Time-variant robust model predictive control under limited capacity communication constraints

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Abstract—This paper is concerned with robust model predictive control (MPC) of an input and state constrained system, when the system and controller are separated by a digital communication channel of limited capacity. To achieve performance and robustness, in the presence of time-variant uncertainty due to a dynamic coding/decoding policy employed to communicate state information across the channel, a time-variant robust MPC policy is employed. The rationale behind the time-variant nature of the control law of this paper is to provide a time-variant adjustment mechanism for achieving a good balance between performance (i.e. small minimal achievable costs) and robustness (i.e. stability and constraint satisfaction) achieved by the controller. The presented theory is illustrated with a simulation example. Some conclusive remarks are made regarding future work and extensions of the results of this paper.

Keywords: Model Predictive Control, Networked Control Systems, Robust Control, Communication Constrained Control.

I. INTRODUCTION

A standard assumption in classical control theory is that data transmission between system and controller can be performed with infinite precision. However with the advent of improved communication technology and motivated by economic considerations, control over digital limited capacity communication networks is becoming more pervasive. Here assumptions of infinite precision on data transmission are no longer valid. Within this framework, the state sensor and controller are no longer assumed to be physically collocated, but are separated via a digital communication channel. For such problems classical control theory can no longer be applied, since data regarding state information is sent via a limited capacity communication channel, such that the controller only observes a transmitted sequence of finite-valued symbols. Current challenges and an overview of communication constrained control are discussed in [3], [5] and [6]. Stabilization of linear systems has been considered in [1],[2],[4],[7] and [8]. Stabilization of nonlinear systems was addressed in [9]. Optimal control within this framework was considered in [10].

This paper is concerned with the control, via a robust MPC policy, of a state and input constrained linear system, when the system and controller are separated by a communication channel of limited capacity, such that quantization of system state must be performed in order to communicate state information across the channel. This is achieved by means of a coding/decoding policy. In this paper, a dynamic (i.e. time-variant) coding/decoding policy is considered. The reader is referred to [1],[2],[5] and [8] for a background on dynamic coding/decoding policies. Static quantization within MPC has previously been considered in [11], [12] and [13]. In essence dynamic coding/decoding policies provide an improved utilization of the channel capacity over static coding/decoding policies, by employing knowledge of the dynamics of the controlled system, in order to effect a decrease in the quantization error bounds (i.e. the sets within which the quantization errors are known to lie in) over time. Thus quantization may be considered an uncertainty of a time-variant nature within the control loop. Herewith the uncertainty bounds (i.e. the sets within which the uncertainty is known to lie in) are time-variant and it is reasonable to expect improved "performance" from the controller, as the quantization errors bounds decrease. In particular, one could consider an adjustment mechanism within the controller, that ensures that the controller is "more robust" when "the uncertainty is high", and "less robust", in order to improve performance, as "the uncertainty decreases".

In this paper, building on previous work, the latter heuristic idea is employed to formulate a robust MPC policy, which, in a time-variant manner, trades off robustness for performance, depending on the known and time-variant uncertainty bounds due to the transmission of the system state across the communication channel. In particular, this paper represents an extension of the results from [26],[14],[15],[16], [17] and [18]. The problem of robust MPC over limited capacity communication channels, as an extension of the results from [14],[15],[16], [17] and [18], was considered in [26]. The aforementioned papers achieve closed loop robustness (i.e. constraint satisfaction and stability), by control of a nominal uncertainty free system, which is controlled within suitably restricted constraints (i.e. state and input constraints on nominal system are tighter than on actual system), based on suitably defined "safety margin sets". Here a safety margin set is a suitably defined set, computed from knowledge of the sets within which the time-variant uncertainty lies. The intuitive idea is that a larger uncertainty results in a larger safety margin set, resulting in a "more robust controller", and a "degradation of performance". Overall, the particular formulation of the control law applied to the actual system ensures that the actual system state and actual system input fall within suitably defined safety margin sets around the state and input of the nominal system respectively. Herewith the actual system satisfies state and input constraints, and stability is achieved. The essential idea exploited in this paper is that smaller safety margin sets are desirable. Thus the aim is to keep the safety margin sets as small as possible for performance, but not too small for the known uncertainty bounds, as to result in instability and/or state and input constraint violations.

The contribution of this paper is an extension of the results of the paper [26] by proposing an improved time-variant safety margin sets update mechanism, which achieves smaller safety margin sets than the controller in [26]. The results of this paper are not confined to uncertainties solely associated with communication channels as considered in this paper, but are sufficiently general and may also be employed to extend the controllers from [14],[15],[16], [17] and [18] by employing the smaller safety margin sets proposed in this paper, and thereby achieve improved performance. Furthermore the MPC controller of this paper results in a relaxation of the restrictive monotonicity constraint on the safety margin sets from the papers [26] and [18], requiring that the safety margin sets be monotonically decreasing with time. This paper concludes with some remarks regarding the proposed method and future work.
II. NOTATION

A. Vector and Matrix Notation

Let $n$ and $m$ be a positive integers. $\mathbb{R}^{n \times m}$ denotes all real valued matrices of dimension $n \times m$. The transpose of $A$ is denoted by $A'$. If $a_{ij}$ denotes the entry of matrix $A$ at row $i$ and column $j$, then $|A|$ denotes the matrix with entry $|a_{ij}|$ at row $i$ and column $j$. With $x = [x_1 \ x_2 \ \ldots \ x_n]' \in \mathbb{R}^n$, the $i$-th entry of $x$ is denoted by $x^{(i)}$, i.e. $x^{(i)} = x_i$. Also $|x|$ denotes the vector whose $i$-th entry is $|x_i|$. If $w \in \mathbb{R}^n$, then the vector inequality $x \leq w$ holds if and only if $x^{(i)} \leq w^{(i)}$ for all $i \in [1,n]$. With $M$ some positive definite symmetric matrix, $\|x\|_M = \sqrt{x'Mx}$ and $\|x\|_\infty = \max_{i \in [1,n]} |x^{(i)}|$ denote the standard square and max norms over $\mathbb{R}^n$ respectively. With $a, b \in \mathbb{Z}$ and $a \leq b$, $[a, b] = \{a, a+1, \ldots, b-1, b\}$ is the finite sequence of integers from $a$ to $b$.

The matrix $I_n$ denotes the identity matrix from $\mathbb{R}^{n \times n}$.

B. Set and Function Notation

The empty set is denoted by $\emptyset$. With $r \in \mathbb{R}^n$ and $r \geq 0$, we define $\mathbb{R}^n(r) = \{x \in \mathbb{R}^n : |x| \leq r\}$. Also $\mathbb{R}^n(\epsilon) = \{x : \|x\|_\infty \leq \epsilon\}$. With $S_1, S_2 \subseteq \mathbb{R}^n$, then $S_1 \cap S_2 = \{a + b : a \in S_1, b \in S_2\}$ denotes the Minkowski sum and $S_1 \oplus S_2 = \{z : \forall b \in S_2, (z + b) \in S_1\}$ denotes the Minkowski difference between $S_1$ and $S_2$. If $\{S_1, \ldots, S_m\}$ is a collection of $m$ sets, with $S_1 \subseteq \mathbb{R}^n$, then $\bigcup_{i=1}^m S_i$, denotes their Minkowski sum defined by $\sum_{i=1}^m S_i = S_1 \oplus (S_2 \oplus \ldots \oplus S_m)$. If $f : S_1 \rightarrow S_2$, then the domain and range of $f$ are defined respectively by $\text{dom}(f) = S_1$ and $\text{ran}(f) = \{y : y = f(x), x \in S_1\}$. With $S \subseteq \mathbb{R}^n$, $\gamma > 0$ a scalar and $L \in \mathbb{R}^{n \times m}$, we have $\gamma S = \{\gamma s : s \in S\}$ and $LS = \{Ls : s \in S\}$. A polyhedral set $P \subseteq \mathbb{R}^n$ is a set that is an intersection of a finite number of half-spaces in $\mathbb{R}^n$. Such a set $P$ can be written in the h-form (i.e. hyperplane form) as $P = \{x : Lx \leq \ell\}$, with $L \in \mathbb{R}^{m \times n}$ and $\ell \in \mathbb{R}^m$ where $m < \infty$, a finite integer, is the number of half-spaces that intersect. We employ the convention to refer to the tuple $(L, \ell)$ as the h-form of set $P$. A set $S$ is bounded, if there exists a scalar $\epsilon$, with $0 < \epsilon < \infty$, such that $S \subseteq \mathbb{R}^n(\epsilon)$. A set is polytopic, if it is bounded and polyhedral. A set $S$ is symmetric, if it holds that $x \in S$ if and only if $-x \in S$. $|S|$ denotes the cardinality of set $S$, i.e. number of elements in $S$, where $|S| < \infty$ for a finite number of elements and $|S| = \infty$ otherwise.

III. THE UNCERTAIN SYSTEM AND THE COMMUNICATION CONSTRAINT

The problem is to control a discrete-time linear time-invariant system $S$ given by:

\[x(t + 1) = Ax(t) + Bu(t) + w(t)\]  \hspace{1cm} (1)

for $t \geq 1$. Here $x(t) \in \mathbb{R}^n$ denotes the system state, $u(t) \in \mathbb{R}^m$ is the controlled input and $w(t) \in \mathbb{R}^n$ denotes an external disturbance. Also $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. It is assumed that the pair $(A, B)$ is controllable. The initial state of the system is given by $x(1)$. The system is subject to input and state constraints of the form:

\[u(t) \in U, \ x(t) \in X, \ t \geq 1\]  \hspace{1cm} (2)

where $U \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$ are convex and polytopic sets, which contain the origin. The disturbance $w(t)$ is assumed bounded, i.e.:

\[w(t) \in W, t \geq 1\]  \hspace{1cm} (3)

where $W \subseteq \mathbb{R}^n$ is a polytopic and convex set, which contains the origin.

In this paper, it is assumed that the state $x(t)$, from the state sensor, is not available directly to the controller $K_t$, but must be communicated over a communication channel of limited capacity, as shown in Figure 1. The assumed model of Figure 1 reflects the case that the system state sensor and controller are not physically collocated, but are in fact at distinct physical locations. Information transfer regarding the system state is performed by a coding/decoding policy by means of the coder $C_t$ located with the state sensor and the decoder $D_t$ located with the controller. Based on information up to time $t$, the coder generates the codeword $s(t)$ for transmission over the channel. The transmitted codeword $s(t)$ is decoded by decoder $D_t$ into a state estimate $\hat{x}(t)$, which is applied to the controller $K_t$, generating the actuation signal $u(t)$. In particular, the following information patterns are imposed on the coder $C_t$, decoder $D_t$ and controller $K_t$:

\[s(t) = C_t(s(t)) \leq C - 2^R, \quad \hat{x}(t) = D_t(s(t)) \leq \hat{x}(t) \leq x(t)\]  \hspace{1cm} (4)

\[u(t) = K_t(\hat{x}(t))\]  \hspace{1cm} (5)

It is also assumed that both coder and decoder, at any time $t \geq 1$, have knowledge of the past and current instances of the control law (i.e. have knowledge of $\{K_t\}_{t \leq t1}$). The following finite bit-rate limited channel capacity constraint is imposed on the communication between system and controller:

**Assumption 1:** For $t \geq 1$, $|\text{ran}(C_t)| \leq C - 2^b$, where $B$ is a given and finite positive integer denoting the available channel capacity in bits. The stated assumption implies that, at each time $t$, the codeword $s(t)$ is chosen from a finite symbol alphabet of at most $C$ many symbols. This defines the restriction of information flow from the state sensor to controller due to the presence of the limited capacity communication channel. The state quantization error $e(t)$ between the state $x(t)$ and state estimate $\hat{x}(t)$ is defined by:

\[e(t) \triangleq x(t) - \hat{x}(t)\]  \hspace{1cm} (7)

The following definition will be required in this paper:

**Definition 1:** A matrix $M \in \mathbb{R}^{n \times n}$ is stable, if all its eigenvalues lie strictly inside the unit circle in the complex plane.

Finally, let $L \in \mathbb{R}^{n \times m}$ be a matrix to be employed in this paper, such that the following assumption holds:

**Assumption 2:** The matrix $A_L \equiv (A + BL)$ is stable.
IV. PRELIMINARIES AND EXISTING RESULTS

A. Uniform Quantization

The coding/decoding policy in this paper employs a uniform quantization of the state space (see e.g. page 105 of [21]). Here, bits are employed in coding the state space dimension, where $B = \sum_{n=1}^{N} B_n$. Define a vector $q \in \mathbb{Z}^n$ with $q(i) = 2^{|b|}$. Let $a \in \mathbb{R}^n$ and $a \geq 0$. The uniform quantization policy considered here partitions the set $\mathbb{B}^n(a) \rightarrow C = 2^B$ hypercubes. With:

$$I_1(a) \equiv \{ x : -a^{(1)} \leq x < -a^{(1)} + 2a^{(1)} \}$$
$$I_2(a) \equiv \{ x : -a^{(1)} + 2a^{(1)} \leq x < -a^{(1)} + 4a^{(1)} \}$$
$$\vdots$$
$$I_{q(i)}(a) \equiv \{ x : a^{(i)} \leq x \leq a^{(i)} \}$$

where $i \in [1,n]$, then for any $x \in \mathbb{B}^n(a)$, there exists a unique integer vector $s \in \mathbb{D} = \{ 1, q(1) \times 1, 1, q(2) \times 1 \times \ldots \times 1, q(n) \}$, such that $x \in I(s) \equiv I_{q(1)}(a) \times I_{q(2)}(a) \times \ldots \times I_{q(n)}(a)$. The quantization map over $\mathbb{B}^n(a)$ is now defined as the map, that, given a vector $x \in \mathbb{B}^n(a)$, returns the unique identifying integer $s$ identifying the quantization cell $I(s)$:

**Quantizer $Q_a$:**

$$Q_s : \mathbb{B}^n(a) \rightarrow \mathbb{D}$$

$s = Q_a(x)$, if $x \in I(s)$

Note that with this choice $\text{ran}(Q_a) = \mathbb{D}$ and $|\text{ran}(Q_a)| = |\mathbb{D}| = 2^B$. The inverse quantizer, which returns the center of the quantization cell, given its unique identifying integer vector, is defined as follows:

**Inverse Quantizer $Q^{-1}_a$:**

$$Q^{-1}_a : \mathbb{D} \rightarrow \mathbb{R}^n$$

$$\hat{x} = Q^{-1}_a(s) = \begin{pmatrix}
-a^{(1)} + a^{(1)}(2a^{(1)} - 1)/q(1) \\
-a^{(2)} + a^{(2)}(2a^{(2)} - 1)/q(2)
\end{pmatrix}$$

Define the following matrix:

$$F(q) \equiv \text{diag}(\frac{1}{q(1)}, \frac{1}{q(2)}, \ldots, \frac{1}{q(n)})$$

where $\text{diag}(\frac{1}{q(1)}, \frac{1}{q(2)}, \ldots, \frac{1}{q(n)})$ denotes a diagonal matrix with diagonal entries $\frac{1}{q(1)}, \frac{1}{q(2)}, \ldots, \frac{1}{q(n)}$. The following simple proposition now holds:

**Proposition 1:** Let $z \in \mathbb{B}^n(a)$, where $a \in \mathbb{R}^n$, $a \geq 0$. With $s = Q_a(z)$ and $\hat{x} = Q^{-1}_a(s)$, the quantization error $e = (z - \hat{x})$ satisfies $e \in \mathbb{B}^n(F(q)a)$.

**Proof:** Given in full version of this paper.

B. Coding/Decoding Policy

The coding/decoding policy proposed in this paper is dynamic, whereby at each time $t$, the difference between the actual state and a predicted state is coded for transmission across the channel (see e.g. [1],[2] and [8]). With $(a(t))_{t=1}^t$ a sequence to be defined below, the coder and decoder are defined as follows under the information patterns from Section III:

**Coder $C_t$:**

$$s(t) = Q_a(x(t) - x_a(t))$$

**Decoder $D_t$:**

$$\hat{x}(t) = Q^{-1}_a(s(t))$$

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Define the following matrix:

$$F(q) \equiv \text{diag}(\frac{1}{q(1)}, \frac{1}{q(2)}, \ldots, \frac{1}{q(n)})$$

where $\text{diag}(\frac{1}{q(1)}, \frac{1}{q(2)}, \ldots, \frac{1}{q(n)})$ denotes a diagonal matrix with diagonal entries $\frac{1}{q(1)}, \frac{1}{q(2)}, \ldots, \frac{1}{q(n)}$. The following simple proposition now holds:

**Proposition 1:** Let $z \in \mathbb{B}^n(a)$, where $a \in \mathbb{R}^n$, $a \geq 0$. With $s = Q_a(z)$ and $\hat{x} = Q^{-1}_a(s)$, the quantization error $e = (z - \hat{x})$ satisfies $e \in \mathbb{B}^n(F(q)a)$.

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The coding/decoding policy proposed in this paper is dynamic, whereby at each time $t$, the difference between the actual state and a predicted state is coded for transmission across the channel (see e.g. [1],[2] and [8]). With $(a(t))_{t=1}^t$ a sequence to be defined below, the coder and decoder are defined as follows under the information patterns from Section III:

**Coder $C_t$:**

$$s(t) = Q_a(x(t) - x_a(t))$$

**Decoder $D_t$:**

$$\hat{x}(t) = Q^{-1}_a(s(t))$$
set such that $\mathcal{F}_f \subseteq \mathbb{R}^n$. The finite horizon optimization problem $P_t(\mathcal{T}_0)$ to be employed as part of the tube MPC law, where $P_t(\mathcal{T}_0)$ is parameterized by time $t$ and an initial state $\mathcal{T}_0 \in \mathbb{R}^n$, over a horizon of length $N$, is defined as follows:

**Finite Horizon Optimization Problem $P_t(\mathcal{T}_0)$:**

$$V_t(\mathcal{T}_0) = \min_{(\mathcal{U}(k))_{k=0}^{N-1}} \sum_{k=0}^{N-1} ||\mathcal{T}(k)||^2_{L^2} + (1/2)||\mathcal{T}(N)||^2_{L^2}$$

subject to the constraints:

$$\mathcal{T}(k) \in \mathcal{U}(k)$$
$$\mathcal{T}(k) \in \mathcal{X}(k)$$
$$\mathcal{T}(N) \in \mathcal{F}_f$$

where:

$$\mathcal{T}(0) = \mathcal{T}_0$$
$$\mathcal{T}(k+1) = A\mathcal{T}(k) + B\mathcal{T}(k)$$
$$0 \leq k \leq N - 1$$

The following definitions will be required subsequently:

**Definition 2:** $P_t(\mathcal{T}_0)$ is feasible, if there exists a sequence $(\mathcal{T}(k))_{k=0}^{N-1}$ such that the constraints given by expressions (17) to (19) are satisfied.

**Definition 3:** Let $t \geq 1$. The region $\mathcal{T}_t \triangleq \{ \mathcal{T}_0 : P_t(\mathcal{T}_0) \text{ is feasible} \}$ is referred to as the nominal feasibility region at time $t$.

The time-variant controller $K_t$, considered in this paper has the form of the robust MPC law from [14], and is defined by:

**Time-Variant Robust MPC Law $K_t$:**

$$u(t) = K_t \hat{x}(t) = \pi(t) + L(\hat{x}(t) - \pi(t))$$
$$\pi(t+1) = A\pi(t) + B\pi(t)$$
$$\pi(t) = \mathcal{Y}_t(\pi(t))$$

where:

- $\mathcal{Y}_t : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{Y}_t$ is defined implicitly by the optimization problem $P_t(\pi(t))$.
- $\mathcal{Y}_t$ is applied in a receding horizon manner (see Chapter 4 of [19]). Namely, given $\pi(t)$ at each time step $t$, we solve the problem $P_t(\pi(t))$, obtain the optimal solution $\{\mathcal{Y}(k)\}_{k=0}^{N-1}$ and only apply the first input $\mathcal{Y}(0)$. In particular:

- $\pi(t) = \mathcal{Y}_t(\pi(t)) = \pi^*(0)$

- $\pi(t)$ and $\mathcal{Y}_t$ are controller internal dynamic variables referred to as the nominal state and nominal input respectively. The system in expression (24) is referred to as the nominal system.

- $L : \mathbb{R}^{m \times n}$ is any matrix satisfying Assumption 2.

**Remark 2:** Note that identical copies of sequences $(\pi(t))_{t=1}^\infty$ and $(\mathcal{Y}(k))_{k=1}^\infty$ are computed separately both inside the actual controller $K_t$ from Figure 1 and also inside the coder $C_t$ due to expression (10), requiring evaluation of the control law to compute $K_{t+1} \hat{x}(t+1)$. 

V. MAIN RESULT: DEFINITION AND STABILITY ANALYSIS OF ROBUST MPC WITH TIME-VARIANT SAFETY MARGINS, SUBJECT TO COMMUNICATION CONSTRAINTS

In this section, the definition of the controller and its stability and robustness properties are established. Since $x(t) = \hat{x}(t) + e(t)$ and $x(t+1) = \hat{x}(t+1) + e(t+1)$ from expression (7), then substituting these two latter expressions into expression (1), it follows that:

$$x(t+1) = Ax(t) + Bu(t) + v(t)$$
$$v(t) = Ax(t) - e(t+1) + w(t)$$
$$v(t) \in \mathcal{V}(t) \triangleq AB^n(b(t)) \oplus B^n(b(t+1)) \oplus \mathcal{W}$$

where the fact was employed that, due to Proposition 2, $e(t+1) \in B^n(b(t+1))$ and $e(t+1) \in B^n(b(t+1))$, since the set $\mathcal{B}^n(b(t+1))$ is symmetric (i.e. $x \in \mathcal{B}^n(b(t+1))$ implies $-x \in \mathcal{B}^n(b(t+1))$). Letting $\delta(t) = \hat{x}(t) - \mathcal{T}(t)$, then by an exactly analogous argument to the argument found in [14], by substituting (23) into (28) and subtracting (24) therefrom, it follows that:

$$\delta(t+1) = A_2 \mathcal{Y}(t) \oplus \mathcal{V}(t), \mathcal{V}(t) \subseteq \mathcal{W}$$

Here $\mathcal{V}(t)$ represents an augmented disturbance consisting of the external disturbance $w(t)$, as well as the state quantization error $e(t)$ over the channel. $\delta(t)$ denotes the difference between the decoded state $\hat{x}(t)$ and the nominal state $\mathcal{T}(t)$, which is described by the linear difference equation in expression (31). Note that Assumption 2 ensures that $A_2$ is a stable matrix, as a result of which $\delta(t)$ will remain bounded, provided that the sets $\mathcal{V}(t)$ remain bounded. Define the set $\mathcal{T}(t)$ recursively as follows:

$$\mathcal{T}(t+1) = A_2 \mathcal{Y}(t) \oplus \mathcal{V}(t), \mathcal{T}(t) \subseteq \mathcal{W}$$

where $0 \subseteq \mathcal{T}(t+1) = A_2 \mathcal{Y}(t) \oplus \mathcal{V}(t), \mathcal{T}(t) \subseteq \mathcal{W}$

$\forall t \geq 1$. The region $\mathcal{T}_t \triangleq \{ \mathcal{T}_0 : P_t(\mathcal{T}_0) \text{ is feasible} \}$ is referred to as the nominal feasibility region at time $t$.

The time-variant controller $K_t$, considered in this paper has the form of the robust MPC law from [14], and is defined by:

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$$u(t) = K_t \hat{x}(t) = \pi(t) + L(\hat{x}(t) - \pi(t))$$
$$\pi(t+1) = A\pi(t) + B\pi(t)$$
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where:

- $\mathcal{Y}_t : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{Y}_t$ is defined implicitly by the optimization problem $P_t(\pi(t))$.
- $\mathcal{Y}_t$ is applied in a receding horizon manner (see Chapter 4 of [19]). Namely, given $\pi(t)$ at each time step $t$, we solve the problem $P_t(\pi(t))$, obtain the optimal solution $\{\mathcal{Y}(k)\}_{k=0}^{N-1}$ and only apply the first input $\mathcal{Y}(0)$. In particular:

- $\pi(t) = \mathcal{Y}_t(\pi(t)) = \pi^*(0)$

- $\pi(t)$ and $\mathcal{Y}_t$ are controller internal dynamic variables referred to as the nominal state and nominal input respectively. The system in expression (24) is referred to as the nominal system.

- $L : \mathbb{R}^{m \times n}$ is any matrix satisfying Assumption 2.

**Remark 2:** Note that identical copies of sequences $(\pi(t))_{t=1}^\infty$ and $(\mathcal{Y}(k))_{k=1}^\infty$ are computed separately both inside the actual controller $K_t$ from Figure 1 and also inside the coder $C_t$ due to expression (10), requiring evaluation of the control law to compute $K_{t+1} \hat{x}(t+1)$.
The set $\Omega = \Omega_1 \ominus B^n(h(1))$ is introduced, which will denote the region of attraction, and which satisfies the following assumption:

**Assumption 7:** $\Omega \neq \emptyset$.

The following lemma will also be required subsequently:

**Lemma 1:** $\delta(t) \in \hat{\mathcal{T}}(t)$ for all $t \geq 1$.

**Proof:** Follows immediately from expressions (31) and (32), and the fact that $\delta(1) = 0$ due to expression (26).

The following assumption will also be required below:

**Assumption 8:** $\mathcal{X}(t) \neq \emptyset$ and $\mathcal{U}(t) \neq \emptyset$ for all $t \geq 1$.

The following theorem now presents the main result of this section and establishes feasibility, stability and constraint satisfaction properties, as well as the "tube interpretation", of the proposed control law:

**Theorem 1:** Let $x(1) \in \Omega$. The following hold for $t \geq 1$:

1) $F(t(\mathcal{T}(t))$ is feasible
2) $\mathcal{X}(t) \in \mathcal{X}(t)$ and $\gamma(t) \in \mathcal{U}(t)$
3) $\lim_{t \rightarrow \infty} \pi(t) = 0$ and $\lim_{t \rightarrow \infty} \pi(t) = 0$
4) $x(t) \in (\mathcal{T}(t) \ominus X_{0}(t)$ and $u(t) \in (\mathcal{T}(t) \ominus U_{0}(t)$
5) $x(t) \in X$ and $u(t) \in U$

**Proof:** Given in the full version of this paper.

### VI. EXAMPLE

Consider the system from expression (1) with:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The constraint sets are defined as:

- $\mathcal{X} = \{[x_1, x_2] : -250 \leq x_1 \leq 250, -50 \leq x_2 \leq 50\}$
- $\mathcal{U} = \{u : -6 \leq u \leq 6\}$
- $\mathcal{W} = \{[w_1, w_2] : -0.15 \leq w_1 \leq 0.15, -0.15 \leq w_2 \leq 0.15\}$

The initial state is chosen as: $x(1) = \{-171.8749, 43.7501\}$. The weighting matrices are chosen as $Q = I_2$, $R = 1$. The prediction horizon is chosen as $N = 15$. The channel capacity is chosen as $B = 10$ bits, with $B_1 = 5$ bits and $B_2 = 5$ bits. The set $\Omega$ is shown in Figure 4. Here $x(1) \in \Omega$. The constants $a_1$ and $d$ are computed as the values such that $B^n(a_1)$ and $B^n(d)$ are the smallest "rectangular" sets, such that $\mathcal{X} \subseteq B^n(a_1)$ and $\mathcal{W} \subseteq B^n(d)$. Due to the "rectangular" nature of the sets $\mathcal{X}$ and $\mathcal{W}$, $a_1 = (250, 50)'$ and $d = (0.15, 0.15)'$. Here with Assumptions 4 and 5 hold. Also $L = K_f = -K$ and $Q_f = P$, where $P$ is the positive definite solution of the Riccati equation $P = A^TPA + Q - K'(R + B'B)K$ and where $K = (R + B'B)^{-1}B'PA$ (see page 121 of [19]). As $(A, B)$ is controllable, $L$ satisfies Assumption 2. The terminal set $\mathcal{X}_f$ is computed as the maximal output admissible set (see e.g. page 121 of [19] or [24]) subject to the state and input constraint sets $\cup_{n = N+1}^\infty \mathcal{X}(t)$ and $\cup_{n = N+1}^\infty \mathcal{U}(t)$ respectively, and for such a choice of $\mathcal{X}_f$ and $K_f$, Assumption 6 is satisfied. For the given bit-rate, Assumptions 7 and 8 also hold. The matrix $|A|F(q)$ is stable with eigenvalues 0.0313 and 0.0313, which results in a bounded quantization error. The simulation is performed by employing the Matlab Multiparametric Toolbox (see [25]). The safety margins here are sufficiently small such that Assumptions 7 and 8 hold, whereas the methods from [18] and [26], due to larger safety margins, violate Assumption 8 for $t = 1$ due to larger safety margins, and are thus inapplicable under the given bit-rate.

The quantization error is shown in Figure 2. Due to the dynamic coding/decoding policy, the quantization error converges quickly to below $10^{-2}$ in the absolute value of each ordinate. The input space is shown in Figure 3. The input space safety margin $T_u(t)$ decreases in correspondence to the drop in quantization error drop from Figure 2. The state space is shown in Figure 4. Here the state $x(t)$ converges towards the origin in the sense of Theorem 1. Figure 5 shows an enlarged plot of the state space over the iterations.

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**Fig. 2.** Plot of absolute values of the ordinates $e^{(1)}(t)$ and $e^{(2)}(t)$ of the quantization error $e(t)$.

**Fig. 3.** Plot of $u(t)$, $\mathcal{T}(t)$ and $\mathcal{U}(t) \ominus T_u(t)$.

**Fig. 4.** Plot of state trajectory $x(t)$ and of the region of attraction $\Omega$.

**Fig. 5.** Plot of $x(t)$, $\mathcal{T}(t)$ and $\mathcal{T}(t) \ominus T_x(t)$ over the first six iterations for $1 \leq t \leq 6$. Also shown is the boundary $\partial \Omega$ of the region of attraction $\Omega$ from Figure 4.
1 \leq t \leq 6. The state \( x(t) \) evolves around the nominal state \( \pi(t) \) within a state space security margin \( T_s(t) \). Furthermore the state space security margin \( T_s(t) \) decreases in correspondence to the drop in quantization error drop from Figure 2. Note that the quantization error converges after about 5 iterations to below \( 10^{-2} \), however the safety margins \( T_s(t) \) and \( T_r(t) \) converge after about 10 iterations. This reflects the lag, due to the corrective linear part \( L(x(t) - \pi(t)) \) of the control law from expression (23), in correctly rejecting the propagation of uncertainty throughout the control loop.

VII. CONCLUSION AND FUTURE WORK

A time-variant robust MPC controller was presented, which employs a time-variant mechanism for update of the safety margins of the finite horizon optimization problem, in order to ensure robustness, but also to improve performance, in the presence of time-variant uncertainty due to communication constraints imposed by the presence of the channel of finite capacity. The proposed method achieves smaller safety margins over previous works, and thereby improves expected system performance. The theory was illustrated with a simple example, for which the safety margin definitions presented in this paper, for the given bit-rate, resulted in a feasible robust MPC controller, unlike with the larger safety margins definitions from previous works, as remarked in Section VI. Future work will focus on efficient computational methods of the proposed controller, and on extensions to more realistic channel models with data drop-outs and data losses.

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