On Uniform Global Asymptotic Stability of Adaptive Systems with Unknown Control Gains

Frédéric Mazenc  Michael Malisoff  Marcio de Queiroz

Abstract—We study adaptive tracking control problems for nonlinear systems in feedback form with multiple inputs and unknown high-frequency control gains. We design controllers that yield uniform global asymptotic stability for the error dynamics, which implies parameter estimation and tracking for the original systems. Our proof relies on a new global strict Lyapunov function construction. We apply our result to a brushless DC motor turning a mechanical load. We use integral input-to-state stability to quantify the effects of time-varying uncertainties on the motor electric parameters.

Key Words—Backstepping, Uncertain Systems, Strict Lyapunov Functions, Uniform Global Asymptotic Stability

I. INTRODUCTION

Lyapunov function methods are at the foundation of much of current research on the stabilization of nonlinear systems; see, e.g., [9], [13], [17], [18]. One important application occurs in adaptive control. Given a nonlinear system

\[ \dot{x} = f(t, x, u) \]

with a smooth reference trajectory \( x_R \) and a vector \( \Theta \) of uncertain constant parameters, the adaptive tracking control problem is as follows: Design a dynamic feedback

\[ u = u(t, \hat{x}, \hat{\Theta}), \quad \hat{\Theta} = \tau(t, x, \hat{x}, \hat{\Theta}) \]

where \( \hat{\Theta} \) is the estimate of \( \Theta \), such that (a) all closed-loop signals \( (x(t), \hat{\Theta}(t)) \) stay bounded and (b) \( x_R - x(t) \to 0 \) as \( t \to \infty \) [3], [8], [12], [19]. Solving this problem does not in general guarantee parameter identification, i.e., we might not have \( \lim_{t \to \infty} \hat{\Theta}(t) = \Theta \). In fact, one does not even know whether \( \hat{\Theta} \) converges to a constant vector in general [11]. Therefore, one cannot prove asymptotic stability for adaptive closed-loop systems without additional assumptions.

When an adaptive tracking controller does not give asymptotic stability, this means that the corresponding closed-loop system does not admit a strict Lyapunov function. Instead, only a nonstrict Lyapunov function can be found in this case. See Section II for the relevant definitions.

Asymptotic stability of adaptive systems normally depends on satisfying a persistence of excitation (PE) condition [17]. That is, a necessary (and sometimes sufficient) condition for identifying the parameters is that the regressor satisfies the PE inequality when integrated along the reference trajectory [7]. The connection between asymptotic stability, parameter identification, and the PE property was originally proven for linear plants but has since been established for certain nonlinear plants. In [20], PE was shown to give asymptotic parameter error convergence under the standard Slotine-Li adaptive controller for robot manipulators, and PE was proven to be necessary and sufficient for uniform global asymptotic stability (UGAS) of a class of nonlinear dynamics that includes the manipulator dynamics; see [15] for additional historical background. Later, [4] showed global exponential stability for a rotational mechanical system by designing the regressor in the adaptive control to establish the PE condition. In [15], we established UGAS for adaptively controlled first-order nonlinear dynamics \( \dot{x} = \omega(x) + u \) under PE, by converting a nonstrict Lyapunov function into an explicit global strict Lyapunov function, thereby solving the adaptive tracking control problem. (See [13] for a more general treatment of this “strictification” approach.)

The main benefit of using the strict Lyapunov-based approach for adaptively controlled nonlinear dynamics is that it provides a general framework for proving UGAS. This note takes the next step towards the generalization started in [15]. Here we consider dynamics in feedback form with unknown high-frequency control gains and multiple inputs. Unknown high-frequency gains naturally occur in electric motors, flight dynamics, and robot manipulators [12]. We explicitly construct a global strict Lyapunov function and a corresponding adaptive controller that give UGAS of the closed loop augmented error dynamics on its domain. This differs from several earlier treatments of adaptive control for dynamics in feedback form such as [12] or with unknown gain such as [2], which do not prove UGAS.

We demonstrate our approach using a brushless DC motor turning a load. Our strict Lyapunov approach leads to an integral input-to-state stability (iISS) robustness analysis under time-varying uncertainties in the motor electric parameters, which arise from variations in the winding resistances. See, e.g., [1], [22], [23] for the importance of iISS. This work is not a straightforward extension of [15]; rather, it provides a new strictification approach that significantly extends the known constructions.

II. DEFINITIONS

Let \( \mathcal{X} \subseteq \mathbb{R}^n \) be any open set containing the origin. A function \( \alpha : [0, \infty) \times \mathcal{X} \to [0, \infty) \) is called positive definite provided \( \inf_{t \geq 0} \alpha(t, x) > 0 \) for all \( x \in \mathcal{X} \setminus \{0\} \) and \( \alpha(t, 0) = 0 \) for all \( t \geq 0 \). A modulus with respect to \( \mathcal{X} \) is any continuous function \( \alpha : \mathcal{X} \to [0, \infty) \) satisfying:
(A) \( \lim_{|\zeta| \to \infty} \alpha(\zeta) = +\infty \) (which holds vacuously if \( X \) is bounded) and (B) \( \lim_{|\zeta| \to \xi^*} \alpha(\zeta) = +\infty \) for each point \( \xi^* \) in the boundary of \( X \) (which holds vacuously if \( X = \mathbb{R}^n \)). Here, \( |\cdot| \) is the usual Euclidean norm. A function \( V \) such that 
\[ \zeta \to \inf V(t, \zeta) \] is a modulus with respect to \( X \) is called proper (on \( X \)). A (global) nonstrict Lyapunov function for a system \( \xi = \mathcal{G}(t, \zeta) \) with state space \( X \) satisfying \( \mathcal{G}(t, 0) = 0 \) for all \( t \in \mathbb{R} \) is any \( C^1 \) function \( V : [0, \infty) \times X \to [0, \infty) \) such that \( V \) is positive definite and proper and \( X \ni \zeta \mapsto W_V(\zeta) = -\sup_{t} [V(t, \zeta) + V(\zeta, \mathcal{G}(t, \zeta))] \) is everywhere nonnegative; if, in addition, \( W_V \) is positive definite, then \( V \) is called a (global) strict Lyapunov function for the system. 

A continuous function \( \gamma : [0, \infty) \to [0, \infty) \) provided it is unbounded and strictly increasing and \( \gamma(0) = 0 \). A continuous function \( \beta : [0, \infty) \times [0, \infty) \to [0, \infty) \) of class \( \mathcal{K} \) (written \( \beta \in \mathcal{K} \)) provided for each fixed \( s \geq 0 \), the function \( \beta(\cdot, s) \) belongs to class \( \mathcal{K}_s \), and for each fixed \( r \geq 0 \), the function \( \beta(r, \cdot) \) is non-increasing and \( \beta(r, s) \to 0 \) as \( s \to \infty \). The UGAS condition for \( \zeta = \mathcal{G}(t, \zeta) \) is the requirement that there exist a modulus \( \Delta \) with respect to \( X \) and a function \( \beta \in \mathcal{K} \) so that 
\[ |\zeta(t)| \leq \beta(\Delta(\zeta(t_0), t-t_0)) \] for all trajectories \( \zeta : [t_0, \infty) \to X \) of the dynamics, all initial times \( t_0 \), and all \( t \geq t_0 \). The iISS condition for a system \( \zeta = \mathcal{H}(t, \zeta, \mathbf{d}) \) with state space \( X \) and measurable essentially bounded functions \( \mathbf{d} : [0, \infty) \to D \) (valued in a given subset \( D \) of a Euclidean space) says that there exist functions \( \beta \in \mathcal{K} \) and \( \alpha, \gamma \in \mathcal{K} \) and a modulus \( \Delta \) with respect to \( X \) so that for each \( \mathbf{d} \) and each initial condition \( \zeta(t_0) = \zeta_0 \), the corresponding trajectory \( \zeta(t; t_0, \zeta_0, \mathbf{d}) \) of \( \zeta = \mathcal{H}(t, \zeta, \mathbf{d}) \) satisfies 
\[ \alpha(\zeta(t; t_0, \zeta_0, \mathbf{d})) \leq \beta(\Delta(\zeta_0), t-t_0) + \int_{t_0}^{t} \gamma(|\mathbf{d}(\xi)|) d\xi \] for all \( t \geq t_0 \). The \( \mathbf{d} \)'s represent uncertainty. Here we assume that the dynamics are sufficiently regular so that all trajectories are uniquely defined on \([t_0, \infty)\) for all initial conditions \( \zeta(t_0) = \zeta_0 \in X \) and all \( \mathbf{d} : [0, \infty) \to D \).

III. MAIN RESULT

Consider the dynamics
\[
\begin{align*}
\dot{x} &= f(x) \\
\dot{z}_i &= g_i(x_i) + k_i(x_i) \cdot \theta_i + \psi_i u_i, & i = 1, 2, \ldots, s
\end{align*}
\] (4)
with unknown constant parameters \( \psi = (\psi_1, \ldots, \psi_s) \in \mathbb{R}^s \) (called high-frequency gains) and \( \theta = (\theta_1, \ldots, \theta_s) \in \mathbb{R}^{p_1 \times \cdots \times p_s} \), and state vector \( \xi = (x, z) \in \mathbb{R}^r \times \mathbb{R}^s \), where \( f : \mathbb{R}^{r+s} \to \mathbb{R}^r \) and \( g_i : \mathbb{R}^{r+s} \to \mathbb{R} \) and \( k_i : \mathbb{R}^{r+s} \to \mathbb{R} \) for \( i = 1, 2, \ldots, s \) are \( C^2 \). The \( \psi_i \)'s, \( r \) and \( s \) are any positive integers, and \( u = (u_1, u_2, \ldots, u_s) \in \mathbb{R}^s \) is the control.

Fix any \( C^2 \) function \( V_R = (x, z) \in [0, \infty) \to \mathbb{R}^r \times \mathbb{R}^s \) having some period \( T > 0 \) that satisfies \( \dot{x}(t) = f(\xi_R(t)) \) everywhere. We refer to \( \xi_R \) as our reference trajectory. The adaptive estimation problem for (4) is to design a closed loop controller \( u \) to simultaneously (I) estimate the parameter vector \( \Theta = (\theta, \psi) \in \mathbb{R}^{p_1 \times \cdots \times p_s} \) and (II) force the state trajectories \( \xi(t) = (x(t), z(t)) \) of (4) to track \( \xi_R(t) \). To solve this problem, it suffices to render the corresponding closed loop augmented error dynamics UGAS; see (10) below. We also wish to explicitly construct a global strict Lyapunov function for the augmented error dynamics, to get \( \alpha, \beta, \gamma, \) and \( \Delta \) in the UGAS and iISS estimates; see below. We set \( \mathcal{F}(t, x) = f(x + \xi_R(t)) - f(\xi_R(t)) \), so \( \mathcal{F}(t, 0) = 0 \) for all \( t \).

We also set \( k_i(\xi) = (k_{i1}(\xi), \ldots, k_{ip_i}(\xi)) \), \( \gamma(\xi(t)) = \dot{\gamma}(\xi_R(t)) \), \( \lambda_i(t) = (\lambda_{i1}(t), \ldots, \lambda_{ip_i}(t)) \) for \( i = 1, \ldots, s \), where \( \dot{\gamma} = (\dot{\gamma}_{i1}, \ldots, \dot{\gamma}_{ip_i}) \) is the unknown parameters \( \theta_i \) and \( \psi_i \) for \( i = 1, 2, \ldots, s \), where \( \dot{\vartheta}_i + \theta_i = \dot{\theta}_i + \theta_i \) and \( \dot{\varphi}_i + \psi_i = \dot{\psi}_i \). The \( C^1 \) functions \( \varpi_{i,j} \) and
\( \dot{\psi}_i \) are to be determined. For any trajectory \( t \mapsto (\xi, \hat{\theta}, \hat{\psi})(t) \) of (8) for which \( |\hat{\theta}_{i,j}(t_0)| < \theta_M \) and \( \hat{\psi}_i(t_0) \in (\psi, \psi) \) for all \( i \in \{1, 2, \ldots, s\} \) and \( j \in \{1, 2, \ldots, p_i\} \), we have \( |\hat{\theta}_i(t) \leq \theta_M \) and \( \hat{\psi}_i(t) \in (\psi, \psi) \) for all \( t \geq t_0 \) in its domain, and all \( i \) and \( j \). Therefore, we can choose
\[
 u_i(t, \xi, \hat{\theta}, \hat{\psi}) = \frac{\nu_i(t, \xi) - \nu_i(\xi, \nu_i) - 2 \hat{\psi}_i(t) - 2 \hat{\psi}_i(t) - \hat{\psi}_i(t)}{\hat{\psi}_i(t)} \tag{9}
\]
for all \( i \in \{1, 2, \ldots, s\} \) as the feedback components in (8) to get the closed loop augmented error dynamics
\[
\begin{align*}
\dot{\theta}_i &= F(t, \xi) \\
\dot{\xi}_i &= v_{f,i}(t, \xi) + k_i(\xi + \xi_R(t)) \cdot \theta_i \\
\hat{\psi}_i &= - \left( \nu_i(t, \xi) + \hat{\psi}_i(t, \xi, \hat{\theta}_i, \hat{\psi}_i) \right), \quad 1 \leq i \leq s, 1 \leq j \leq p_i \\
\hat{\psi}_i &\leq - \left( \frac{\nu_i(t, \xi)}{\theta_i} \hat{\psi}_i \right), \quad 1 \leq i \leq s,
\end{align*} \tag{10}
\]
since the \( \theta_i \)'s and \( \hat{\psi}_i \)'s are constant. The state space for (10) is
\[
\mathcal{X} = \mathbb{R}^{r+s} \times \left( \prod_{i=1}^s \left\{ (\theta_{i,j} - \theta_M, \theta_{i,j} + \theta_M) \right\} \times \left( \prod_{i=1}^s \left( \hat{\psi}_i - \hat{\psi}_i, \hat{\psi}_i + \hat{\psi}_i \right) \right) \right) \subseteq \mathbb{R}^{r+s+p_1+ \ldots + p_s}.
\]
We choose the \( C^1 \) functions
\[
\begin{align*}
\omega_{i,j} &= - \frac{\partial \nu_i}{\partial \theta_i}(t, \xi) k_{i,j}(\xi + \xi_R(t)) \quad \text{and} \\
\Omega_i &= - \frac{\partial \nu_i}{\partial \xi}(t, \xi) u_i(t, \xi, \hat{\theta}_i, \hat{\psi}_i),
\end{align*} \tag{11}
\]
for \( 1 \leq i \leq s \) and \( 1 \leq j \leq p_i \), and we do our analysis on the resulting closed loop system (10). The choices (11) in (10) and Assumption 3 (applied with \( X = \hat{x} \) and \( Z = \hat{z} \)) guarantee that along all trajectories of (10), the time derivative of \( V_1 : [0, \infty) \times \mathcal{X} \rightarrow [0, \infty) \) defined by
\[
\begin{align*}
V_1(t, \xi, \hat{\theta}, \hat{\psi}) &= V(t, \xi) + \sum_{i=1}^s p_i \int_0^{\hat{\theta}_i} \frac{m}{\theta_M - m - \hat{\theta}_i} \, dm \\
&\quad + \sum_{i=1}^s \int_0^{\hat{\psi}_i} \frac{m}{\hat{\psi}_i - m + \hat{\psi}_i} \, dm \tag{12}
\end{align*}
\]
satisfies \( \dot{V}_1 \leq -W(\hat{\xi}) \).

We convert \( V_1 \) into a strict Lyapunov function \( V^\sharp \) for (10). We use the real valued functions
\[
\begin{align*}
\Psi_i(t, \xi, \hat{\theta}, \hat{\psi}) &= - \frac{d}{\partial \theta_i}(\hat{\psi}_i(t)) \quad \text{and} \\
\Omega_i(t, \xi, \hat{\theta}, \hat{\psi}) &= \Omega_i(t, \xi, \hat{\theta}, \hat{\psi}) \tag{13}
\end{align*}
\]
where \( \lambda_i(t) = (k_i(\xi_R(t)), \hat{\psi}_i(t_0)) \) and \( \Omega_i(t) = \frac{\partial \nu_i}{\partial \xi}(\hat{\theta}_i, \hat{\psi}_i) \Omega_i(t) \). According to Lemma 1, \( \alpha_i(\hat{\theta}_i, \hat{\psi}_i) \) and \( \alpha_i(\hat{\theta}_i, \hat{\psi}_i) \) are such that
\[
\frac{d}{dt}(\hat{\psi}_i(t)) \leq -c_0 \frac{\theta_i(t)}{\theta_M} + G_3(|\theta_i(t)|) + G_3(|\hat{\psi}_i(t)|).
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\]
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\[
\frac{d}{dt}(\hat{\psi}_i(t)) \leq -c_0 \frac{\theta_i(t)}{\theta_M} + G_3(|\theta_i(t)|) + G_3(|\hat{\psi}_i(t)|).
\]

We can simplify the dynamics (10) by
\[
\dot{\hat{\psi}}_i = \frac{d}{dt}(\hat{\psi}_i(t)) \leq -c_0 \frac{\theta_i(t)}{\theta_M} + G_3(|\theta_i(t)|) + G_3(|\hat{\psi}_i(t)|).
\]

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\]

IV. SKETCH OF THEOREM 1

The complete proof of Theorem 1 in [16], here is a sketch. We begin by writing the \( \hat{\xi} \) dynamics from (10) as
\[
\dot{\hat{\xi}} = \frac{\partial \psi}{\partial \psi} \lambda_i(t) \alpha_i(\hat{\theta}_i, \hat{\psi}_i) + \rho_i(t, \theta, \hat{\psi}, \xi) \tag{18}
\]
where
\[
\rho_i(t, \theta, \hat{\psi}, \xi) = \left[ k_i(\xi_R(t)) - k_i(\xi_R(t)) \right] \cdot \frac{\hat{\theta}_i \hat{\psi}_i - \hat{\psi}_i}{\hat{\psi}_i} + \frac{\hat{\psi}_i}{\psi} \cdot v_{f,i}(t, \xi) - \frac{\hat{\psi}_i}{\psi} \cdot g_i(\xi_R(t)) - g_i(\xi_R(t)) \tag{19}
\]
and we use \( \hat{\theta}_i \hat{\psi}_i - \hat{\psi}_i \hat{\theta}_i = \hat{\psi}_i \hat{\theta}_i - \hat{\psi}_i \hat{\theta}_i \). Assumption 1 gives everywhere positive increasing functions \( G_1, G_2 \in C^1 \) such that
\[
\begin{align*}
\frac{\partial \alpha_i}{\partial \theta_i} \hat{\theta}_i + \frac{\partial \alpha_i}{\partial \psi_i} \hat{\psi}_i &\leq |\hat{\xi} G_1(|\hat{\xi}|)| \quad \text{and} \\
\rho_i(t, \theta, \hat{\psi}, \xi) &\leq |\hat{\xi} G_2(|\hat{\xi}|) | \tag{20}
\end{align*}
\]
Using our properties of \( G_1 \) and \( G_2 \), and recalling Assumption 2 and the equality in (16), we get
\[
\frac{\partial \alpha_i}{\partial \theta_i} \hat{\theta}_i + \frac{\partial \alpha_i}{\partial \psi_i} \hat{\psi}_i \leq \frac{\partial \alpha_i}{\partial \theta_i} \hat{\theta}_i + \frac{\partial \alpha_i}{\partial \psi_i} \hat{\psi}_i + |\hat{\xi} G_1(|\hat{\xi}|)| + |\hat{\xi} G_2(|\hat{\xi}|) | \tag{21}
\]
where
\[ G_3(r) = \max_i \left\{ r \max_i |\dot{\lambda}_i(t)| \right\} \]
\[ + 2r \max_i \left( |\lambda_i(t)| + \frac{1}{T_r} |\dot{\gamma}_i(t)| \right) \{ G_1(r) + G_2(r) \}. \]  (22)

Using the triangular inequality \( G_3|\alpha_i| \leq c_0|\alpha_i|^2/(27 \psi^2) + T \sqrt{G_3^2 + (2c_0) \psi^2} \) on (21), and recalling that \( V^i \) is proper on \( \mathbb{R}^{n+2} \), we can find an everywhere positive \( C^1 \) function \( G_4 \) such that
\[ \Upsilon_i \leq -c_0 \frac{G_4(V_1)}{|\alpha_i(\theta_i, \psi_i)'|^2 + |\dot{\xi}_i|^2}. \]  (23)

Finally, since Assumption 3 guarantees positive definite quadratic lower bounds for \( V_1 \) and \( W \) in some neighborhood \( N \) of the origin, and since \( V_1 \leq -W \) along all trajectories of (10), simple calculations provide a function \( K \in C^1 \cap K_{\infty} \) such that (17) is positive definite and bounded from below by a positive definite quadratic function of \( Y = (\xi, \theta, \psi) \), and such that
\[ \dot{V}^2 \leq -c_0 \frac{G_4(V_1)}{\sum_{i=1}^s |\alpha_i(\theta_i, \psi_i)'|^2 + |\dot{\xi}_i|^2}. \]  (24)
along all trajectories of (10). The right side of (24) is negative definite in \( Y \in \mathcal{X} \), so \( V^2 \) is a global strict Lyapunov function for (10). The UGAS property now follows from standard arguments [16]. \( \Box \)

Remark 2: The importance of Theorem 1 is that it reduces the search for a global strict Lyapunov function to the construction of \( K \in C^1 \cap K_{\infty} \) from (17). The only requirements for \( K \) are that (17) must be proper and positive definite, and (24) must hold. In Section V-D, we illustrate how \( K \) can be constructed easily, when the data from Assumptions 1-3 are available. Note too that the explicit formula for \( V^2 \) in (17) leads to closed form expressions for the functions \( \beta \) and \( \Delta \) in the UGAS estimate [16]. \( \Box \)

V. APPLICATION OF THEOREM 1 TO DC MOTOR

A. Model

Consider the four-dimensional dynamics
\[
\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -B_1 y_2 - N \sin(y_1) + K_r [K_B \xi_1 + 1] \xi_2 \\
\dot{\xi}_1 &= H_1(y, \xi) \beta_1 + \gamma_1 u_i, \quad i = 1, 2
\end{align*}
\]  (25)

for a brushless DC motor turning a mechanical load, where \( H_1(y, \xi) = (-\xi_1 - \xi_2 \xi_2, -y_2) \) and \( H_2(y, \xi) = (-\xi_2, y_2) \) and \( B, M, N, K_r, \) and \( K_B \) are positive constants [5]. Here, \( B \) is the viscous friction coefficient, \( M \) is the mechanical inertia of the system, \( N \) is related to the load mass and gravitational constant, and \( K_r \) and \( K_B \) are torque transmission coefficients. The unknown vectors \( \beta_1 \in \mathbb{R}^3 \) and \( \beta_2 \in \mathbb{R}^2 \) and unknown scalars \( \gamma_1 \) and \( \gamma_2 \) are the motor electric parameters, depending on the winding resistance, winding inductances, and the number of permanent magnet rotor pole pairs. The \( u_i \)'s are the controls. We choose the values \( B = 0.035 \text{ Nm-s/rad}, \ M = 0.048 \text{ kg-m}^2, \ N = 1.7195 \text{ kg-m}^2/s^2, \ K_r = 0.506 \text{ Nm/A} \) and \( K_B = 1.01 \times 10^{-3} \text{ Nm/A}^2 \) for the mechanical constants from [5].

To simplify later calculations, we use
\[\tau = \frac{B_1}{B}, \quad x_1 = y_1, \quad x_2 = \frac{M_B}{B}, \quad z_1 = K_B \xi_1, \]
\[z_2 = \frac{M_B^2 K_B \xi_2}{B}, \quad \alpha = \frac{B M_B}{B}, \quad \theta_1 = \frac{K_r M_B}{B}, \]
\[\psi_1 = \frac{K_r M_B}{B}, \quad \theta_2 = \frac{K_r M_B^2}{B}, \quad \text{and} \quad \psi_2 = \frac{K_r M_B^3}{B}. \]

We also choose \( \xi = (x, z) \),
\[k_1(\xi) = \left( -\frac{\alpha B}{B}, -\frac{\alpha^2 B}{B} z_2, -\frac{\alpha B}{B} x_2 \right) \]
\[k_2(\xi) = \left( -\frac{\alpha^2 B}{B} z_2, -\frac{\alpha B}{B} x_2 \right). \]  (26)

Then \( k_i(\xi) = H_i(y, \xi) \) for \( i = 1, 2 \) so if we now denote the time derivative with respect to \( \tau \) by a dot, then (25) becomes
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_2 - \frac{\alpha B}{B} \sin(x_1) + [z_1 + 1] z_2 \\
\dot{z}_1 &= k_1(\xi) \theta_1 + \psi_1 u_i, \quad i = 1, 2
\end{align*}
\]  (27)

which is a special case of (4) with \( r = s = 2, \ p_1 = 3, \ p_2 = 2, \ g_1 = 0 \) for \( i = 1, 2 \), and \( f(\xi) = (x_2, -x_2 - \pi \sin(x_1) + [z_1 + 1] z_2)^\top \). We apply Theorem 1 to (27), with \( T = 2\pi \).

B. Reference Trajectory

Since \( f(\xi) = (x_2, -x_2 - \pi \sin(x_1) + [z_1 + 1] z_2)^\top \), there are many possible choices for \( C^2 \) reference trajectories \( \xi_R = (x_{R,1}, x_{R,2}, z_{R,1}, z_{R,2}) \) that satisfy our requirement \( x_R = f(\xi_R) \). For concreteness, we choose
\[
\begin{align*}
x_{R,1}(t) &= -\cos(t), \quad x_{R,2}(t) = \sin(t), \quad z_{R,1}(t) = -\frac{2+3 \cos(t)}{2} \cos(t) + \sin(t) - \pi \sin(\cos(t)), \quad z_{R,2}(t) = \frac{2+3 \cos(t)}{2} \cos(t) + \sin(t) - \pi \sin(\cos(t))
\end{align*}
\]  (28)
in all of what follows. Our formulas (28) easily give
\[\max_{t \geq 0, i = 1, 2} \max \{ |K_R(t)|, |\dot{z}_{R,i}(t)|, |\ddot{z}_{R,i}(t)| \} \leq R_e, \]  (29)
where \( R_e = 30 + 5\pi \).

C. Verifying the Assumptions of Theorem 1

To check Assumption 2, we find the eigenvalues of the matrices \( P_i \) defined in (5). With the above values of \( B, M, N, K_r, \) and \( K_B \), Mathematica gives the eigenvalues \( \{1.41789 \times 10^6, 1921.77, 3.05671, 0.0755335\} \) for \( P_1 \) and \( \{13963.1, 13333.7, 0.3322\} \) for \( P_2 \). Therefore, the \( P_i \)'s are positive definite, and we can satisfy Assumption 2 with the lower bound \( c_0 = 0.07 \) for their eigenvalues.

To check the assumptions of Theorem 1, it remains to build the Lyapunov function \( V \) and feedback \( v_f \) for the auxiliary system (7), as required by Assumption 3. In this case, the auxiliary system is the special case of the dynamics
\[
\begin{align*}
X_1 &= X_2 \\
X_2 &= -X_2 - \alpha \sin(X_1 + x_{R,1}(t)) + \beta \sin(x_{R,1}(t)) + Z_1 Z_2 + z_{R,2}(t) Z_1 \\
\dot{Z}_i &= v_f, \quad i = 1, 2
\end{align*}
\]  (30)
when the constants \( \varepsilon_i \) are zero, but we study the more general setting (30) with possibly nonzero \( \varepsilon_i \)'s since it will facilitate our robustness analysis in Section V-E. The following is shown in [16], using backstepping:
Lemma 2: Let $\epsilon_1$ and $\epsilon_2$ be any scalars, and let $\bar{\ell} > 0$ be any constant so that
\[
\bar{\ell} \geq 2 + \frac{128(\pi + 1) \max_i |\epsilon_i|}{m},
\]
where $m = \min_{i(2\pi + 1)}$. Then the time derivative of
\[
V(t, X, Z) = \bar{\ell}^2 (X_1^2 + X_1 X_2 + X_2^2) + \frac{1}{2} Z_2^2 + 8 \left[ Z_2 - \sqrt{\pi} \sin(x_1 + x_1, t) \sqrt{\pi} \sin(x, t) - 0.5 X_1 \right]^2
\]
along all trajectories of (30), where $X = (X_1, X_2)$ and $Z = (Z_1, Z_2)$, in closed loop with the feedbacks
\[
v_f,1(t, X, Z) = -\bar{\ell} Z_1 - \frac{\bar{\ell}}{2} [X_1 + 2 X_2] [Z_2 + z_{R,2}(t)]
\]
\[
v_f,2(t, X, Z) = \frac{1}{2} \left[ \sqrt{\pi} \cos(x_1 + x_1, t) - 0.5 \right] X_2 + \frac{\sqrt{\pi} \cos(x_1 + x_1, t) - 0.5 \pi \sin(x_1 + x_1, t)}{(x_1 + x_1, t) + 1} \right] Z_2
\]
\[
\frac{\sqrt{\pi} \cos(x_1 + x_1, t) - 0.5 \pi \sin(x_1 + x_1, t)}{(x_1 + x_1, t) + 1} \right] Z_2
\]
\[
\Rightarrow V(t, X, Z) \leq -0.5 \bar{\ell} m^2 |X(X, Z)|^2.
\]
Also,
\[
\frac{1}{\pi(0.5 + \pi)} |X(X, Z)|^2 \leq V(t, X, Z),
\]
\[
\max_i |v_f,1(t, X, Z)| \leq 9\bar{\ell}(1 + R_u)(\pi + 1)(1 + |X(X, Z)|)
\]
\[
\max_i |v_f,2(t, X, Z)| \leq 32 (1 + \pi) |X(X, Z)|
\]
\[
\max_i |\partial v_f| (t, X, Z) \leq 32 (1 + \pi) |X(X, Z)|
\]
for all $(t, X, Z) \in [0, \infty) \times \mathbb{R}^4$. The lower bound on $V$ from (36) gives
\[
V_1(t, Y) \geq L |Y|^2,
\]
where
\[
L = \min \left\{ \frac{1}{\pi(0.5 + \pi)} k \quad \frac{2\bar{\ell} m^2}{2 \bar{\ell} m^2 - \psi^2} \right\}
\]
for all $Y = (\xi, \dot{\theta}, \psi) \in X$.

D. Global Strict Lyapunov Function Construction

The strict Lyapunov function $V_1$ for the corresponding error dynamics (10) is now given by (17) with $T = 2\pi$, $c_0 = 0.07$, and any function $K$ satisfying the two requirements from Remark 2. Here we derive a formula for $K$ by constructing the functions $\mathcal{G}_i$ from the proof of Theorem 1; see [16] for more details on the constructions.

We use the bound (29) and the constants
\[
\bar{K} = \max \left\{ 1, \frac{1}{K_b}, \frac{B^3}{M^3 K_b}, \frac{B^2}{M^2 K_b} \right\}
\]
and
\[
\gamma_* = \left\{ 2\bar{K} R_*^2 + \frac{4\pi^2 R_*^4}{\psi} \right\} \max \left\{ 1 + 2 \theta M, 2 \psi^2 \right\}.
\]
The integrals $\Omega_i$ defined in (14) with $T = 2\pi$ satisfy $\max_i |\Omega_i(t)| \leq 8k^2 R_*^2 \pi^2$. Our formulas (26) for the $k_i$'s give $\max_i |k_i(\xi)| \leq 3\sqrt{1 + |\xi|^2}$. Therefore, (15) and (36) give the global bounds
\[
\max_i |\xi_i(t, Y)| \leq \gamma_* |Y|^2,
\]
\[
\max_i |\xi_i(t)| \leq |\xi_i(t)| \theta_i(\xi_i(t))
\]
\[
\max_i |\psi_i(t)| \leq |\xi_i(t)| \psi_i(\xi_i(t)).
\]
where $\theta_i(r) = 288K \left( 1 + \pi \right) \theta M \left( 1 + 2 \left[ r^2 + R_*^2 \right] \right)$ and
\[
\psi_i(r) = \frac{32}{27} \psi (1 + \pi) \left[ 9(1 + R_u)(\pi + 1)(1 + r) + 9k \theta M \left( 1 + 2 \left[ r^2 + R_*^2 \right] \right) + R_u \right].
\]
Using the bound for $D_{\alpha_i}$ from (15), we can then take $\mathcal{G}_1(r) = (3(\overline{v} + \theta_M) + 1) (\theta_i(r) + \psi_i(r))$.

To find the other $\mathcal{G}_i$'s, first apply the Mean Value Theorem to the $k_i$'s to get $\max_i \xi_i(x_i, \theta_i, \psi_i, \xi_i) \leq \xi_i(\mathcal{G}_1(\xi_i))$, with the choice $\mathcal{G}(r) = \left\{ 6K (2r + 2R_u + 1) \psi_M + 9k \bar{K} (1 + r) (1 + R_u) (1 + \pi) / \psi \right\}$. By bounding $\max_i \xi_i(\tilde{\xi}(t))$, it follows from (30) that (21) and (23) hold with $\gamma\mathcal{G}(r) = 2\mathcal{G}(r)$ and
\[
\mathcal{G}_4(r) = \frac{4}{\pi} \left[ \frac{\psi}{G} \right] \left( \sqrt{r/L} \right),
\]
where
\[
\gamma\mathcal{G}(r) = \left[ \frac{2K R_*^2 + 4\pi^2 R_*^4}{\psi} \right] (\mathcal{G}_1(r) + \mathcal{G}_2(r)) + \bar{K} \sqrt{r}. \]

Therefore, choosing
\[
K(r) = \frac{4}{\pi} \gamma\mathcal{G}(r) + \frac{1}{2r} \int_0^r \left( \frac{c_0}{\psi} + 4\gamma\mathcal{G}(s) \right) ds,
\]
and recalling (39) and the lower bound for $V_1$ from (36), we get $V_1^2(t, Y) \geq K(L + Y)^2 - 2\gamma\mathcal{G}_1(\tilde{\xi}) \leq 2\gamma\mathcal{G}_1(\tilde{\xi})$ (using the $4\gamma\mathcal{G}_1$ term from (40)), and the necessary decay estimate
\[
V_1^2 \leq K'(V_1) V_1 - \frac{c_0}{\psi} \sum_{i=1}^2 |\alpha_i(\xi_i, \psi_i)|^2 + 2\gamma\mathcal{G}_1(\tilde{\xi}) \leq - \frac{c_0}{\psi} \sum_{i=1}^2 |\alpha_i(\xi_i, \psi_i)|^2 + 2\gamma\mathcal{G}_1(\tilde{\xi})
\]
follows from (23), (35), and the integral term in (40) because $\dot{V}_1 \leq -0.5 \bar{\ell} m^2 |\xi|^2$ along the system trajectories.

E. Robustness

To demonstrate the value of our strict Lyapunov function construction, we prove an iISS result for cases where there are time-varying additive uncertainties on the parameters $\theta_1$ and $\theta_2$ in (27). This gives the perturbed motor dynamics
\[
\begin{aligned}
\dot{\hat{x}}_1 &= x_2 \\
\dot{\hat{x}}_2 &= -x_2 - \pi \sin(x_1) + [z_1 + 1] z_2 \\
\dot{\hat{z}}_1 &= k_1(\xi) \theta_1 - \frac{1}{k_b} z_1 \delta_1 + \psi_1 u_1 \\
\dot{\hat{z}}_2 &= k_2(\xi) \theta_2 - \frac{1}{k_b} z_2 \delta_2 + \psi_2 u_2 ,
\end{aligned}
\]
where the $\delta_i$'s are measurable essentially bounded functions.

The above uncertainties capture the well-known variation in the winding resistances during the motor operation (e.g., ohmic heating may cause 100% resistance variation from the nominal value [14]). The functions $k_i$ are from (26). We restrict to those uncertainties $\delta = (\delta_1, \delta_2)$ for which $|\delta_i| \leq \delta$, where $\delta > 0$ is an arbitrary constant.

We pick $V$ and $v_f$ as in Lemma 2, where $\bar{\ell}$ is any fixed constant such that
\[
\bar{\ell} \geq 2 + \frac{128(\pi + 1) \delta}{m},
\]
and $\bar{\ell}$ is from (38). We also take
\[
W(X, Z) = 0.25 \ell m^2 |X(X, Z)|^2
\]
and
\[
C_1 = \left\{ 64 R \bar{K} (1 + \pi)^2 / \ell \right\}.
\]
Applying Lemma 2 with $\epsilon_1 = -\delta_1 / K_b$ and $\epsilon_2 = -B^2 \delta_2 / (M^2 K_r)$ gives a proper positive definite function
\[ V \text{ whose time derivative along the trajectories of} \]
\[ \begin{align*}
\dot{X}_1 &= X_2 \\
\dot{X}_2 &= -X_2 - \pi \sin(X_1 + x_{R1}(t)) + \sum_{i=1}^{\ell_m} \left( R_{i,2} \right) Z_2 + z_{R2}(t) Z_1 \\
&\quad + z_{R1}(t) + 1 \right] Z_2 \\
\dot{Z}_1 &= v_{f,1}(t, X, Z) - \frac{1}{R_{k_b}} \left( Z_1 + z_{R1}(t) \right) \delta_1 \\
\dot{Z}_2 &= v_{f,2}(t, X, Z) - \frac{R_{k_b}}{R_{k_0}} \left( Z_2 + z_{R2}(t) \right) \delta_2
\end{align*} \]  

(43)

satisfies

\[ \begin{align*}
\dot{V} &\leq -0.5\ell_{\mathcal{M}}(X, Z)^2 \\
&\quad + |\theta| \sum_{i=1}^{\ell_m} |\partial V_X(t, X, Z)z_{R,i}(t)| \\
&\leq -0.5\ell_{\mathcal{M}}(X, Z)^2 \\
&\quad + \{(|X|, Z) \} |4\delta R_0| |\theta(1 + i)| \\
&\leq -W(X, Z) + C_1 |\delta|^2.
\end{align*} \]  

(44)

The first inequality used our choices of \( \delta \) and \( \tilde{k} \), the second was by our choice of \( R_\alpha \) and (36), and the last inequality used the triangle inequality \( a \delta \leq 0.25\ell_{\mathcal{M}}(X, Z)^2 + C_1 |\delta|^2 \) where \( a \) and \( \delta \) are the terms in braces in (44). Reasoning as we did for (10) shows that the time derivative of (12), along all trajectories of the corresponding perturbed error dynamics

\[ \begin{align*}
\dot{\hat{z}}_1 &= v_{f,1}(t, \tilde{z}, \tilde{\theta}) - \frac{1}{R_{k_b}} (\tilde{z}_1 + z_{R1}(t)) \delta_1 \\
&\quad + k_{1}(\tilde{z} + \xi(t)) \tilde{\theta}_1 + \tilde{\psi}_1 M(t, \tilde{z}, \tilde{\theta}, \tilde{\psi}) \\
\dot{\tilde{\psi}}_i &= - \left( \tilde{\partial}_{\tilde{z}_i} - \tilde{\partial}_{\hat{z}_i} \right) \tilde{\psi}_i, 1 \leq i \leq s, 1 \leq j \leq p_i \\
\dot{\tilde{\theta}}_{i,j} &= \tilde{\tilde{Z}}_{i,j} \omega_{i,j} u_{i,j} + \tilde{\psi}_2 u_{2}(t, \tilde{z}, \tilde{\theta}, \tilde{\psi})
\end{align*} \]  

(45)

where \( u_{i,j} \), \( \omega_{i,j} \), and \( \tilde{U}_i \) are defined in (9) and (11) as before, satisfies \( V \tilde{L} \leq -W + C_1 |\delta|^2 \). Taking \( V \tilde{L} \) from Theorem 1 and the state space \( \mathcal{X} \), we prove the following in [16].

Theorem 2: We can explicitly construct functions \( \beta \in \mathcal{K} \mathcal{L} \) and \( \alpha, \gamma \in \mathcal{K}_\infty \) so that for each uncertainty \( \delta = (\delta_1, \delta_2) : [0, \infty) \to \mathbb{B}_2 \) and each initial time \( t_0 \), the corresponding trajectories \( Y = (\tilde{z}, \tilde{\theta}, \tilde{\psi}) : [t_0, \infty) \to \mathcal{X} \) of (45) all satisfy

\[ \alpha(|Y(t_0)|) \leq \beta(\max_{t \geq t_0} \dot{V}_L(t, Y(t_0)), t - t_0) + \int_{t_0}^{t} \gamma(|\delta(q)|) dq \]  

(46)

for all \( t \geq t_0 \). Hence, (45) is iISS with respect to disturbances \( \delta = (\delta_1, \delta_2) : [0, \infty) \to \mathbb{B}_2 \).

VI. CONCLUSION

Constructing strict Lyapunov functions is a central problem that is of considerable interest [13]. Here we gave a new global strict explicit Lyapunov function construction for adaptive tracking problems with unknown high frequency control gains, leading to a UGAS proof for the augmented error dynamics. Our work significantly improved on the known Lyapunov constructions. We applied our work to a brushless DC motor driving a mechanical load. We used iISS to quantify the effects of time-varying parametric uncertainty on the tracking and parameter estimation.

REFERENCES