by a single control input is not trivial. To address these challenges, a nonlinear backstepping based control law is proposed to regulate the ammonia coverage ratio of the upstream cell to a desired (high) value in order to maintain high SCR NOx conversion efficiency, and to keep the ammonia coverage ratio of the downstream cell below a low upper limit for high ammonia adsorption capability. Simulation studies based on the first phase of the FTP75 cycle showed that the proposed controller can regulate the ammonia coverage ratios in each cell well and can satisfy the control requirements.

I. INTRODUCTION

DIESEL engines are noted for their better fuel economy and power density compared to corresponding gasoline engines. While Diesel engines have been and will continue to be the dominant power sources of medium- and heavy-duty vehicles, they are rapidly expanding in the light-duty vehicle applications as well. However, Diesel engine emissions control, especially NOx control, still represents a major challenge. Among the current emissions control technologies, selective catalytic reduction (SCR) is one of the most promising approaches for Diesel engine NOx emissions control [16]. SCR utilizes ammonia as the reductant to reduce NOx by catalytic reactions inside a catalyst. However, since both tailpipe NOx and ammonia emissions are undesirable, ammonia dosing control is quite challenging, particularly for vehicle applications where transient operations are very common. Urea over-dosing can cause tailpipe ammonia slip and also can increase use-cost, but insufficient ammonia injection will result in low NOx reducing efficiency, and lead to higher tailpipe NOx emissions. As shown in [2][3], a promising way of simultaneously minimizing the NOx emissions and limiting the ammonia slip is to control the ammonia surface coverage inside the SCR catalyst at an optimal value. The ammonia surface coverage ratio $\theta_{NH_3}$ is defined in Eq. (1), where $\theta$ is ammonia storage capacity and $M_{NH_3}$ is the amount of ammonia stored inside a SCR catalyst.

$$\theta_{NH_3} = \frac{M_{NH_3}}{\theta}.$$  \hspace{1cm} (1)

Currently there is no commercial sensor for directly measuring ammonia coverage ratio in a SCR. In [4], a method has been proposed to measure this value by lab equipment. Ammonia coverage ratio observers based on available sensors have been proposed in [5][6][7]. Due to the complex chemical reactions inside the catalyst and the highly dynamic engine exhaust gas conditions, systematic control of SCR ammonia coverage ratio is not trivial. Continuous stirred tank reactor (CSTR) modeling approach has been used to develop control-oriented SCR models [10]. CSTR assumes all the states inside the reactor are homogeneous. Such an assumption may work well for short SCRs where the state profile gradients along the SCR axial direction may not be significant. However, for long SCR cans in medium- and heavy-duty engine applications, the state distribution gradients along the SCR axial direction could be considerable, and consequently the homogeneity assumption of the single-cell CSTR may not be valid. As the authors of [1] pointed out, more than one CSTR models in series can better represent the main dynamics of a SCR.

In [8], the authors proposed a nonlinear controller utilizing backstepping technique to control the ammonia surface coverage ratio of the last cell in a SCR with multiple cells. The control objective of the study was to regulate the ammonia coverage of the downstream cell such that the ammonia slip can be limited under a constraint. It was demonstrated by simulations that the backstepping-based control strategy can achieve the objective fairly well. However, regulating the downstream cell ammonia coverage ratio can only limit the tailpipe ammonia slip. The NOx conversion efficiency was not considered in the controller. In this paper, we further extend the backstepping control law to control the ammonia coverage ratio of a two-cell SCR system as a staircase profile. It was shown in [4] that, in principle, NOx and ammonia emission minimizations can be achieved via controlling the coverage ratio of the SCR upstream portion to a higher value and a lower value in the downstream portion. In this paper, a backstepping based nonlinear control law was designed to regulate the ammonia coverage ratio of the upstream cell to a high desired value in order to convert most NOx emissions to nitrogen molecules. At the same time, the coverage ratio of the downstream cell is constrained under a low upper bound in order to maintain high ammonia adsorption capacity and prevent undesirable tailpipe ammonia slip, particularly during abrupt transients. It is also desirable to have the ammonia coverage ratio of the downstream cell close to the upper limit for better NOx...
conversion. Compared with the conservative approaches that keep the entire SCR cell ammonia level low to prevent transient excessive ammonia slip [17], such a two-cell SCR architecture with a staircase ammonia coverage ratio profile control has the potential of achieving the same level of tailpipe NO\textsubscript{x} and ammonia emissions with smaller total catalyst volume (lower costs).

The rest of this paper is organized as follows. A brief introduction of the SCR model is presented in the following section. After that, the two-cell SCR staircase ammonia coverage ratio profile controller design is described and analyzed, followed by simulation results based on the first phase of the FTP75 driving cycle. Conclusive remarks and future work are summarized in the end.

II. MODELING OF SELECTIVE CATALYTIC REDUCTION

A. SCR Operation Principles

SCR NO\textsubscript{x} reduction reaction mainly includes three processes. In the first part, ammonia is injected at upstream of the SCR catalyst as the source of reductant. The ammonia inside the catalyst is then adsorbed on the catalyst surface. The adsorbed ammonia then catalytically reacts with NO\textsubscript{x} and converts them to nitrogen molecules and water. The dominant chemical reactions and the corresponding reaction rates are briefly explained below.

1) NH\textsubscript{3} Adsorption/Desorption

The ammonia entered the SCR catalyst can be adsorbed on the SCR substrate. The adsorbed NH\textsubscript{3}\textsuperscript{*} can also be desorbed from the substrate as shown in the following reaction equation.

\[
\text{NH}_3 + \theta \leftrightarrow \text{NH}_3^*.
\] (2)

The rate of the NH\textsubscript{3} adsorption/desorption can be expressed by the following equations [10].

\[
R_{ads} = k_{ads} \exp \left( -\frac{E_{ads}}{RT} \right) \theta_{NH_3}(1 - \theta_{NH_3}),
\] (3)

\[
R_{des} = k_{des} \exp \left( -\frac{E_{des}}{RT} \right) \theta_{NH_3^*}.
\] (4)

Also, at temperatures higher than 450 deg C or so, the adsorbed NH\textsubscript{3} can be oxidized to NO by the following reaction.

\[
\text{NH}_3^* + 1.25O_2 \rightarrow \text{NO} + 1.5H_2O,
\] (5)

\[
R_{oxi} = k_{oxi} \exp \left( -\frac{E_{oxi}}{RT} \right) \theta_{NH_3},
\] (6)

2) NO\textsubscript{x} Reduction

The adsorbed NH\textsubscript{3} can then catalytically react with NO\textsubscript{x} to become nitrogen according to the following reactions.

\[
4\text{NH}_3^* + 4\text{NO} + O_2 \rightarrow 4\text{N}_2 + 6H_2O,
\] (7)

\[
2\text{NH}_3^* + \text{NO} + \text{NO}_2 \rightarrow 2\text{N}_2 + 3H_2O,
\] (8)

\[
4\text{NH}_3^* + 3\text{NO}_2 \rightarrow 3.5\text{N}_2 + 6H_2O.
\] (9)

Because more than 90% of the Diesel exhaust gas NO\textsubscript{x} is usually composed of NO [11], assuming there is no DOC/DPF placed upstream of the SCR to convert part of the NO to NO\textsubscript{2}, reaction in Eq. (7) is considered the dominant reaction in NO\textsubscript{x} reduction. The reaction rate is described below [5]. In the presence of DOC/DPF, part of NO will be converted to NO\textsubscript{2} and the reaction described in Eq. (8) will be faster. Such effects are addressed in authors’ work [18].

\[
R_{\text{red}} = k_{\text{red}} \exp \left[ -\frac{E_{\text{red}}}{RT} \right] C_{\text{NO}} \theta_{\text{NH}_3},
\] (10)

B. SCR Model

Based on the molar balance, the main SCR dynamics can be described by the following equations [10]:

\[
\dot{C}_{\text{NO}} = \theta(R_{\text{oxi}} - R_{\text{red}}),
\] (11)

\[
\dot{\theta}_{\text{NH}_3} = R_{\text{ads}} - R_{\text{des}} - R_{\text{red}} - R_{\text{oxi}},
\] (12)

\[
\dot{\theta}_{\text{NH}_3} = \theta(R_{\text{des}} - R_{\text{ads}}).
\] (13)

The above dynamic equations can be used to develop the SCR model using the CSTR approach or mass conservation law. Several SCR models for vehicle applications have been proposed in the literature [6][10][12][13]. In this paper, we adopted the model presented in [10]. The nonlinear model in state variable form is shown in Eq. (14).

\[
\begin{bmatrix}
\dot{C}_{\text{NO}} \\
\dot{\theta}_{\text{NH}_3}
\end{bmatrix} =
\begin{bmatrix}
-C_{\text{NO}} \left( \theta R_{\text{oxi}} \theta_{\text{NH}_3} + \frac{F}{V} \right) + r_{\text{NH}_3,\text{oxi}} \theta_{\text{NH}_3} \\
-\theta_{\text{NH}_3} \left( R_{\text{ads}} C_{\text{NH}_3} + r_{\text{des}} C_{\text{NO}} + r_{\text{oxi}} + r_{\text{ads}} C_{\text{NH}_3} \right)
\end{bmatrix} +
\begin{bmatrix}
0 \\
C_{\text{NH}_3,\text{in}} + \frac{F}{V} \dot{C}_{\text{NO,in}}
\end{bmatrix},
\] (14)

where \( r_x = k_x e^{-\frac{E_x}{RT}} \), \( x = \text{ads, des, oxi, red} \).

![Figure 1. Two-cell SCR structure.](image)

However, the CSTR model assumes all the states are uniform throughout the SCR catalyst, which may not be valid for large SCR catalysts used in medium- and heavy-duty Diesel engine vehicles. The authors in [1] pointed out that to better capture the SCR dynamics and state distribution gradients along the SCR axial direction, a SCR catalyst should be divided into multiple cells. The SCR cell means a single CSTR model as presented in Eq. (14). Physically, a SCR cell can be a SCR brick or a section of the catalyst. The SCR model to be studied in this paper is a two-cell SCR model which can be represented by Eq. (15) and Figure 1.

\[
\begin{bmatrix}
\dot{C}_{\text{NO,i}} \\
\dot{\theta}_{\text{NH}_3,i}
\end{bmatrix} =
\begin{bmatrix}
-C_{\text{NO,i}} \left( \theta R_{\text{oxi,i}} \theta_{\text{NH}_3,i} + \frac{F}{V} \right) + r_{\text{NH}_3,\text{oxi,i}} \theta_{\text{NH}_3,i} \\
-\theta_{\text{NH}_3,i} \left( R_{\text{ads,i}} C_{\text{NH}_3,i} + r_{\text{des,i}} C_{\text{NO,i}} + r_{\text{oxi,i}} + r_{\text{ads,i}} C_{\text{NH}_3,i} \right)
\end{bmatrix} +
\begin{bmatrix}
0 \\
C_{\text{NH}_3,i,\text{in}} + \frac{F}{V} \dot{C}_{\text{NO,in,i}}
\end{bmatrix},
\] (15)

\( i = 1,2 \).
where $C_{NH_3}$ and $C_{NO_x}$ are the controlled ammonia concentration and engine exhaust gas NO concentration into the SCR catalyst, and $C_{NH_3,1}$ and $C_{NO_x,1}$ are the tailpipe ammonia and NO concentrations.

III. Two-Cell SCR Controller Design

According to Eq. (10), higher ammonia coverage ratio can always achieve a higher NOx conversion efficiency, but will have a higher tendency of ammonia slip during transients as can be seen from Eq. (4). On the other hand, lower coverage ratio can achieve higher ammonia adsorption rate according to the reaction in Eq. (3). Based on these reasons, lower coverage ratio at the downstream part is desired to limit the tailpipe ammonia slip and higher coverage ratio is desired at upstream part to increase NOx conversion efficiency. Theoretically, it is possible to find an optimal coverage ratio profile along the axial length of the catalyst. By controlling the coverage ratio to this profile, the combination of the NOx and ammonia tailpipe emissions can be minimized in a desired fashion. But practically, it is extremely difficult to accomplish because controlling coverage ratio to a smooth "profile" requires many cells/sensors or a SCR model with partial differential equations which are difficult to handle from control design point of view. To better represent the SCR state distribution along the axial direction as well as maintain a tractable level of complexity, we propose a two-cell based control strategy. The upstream cell serves as the NOx conversion cell (NCC) whose ammonia coverage ratio is controlled to a high value to have high NOx conversion efficiency. The target value is the maximum possible ammonia coverage ratio at which the ammonia slip of the NCC will not excessively exceed a prescribed value during normal operations. The downstream cell serves as an ammonia reservoir cell (ARC) which holds a lower coverage ratio and intends to capture the ammonia slip from the upstream NCC, particularly during transients.

To achieve the above control objectives, a backstepping-based nonlinear control law was designed. The goal of the controller is to regulate the ammonia coverage ratio of the NCC at an optimal value, with the constraint that the coverage ratio of the ARC does not exceed a prescribed low level. A schematic presentation is shown in Figure 2.

![Fig. 2 Schematic presentation of the two-cell based SCR control.](image)

Considering the control input as $C_{NH_3,3}$ and the controlled variables as $\theta_{NH_3,1}$ and $\theta_{NH_3,2}$, the control problem based on the model in Eq. (15) is formulated below. The objective of the controller is to regulate $\theta_{NH_3,2}$ to the optimal value $\theta^*_2$ and, at the same time, keep $\theta_{NH_3,1}$ under the upper limit $\theta^*_1$.

The related dynamics are listed below.

\[
\dot{\theta}_{NH_3,1} = -\theta_{NH_3,1}(r_{ads,1}C_{NH_3,1} + r_{des,1} + r_{red,1}C_{NO_x,1} + r_{ox,1}) + r_{ads,1}C_{NH_3,1'}, \\
\dot{\theta}_{NH_3,1'} = -\theta_{NH_3,1'}(1 - \theta_{NH_3,1}) + F \frac{v_1}{v_1}, \\
\dot{\theta}_{NH_3,2} = -\theta_{NH_3,2}(r_{ads,2}C_{NH_3,2} + r_{des,2} + r_{red,2}C_{NO_x,2} + r_{ox,2} + r_{ads,2}C_{NH_3,2'}),
\]

Consider the terms of reaction rates, ammonia capacities, flow rates, and NO concentrations as known bounded time-varying signals denoted as a vector $\delta_i(t), i = 1, 2$. Equation (16) can then be rewritten as:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, \delta_1(t)) + g_1(x_1, \delta_1(t))x_2, \\
\dot{x}_2 &= f_2(x_1, x_2, \delta_1(t)) + g_2(\delta_1(t))x_3, \\
\dot{x}_3 &= f_3(x_3, x_4, \delta_2(t)) + g_3(\delta_2(t))u, \\
\dot{x}_4 &= f_4(x_4, \delta_2(t)) + g_4(x_4, \delta_2(t))x_3,
\end{align*}
\]

where we have

\[
[x_1, x_2, x_3, x_4]^T = [\theta_{NH_3,1}, C_{NH_3,1}, C_{NH_3,2}, \theta_{NH_3,2}]^T, \\
0 < [x_{1,\min}, x_{2,\min}, x_{3,\max}, x_{4,\max}]^T \leq [x_{1,\max}, x_{2,\max}, x_{3,\max}, x_{4,\max}]^T, \\
\delta_i(t) \in C^1, \\
0 < [r_{ads,\min}, r_{des,\min}, r_{red,\min}, r_{ox,\min}, F(t), \theta(t), C_{NO_x,\min}]^T \leq \delta_i(t), \\
\leq [r_{ads,\max}, r_{des,\max}, r_{red,\max}, r_{ox,\max}, F_{\max}, \theta_{\max}, C_{NO_x,\max}]^T.
\]

Eq. (20)-(22) lead to:

\[
f_i < 0, f_0 < 0, g_1 > 0, g_2 > 0, g_3 > 0, g_4 > 0.
\]

It is desirable to achieve $x_4 = \theta^*_2$ and satisfy the constraint $x_1 \leq \theta^*_1$.

The control law is proposed as:

\[
u = -K_3 \text{sign}(x_3 - \theta^*_3) - f_1 + \theta_2 - g_1(x_2 - \theta_1) - g_4(x_4 - \theta^*_2),
\]

where

\[
\begin{align*}
\theta_2 &= -K_2(x_2 - \theta_1) - f_2 + \theta_1 - g_1(x_1 - \theta^*_1), \\
\theta_1 &= -K_1 \left( \frac{x_1 - \theta^*_1}{C} \right), \\
K_1 &= \frac{2g_4(1 - \theta^*_1)}{\text{sign}[(x_1 - \theta^*_1)(x_4 - \theta^*_2)] + 1}.
\end{align*}
\]

3005
\[ (C, K_3) > 0, \]  
\[ K_2 > \frac{|f_2|_{\text{max}} + g_{1,\text{max}}}{x_{2,\text{min}}} \]  
\[ K_3 > g_{2,\text{max}} |x_2 - \Theta_1|_{\text{max}}, K_3 > g_{4,\text{max}} |x_4 - \Theta_2|_{\text{max}}, \]  

The boundedness of \( \Phi_1 \) and \( \Phi_2 \) is ensured by the conditions in Eq. (20)-(22).

**Proof:** The proof is based on the backstepping technique. Two cases will be considered: \( x_1 > \Theta_1^* \) and \( x_1 \leq \Theta_1^* \). In the first case, since \( x_1 \leq \Theta_1^* \) is the constraint that has to be satisfied, we will prove that \( x_4 \) can convert to \( \Theta_1^* \) regardless of \( x_4 \). In the second case, since the constraint is satisfied, we want to prove that \( x_4 \) can converge to the optimal value \( \Theta_2^* \).

1) **Case 1:** \( x_1 > \Theta_1^* \)

Considering the first equation in Eq. (17) as the first part of the system, we want to prove that \( x_1 \) can converge to the constraint value \( \Theta_1^* \) by the virtual control input of \( x_2 \).

The positive definite Lyapunov function candidate \( V_1 \) is defined as:

\[ V_1 = \frac{1}{2} \hat{x}_1^2, \]  
where \( \hat{x}_1 = x_1 - \Theta_1^* \) and \( 0 < \Theta_1^* \leq 1 \).

\[ \dot{V}_1 = \dot{x}_1 \hat{x}_1 = \hat{x}_1 f_1 + \hat{x}_1 g_1 x_2. \]  
Combining Eq. (32) and Eq. (16):

\[ \dot{V}_1 = -\left( r_{\text{des},1} + r_{\text{red},1} C_{N0,1} + r_{\text{axi},1}\right) \hat{x}_1 (\hat{x}_1 + \Theta_1^*) + r_{\text{ads},1} (1 - (\hat{x}_1 + \Theta_1^*)) \hat{x}_1 x_2. \]  
By selecting the virtual control input of \( x_2 \) as \( \Phi_1 \) in Eq. (26) and combining Eq. (26) and Eq. (27), we have:

\[ x_{2,\text{des}} = \Phi_1 = -K_1 \left( \frac{x_1 - \Theta_1^*}{1 - x_1} \right) \]

\[ \frac{2g_4(1 - \Theta_1^*)}{C} \left( \frac{|x_1 - \Theta_1^*|}{1 - x_1} \right) + 1 \right] \]  
which leads to

\[ V_1 = -\left( r_{\text{des},1} + r_{\text{red},1} C_{N0,1} + r_{\text{axi},1}\right) \hat{x}_1 (\hat{x}_1 + \Theta_1^*) \]

\[ -\frac{2g_4(1 - \Theta_1^*)}{C} \left( \frac{|x_1 - \Theta_1^*|}{1 - x_1} \right) + 1 \right] \]  
Since \( \hat{x}_1 > 0 \), we have:

\[ (r_{\text{des},1} + r_{\text{red},1} C_{N0,1} + r_{\text{axi},1}) \hat{x}_1 (\hat{x}_1 + \Theta_1^*) \]

\[ > (r_{\text{des},1} + r_{\text{red},1} C_{N0,1} + r_{\text{axi},1}) \hat{x}_1^2 \]

\[ > 0, \]  
and

\[ \frac{C}{2g_4(1 - \Theta_1^*)} \left( \frac{|x_1 - \Theta_1^*|}{1 - x_1} \right) \]

\[ + 1 \right] \frac{|x_1 - \Theta_1^*|}{1 - x_1} \hat{x}_1^2 \geq 0 \]

because \( \frac{C}{2g_4(1 - \Theta_1^*)} \left( \frac{|x_1 - \Theta_1^*|}{1 - x_1} \right) \geq 0 \). So

\[ V_1 \leq -\left( r_{\text{des},1} + r_{\text{red},1} C_{N0,1} + r_{\text{axi},1} \right) \hat{x}_1 (\hat{x}_1 + \Theta_1^*) \]

\[ \leq -\left( r_{\text{des},1} + r_{\text{red},1} C_{N0,1} + r_{\text{axi},1} \right) \hat{x}_1^2 = -W_1, \]  
where \( W_1 \) is continuous and positive definite. Thus \( \dot{V}_1 \leq -W_1 \) guarantees the convergence of \( x_1 \) to \( \Theta_1^* \).

The next step we consider the second equation in Eq. (17) as the second backstepping step, in which we want to prove that \( x_2 \) can converge to the desired value \( x_{2,\text{des}} \) as described in Eq. (34) by designing the virtual control of \( x_3 \) as the control law \( \Phi_2 \) in Eq. (25). Selecting \( x_2 = x_3 - \Phi_2 \). The positive definite Lyapunov function candidate is selected as:

\[ V_2 = V_1 + \frac{1}{2} z_2^2 = \frac{1}{2} \hat{x}_1^2 + \frac{1}{2} z_2^2. \]  
And the time derivative of \( V_2 \) is:

\[ \dot{V}_2 = \hat{x}_1 \dot{x}_1 + z_2 \dot{z}_2. \]  
Since \( z_2 = \Phi_1 + z_3 \), from Eq. (32) and Eq. (38) we have

\[ \dot{V}_2 \leq -W_1 + \hat{x}_1 g_1 z_2 + z_3 \left( \hat{x}_2 - \hat{o}_1 \right) \]

\[ = -W_1 + z_2 \left( g_1 \hat{x}_1 + f_3 + g_2 x_3 - \hat{o}_1 \right). \]  
By letting \( x_3 \) as the virtual control and \( x_3 = \Phi_2 \), the above equation becomes:

\[ \dot{V}_2 \leq -W_1 - K_2 z_2^2 = -W_2, \]  
where \( W_2 \) is positive definite. Thus the \( \dot{V}_2 \) in Eq. (42) guarantees the convergence of \( x_2 \) to \( x_{2,\text{des}} \).

In the third backstepping step, we consider the dynamics of the third equation of Eq. (17), and we want to prove that by applying the control law \( u \) in Eq. (24), \( x_3 \) can converge to the desired value \( x_{3,\text{des}} \). Let \( z_3 = x_3 - \Phi_2 \). The positive definite Lyapunov function candidate is selected as:

\[ V_3 = \frac{1}{2} z_3^2 \]

Since \( x_3 = \Phi_2 + z_3 \) and based on Eq. (41), the time derivative of \( V_3 \) is:

\[ \dot{V}_3 = z_2 \dot{z}_3 \leq -W_2 + z_3 g_2 z_3 + z_3 \left( \hat{x}_2 - \hat{o}_2 \right) \]

\[ = -W_2 + z_3 \left( g_2 z_2 + f_3 + g_3 u - \hat{o}_2 \right). \]  
Combining the \( u \) in Eq. (24) and Eq. (44), we have

\[ \dot{V}_3 \leq -W_2 - z_3 \left( K_3 \text{sign}(x_3) + g_4(x_4 - \Theta_2^*) \right) \]

\[ = -W_3. \]  
Since \( K_3 > g_{4,\text{max}} |x_4 - \Theta_2^*|_{\text{max}}, W_1 \) is positive definite and the control law renders the convergences of \( \hat{x}_1, x_2, z_3 \).

**Remarks:** Based on the Lyapunov functions in Eqs. (31), (39), and (43), we showed that by the control law in Eq. (24), \( x_3 \) converges to \( x_{3,\text{des}} \), \( x_2 \) converges to \( x_{2,\text{des}} \), and \( x_1 \) converges to the desired value \( \Theta_1^* \). In this case, as the ammonia coverage ratio in the ARC is higher than the upper limit, to prevent undesirable tailpipe ammonia slip, the control law will reduce the ARC ammonia coverage ratio to the limit regardless of the ammonia coverage ratio in NCC.

2) **Case 2:** \( x_1 \leq \Theta_1^* \)

This is the case when the ammonia coverage ratio of the ARC is below the upper limit. In the first backstepping step, dynamics of the forth equation of Eq. (17) is considered. Selecting a positive definite Lyapunov function candidate \( V_4 \) as:

\[ V_4 = \frac{1}{2} \hat{x}_4^2, \]  
where \( \hat{x}_4 = x_4 - \Theta_2^* \). The time derivative of \( V_4 \) is:
\[ \dot{V}_4 = \dot{x}_4 (f_4 + g_4 x_3), \]  
(47)

where \( f_4 < 0 \) and \( g_4 > 0 \). The virtual control input of \( x_3 \) is selected as \( \varnothing_2 \). Then Eq. (47) becomes:

\[ \dot{V}_4 = \dot{x}_4 (f_4 + g_4 \varnothing_2) \]

\[ = \dot{x}_4 \left[ f_4 - K_2 (x_2 - \varnothing_2) - f_2 - \varnothing_1 - g_4 (x_1 - \theta_1^*) \right] + g_2 \frac{K_2 (x_2 - \varnothing_2)}{g_2} \]

(48)

Two cases are discussed below.

**Case 2.1: \( x_4 \geq \theta_2^* \)**

In this case, since \( x_1 \leq \theta_1^* \) and \( x_4 > \theta_2^* \), \( \text{sign}[(x_1 - \theta_1^*)(x_4 - \theta_2^*)] + 1 = 0 \) and thus \( K_1 = 0 \) and \( \varnothing_1 = \varnothing_2 = 0 \). \( V_4 \) becomes:

\[ \dot{V}_4 = \dot{x}_4 \left( f_4 - 4 K_2 x_4 + f_2 + g_4 (x_1 - \theta_1^*) \right) \]

(49)

Since \( f_4 < 0 \), \( g_1, g_2, g_4 > 0 \), and \( K_2 > \frac{f_2 \max + g_1 \max}{g_2 \min} \), we have:

\[ \dot{V}_4 \leq \dot{x}_4 \left( f_{4, \max} - g_4 \min \frac{K_2 x_2 - f_2 - g_1}{g_2 \max} \right) \]

\[ \leq \dot{x}_4 \left( f_{4, \max} - g_4 \min \frac{f_2 \max + g_1 \max + f_2 - g_1}{g_2 \max} \right), \]

(50)

which is negative definite and guarantees that \( x_4 \) can converge to \( \theta_2^* \).

**Case 2.2: \( x_4 < \theta_2^* \)**

Since \( x_4 < \theta_2^* \), \( \dot{x}_4 < 0 \), for \( V_4 \) being negative definite, according to Eq. (48), we need

\[ f_4 + g_4 \varnothing_2 > 0. \]

(51)

Combining Eqs. (51), (25), (26), and (27), we have:

\[ f_4 + g_4 \varnothing_2 = f_4 + \frac{g_4}{g_2} \left[ \frac{-K_2 (x_2 + K_1 x_1 - \theta_1^*) - f_2}{g_2} \right] - K_1 \frac{(1 - x_1^2)}{g_4 (1 - \theta_1^*)} \]

\[ - g_4 (x_1 - \theta_1^*) \]

(52)

Since \( x_1 \leq \theta_1^* \) and \( x_4 < \theta_2^* \), \( K_1 = \frac{C}{g_4 (1 - \theta_1^*)} \). Based on Eq. (52), Eq. (51) can be satisfied if:

\[ x_1 - \theta_1^* < \frac{1}{K_2 C + g_1 g_2 (1 - \theta_1^*) (1 - x_1)} \left[ f_4 - g_4 K_2 x_2 \right] \]

\[ - g_4 f_2 - \frac{C (f_1 + g_1 x_2) (1 - \theta_1^*)}{g_2 (1 - x_1)^2} \]

(53)

If \( \epsilon \geq 0 \), since \( x_1 \leq \theta_1^* \), Eq. (51) holds and \( \dot{x}_4 \) converges to zero. In the case \( \epsilon < 0 \), if \( K_2 \) and \( C \) are chosen large enough, \( \epsilon \) can be very close to zero from the left. It means Eq. (58) can be satisfied for \( x_4 \) being less than \( \theta_1^* - |\epsilon| \).

For example, nominal values of \( K_2 = C = 100 \) lead to \( \epsilon = -0.03 \), which is sufficiently small. We conclude here that \( x_4 \) can converge to \( \theta_1^* \) if \( x_1 \) is less than \( \theta_1^* - |\epsilon| \).

In the second backstepping step, we consider the dynamics of the third equation of Eq. (17). A Lyapunov function candidate \( V_5 \) is designed as:

\[ V_5 = V_4 + \frac{1}{2} x_3^2, \]

(54)

From Eqs. (48), and the control law \( u \) in Eq. (24), we have

\[ V_5 = \dot{V}_4 + z_3 \dot{z}_3 \leq -W_4 + \dot{x}_4 g_4 z_3 + z_3 (x_3 - \varnothing_2) \]

\[ = -W_4 \]

\[ + z_3 (\dot{x}_4 g_4 + f_3 + g_3 u - \varnothing_2) \]

\[ = -W_4 - z_3 (K_3 \text{sign}(x_3) + g_2 z_2), \]

(55)

where, \( W_4 \) is a positive term representing both Case 2.1 and Case 2.2. Since \( K_3 > g_2 \max |x_2 - \varnothing_1 \max |, V_5 \) is negative and thus \( x_3 \) converges to \( \varnothing_2 \) and \( x_4 \) converges to \( \theta_2^* \), if \( x_1 \) is less than \( \theta_1^* - |\epsilon| \).

**Remarks:** Here, the \( \epsilon \) can be very small as \( K_2 \) and \( C \) are designed to be large enough. Physically, the region \( \theta_1^* > x_1 > \theta_1^* - |\epsilon| \) can be considered as a buffering region which can prevent \( x_1 \) from overshooting the upper limit \( \theta_1^* \).

**IV. SIMULATION STUDIES**

Simulation studies were conducted based on a full-vehicle model which includes models for engine, vehicle, and aftertreatment systems. Figure 3 shows a representative simulation result of the proposed control law. The simulation was done based on the first phase of a FTP 75 cycle. The constraint value of the ARC \( \theta_1^* \) was set to 0.15 at initial and then stepped up to 0.3 at the 250th second. The set value of the NCC \( \theta_2^* \) was 0.3 at initial and stepped up to 0.5 at the 250th second. The ammonia concentration was constrained under 0.01 mol/m³. As can be seen, the ammonia coverage ratio of the NCC tracked the desired value \( \theta_2^* \) very well and the ARC constraint is satisfied except the first 20 seconds. The ammonia coverage ratio offsets in the first 20 seconds in the two cells are due to that the area was over dosed at the start of the cycle and it led to an overshoot at the ammonia reservoir cell. Because the vehicle at the first 20 seconds of the FTP75 cycle is in idle, the NOx emissions from the engine are very low. Since NOx gas is the only factor that has the ability to decrease the ammonia coverage ratios, the ammonia coverage ratio of the ARC remained almost constant with the absence of sufficient NOx gases. Intuitively, during this time period, in order to make the constraint of \( x_1 \) smaller than \( \theta_1^* \) to be satisfied, the controller should stop ammonia injection as it did in Figure 3. Theoretically, this is the situation of Case 1 where we proved the control law tries to enforce \( x_1 \) to satisfy the constraint regardless of \( x_4 \). Besides the first 20 seconds, we can see the control law can regulate the coverage ratio of the NCC to the desired values and keep the ARC ammonia coverage ratio under the constraint values simultaneously.

Figure 4 and Figure 5 show the comparisons of the tailpipe NOx and ammonia emissions with a PID controller controlling the tailpipe NOx emissions, and a PID controller controlling the tailpipe ammonia slip, respectively. The SCR catalyst volume used in these two cases was the same as the total volume of the two-cell case. As it shows, the PID NOx controller can regulate the average NOx emissions to the same level as the two-cell based controller (0.07 g/mile). However, the ammonia slip is much higher (152 ppm compared to 10 ppm). On the other hand, the PID ammonia controller achieves the same level of average tailpipe ammonia emission (10 ppm), but the peak value (151 ppm compared with 50 ppm) and NOx emission (0.28...
g/mile compared to 0.07 g/mile) are much higher than those of the two-cell based controller.

Figure 3. Two-cell SCR NH₃ coverage ratios.

Figure 4 Tailpipe emission NOₓ concentration.

Figure 5 Tailpipe emission ammonia concentration.

V. CONCLUSIONS AND FUTURE WORK

A two-cell SCR architecture with a staircase ammonia coverage ratio profile control is proposed for high NOₓ conversion efficiency and low ammonia slip. A backstepping based nonlinear SCR control law is designed to control the ammonia coverage ratios of the two-cell SCR system. The control law can regulate the ammonia coverage ratio of the upstream cell to a relatively high value for a high NOₓ conversion efficiency and constrain the coverage ratio of the downstream cell under an upper limit to maintain high ammonia adsorption capability and prevent excessive ammonia slip during transient operations. Based on the first phase of the FTP 75 cycle, simulation results verified that the designed controller can regulate the coverage ratios of the different cells to satisfy the proposed SCR control requirements. Comparisons with conventional controllers also showed the advantages of the proposed SCR control strategy in simulations.

In this paper, the set values for the NCC ammonia coverage ratio and ARC upper limit are not optimal. Selecting the optimal set values for the two cells will be a future study. Also, the urea-ammonia dynamics and urea injection rate constraint are not explicitly considered in the control law. The former can be addressed by augmenting the system and backstepping, and the later will be taken into account by predictive control. Model uncertainties will be incorporated.

ACKNOWLEDGMENT

The authors would like to thank the partial financial support provided by the members of The Ohio State University Center for Automotive Research (OSU-CAR) Industrial Consortium.

REFERENCES