Construction of Lyapunov Functions for Networks of iISS Systems: An Explicit Solution for a Cyclic Structure

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Abstract—This paper addresses the problem of constructing Lyapunov functions to establish integral input-to-state stability (iISS) for networks of nonlinear systems. Since the iISS small-gain theorem developed recently has been restricted to interconnection of two subsystems, this paper formulates the problem for a network consisting of more than two iISS subsystems interconnected in a general way. In the case of a cyclic interconnection structure, this paper explicitly shows a solution from which an iISS Lyapunov function of the cyclic network is composed immediately. This paper aims at improving the author’s iISS small-gain technique for deriving a general stability criterion linked directly to a general procedure for explicitly constructing a Lyapunov function for a broader class of iISS networks.

I. INTRODUCTION

This paper continues the development of the methodology of constructing Lyapunov functions to establish the stability of interconnected systems based on integral input-to-state stable (iISS) properties of subsystems. The class of iISS systems this paper deals with is much broader than the class of input-to-state stable (ISS) systems which include finite $L^p$-gain systems as a subset [3]. Until recently, the small-gain method was essentially restricted to ISS systems and input-output stable systems with finite ISS-type gain. One of the fundamentals in the small-gain methodology is the ISS small-gain theorem [15] equipped with its Lyapunov interpretation [14]. The formulation given there was presented for interconnection of two subsystems. To deal with more than two subsystems, one may think of repeated application of the two subsystems argument. It, however, causes unnecessary conservativeness and often necessitates additional assumptions which are not fulfilled by practical applications. More importantly, an ad hoc aggregation process renders the construction of a Lyapunov function too complicated to be explicit. Very recently, the ISS small-gain theorem has been extended to general networks [5], [16]. The author of this paper also has attempted to extend the generalized ISS small-gain method to the iISS case [11]. However, the stability condition covering iISS subsystems in [11] is readily computable as a small-gain criterion only in the case of two subsystems. Explicit solutions to the stability condition for multiple subsystems are not given in [11], so that Lyapunov functions have not yet been constructed in the case of multiple subsystems. The main purpose of this paper is to make an important step in the direction of developing an explicit procedure for constructing Lyapunov functions of general networks consisting of more than two iISS subsystems.

It would be natural to consider the question as to whether the small-gain argument is applicable to a feedback loop involving more than two subsystems. As in the straightforward $L^p$-gain case, the ISS small-gain argument is still valid and the loop is ISS if the loop gain consisting of all the subsystems is less than the identity map. The question remains unanswered in the case of iISS subsystems since individual gains are not bounded in magnitude. It is known that one of the two iISS subsystems composing a loop is necessarily ISS for guaranteeing the stability of the loop based on gain-type information [12], [13]. This fact brings us to the question as to how many non-ISS subsystems are allowed in a loop comprising more than two subsystems. Providing an answer to this question is practically important since many applications in biological, logistic and communication systems involve loops in their large networks. Recently, applications in biological systems have motivated researchers to develop stability criteria for interconnected systems in a cyclic structure (i.e., a loop), and passivity-based criteria and associated Lyapunov functions were presented [2]. The passivity criteria make use of the sign properties of the interconnection terms. This paper pursues an alternative approach making use of the magnitude of the interconnection terms in the framework of iISS, which is considered as an extension of the classical linear gain methodology [1]. Components of biological systems often exhibit iISS properties although they have neither finite $L^p$ nor ISS gain. The iISS small-gain method is useful in analyzing those systems [7], [8].

This paper pursues a construction procedure resulting in smooth Lyapunov functions directly. This contrasts with the non-smooth ones [14], [4], [16] available for ISS systems. The smooth construction is not only advantageous in applications, but also is the key to the coverage of iISS systems [11].

Throughout this paper, we shall use the following notation. Let $|·|$ denote the Euclidean norm. A continuous function $\omega : \mathbb{R}_+ := [0, \infty) \rightarrow \mathbb{R}_+$ is said to be positive definite and denoted by $\omega \in \mathcal{P}$ if it satisfies $\omega(0) = 0$ and $\omega(s) > 0$ holds for all $s > 0$. A function is of class $\mathcal{K}$ if it belongs to $\mathcal{P}$ and is strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and is unbounded. The symbol $\text{Id}$ denotes the identity maps. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. Due to the space limitation, proofs of theorems and lemmas except the main theorem are omitted.
II. NETWORK OF iISS SYSTEMS

Consider a network $\Sigma$ consisting of $n$ subsystems $\Sigma_i$, $i = 1, 2, ..., n$ where $n \geq 2$. Let $x = [x_1^T, ..., x_n^T]^T \in \mathbb{R}^N$ denote the state vector of $\Sigma$, where the state vector of each subsystem is $x_i \in \mathbb{R}^N_i$, and $N := \sum N_i$ holds. Suppose that the dynamics of the $i$-th subsystem $\Sigma_i$ is governed by

$$
\Sigma_i : \dot{x}_i = f_i(x_1, ..., x_n, r),
$$

where $r \in \mathbb{R}^M$ and $f_i : \mathbb{R}^{N_i+M} \to \mathbb{R}^{N_i}$. For each $i \in \{1, 2, ..., n\}$, the subsystem (1) is assumed to have a unique solution $x_i(t)$ for any given initial condition $x_i(0) \in \mathbb{R}^{N_i}$ and any $L^\infty$-inputs $x_j : [0, \infty) \to \mathbb{R}^{N_i}, j \neq i$, and $r : [0, \infty) \to \mathbb{R}^M$. For instance, this can be guaranteed by the local Lipschitz condition on the $f_i$. Let $f = [f_1^T, ..., f_n^T]^T : \mathbb{R}^{N+n} \to \mathbb{R}^N$. The overall network $\Sigma$ can be written as

$$
\Sigma : \dot{x} = f(x, r).
$$

Instead of the knowledge of $f$, this paper assumes that each subsystem $\Sigma_i$ satisfies a dissipation inequality and we use that information for verifying the stability of $\Sigma$.

**Assumption 1:** For each $i = 1, 2, ..., n$, there exist a $C^1$ function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}$, and continuous functions $\alpha_i \in \mathcal{K}$, $\sigma_{ij}, \tau_{ij} \in \mathcal{K}_\infty$ such that

$$
\alpha_i(|x_i|) \leq V_i(x_i) \leq \tau_i(|x_i|), \quad x_i \in \mathbb{R}^{N_i}
$$

holds along the trajectories $x_i(t)$ for all $x_j \in \mathbb{R}^{N_i}, j \neq i$ and all $r \in \mathbb{R}^M$, where $\sigma_i, \tau_i \equiv 0, i = 1, 2, ..., n$.

The inequality (4) is called a dissipation inequality. This assumption means that each subsystem $\Sigma_i$ with the inputs $x_j$, $j \neq i$ and $r$ is integral input-to-state stable (iISS), and that $V_i$ is an iISS Lyapunov function for the disconnected subsystem $\Sigma_i$ [3]. Under a stronger assumption $\alpha_i \in \mathcal{K}_\infty$, the subsystem $\Sigma_i$ is input-to-state stable (ISS), and the function $V_i$ is an ISS Lyapunov function [17]. By definition, the set of iISS systems is essentially larger than the set of ISS systems. Note that the function $V_i$ is said to be an iISS Lyapunov function even when $\alpha_i \in \mathcal{P}$ [3]. Nevertheless, to allow for feedback loops in $\Sigma$, this paper assumes $\alpha_i \in \mathcal{K}$ which is a strict subset of $\mathcal{P}$. It has been proved that a feedback interconnection of iISS systems defined with dissipation inequalities (4) is iISS only if $\alpha_i \in \mathcal{K}$ holds for all $i$ [9]. The goal of this paper is to construct an iISS Lyapunov function $V(x)$ of the network $\Sigma$ with respect to input $r$ and state $x$, and to find a condition under which the construction is possible.

III. SUM-TYPE LYAPUNOV FUNCTION

Define $A, \Gamma, \Lambda, D : s \in \mathbb{R}^n_+ \mapsto z \in \mathbb{R}^n_+$ by

$$
z = A(s) = \begin{bmatrix}
\alpha_1 \circ \tau_1^{-1}(s_1) \\
\alpha_2 \circ \tau_2^{-1}(s_2) \\
\vdots \\
\alpha_n \circ \tau_n^{-1}(s_n)
\end{bmatrix} = \begin{bmatrix}
\alpha_1 & 0 & \cdots & 0 \\
0 & \alpha_2 & \cdots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_n
\end{bmatrix}
\begin{bmatrix}
\tau_1^{-1}(s_1) \\
\tau_2^{-1}(s_2) \\
\vdots \\
\tau_n^{-1}(s_n)
\end{bmatrix}
$$

where $r \in \mathbb{R}^M$ and $f_i : \mathbb{R}^{N_i+M} \to \mathbb{R}^{N_i}$. For each $i \in \{1, 2, ..., n\}$, the subsystem (1) is assumed to have a unique solution $x_i(t)$ for any given initial condition $x_i(0) \in \mathbb{R}^{N_i}$ and any $L^\infty$-inputs $x_j : [0, \infty) \to \mathbb{R}^{N_i}, j \neq i$, and $r : [0, \infty) \to \mathbb{R}^M$. For instance, this can be guaranteed by the local Lipschitz condition on the $f_i$. Let $f = [f_1^T, ..., f_n^T]^T : \mathbb{R}^{N+n} \to \mathbb{R}^N$. The overall network $\Sigma$ can be written as

$$
\Sigma : \dot{x} = f(x, r).
$$

Instead of the knowledge of $f$, this paper assumes that each subsystem $\Sigma_i$ satisfies a dissipation inequality and we use that information for verifying the stability of $\Sigma$.

**Assumption 1:** For each $i = 1, 2, ..., n$, there exist a $C^1$ function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}$, and continuous functions $\alpha_i \in \mathcal{K}$, $\sigma_{ij}, \tau_{ij} \in \mathcal{K}_\infty$ such that

$$
\alpha_i(|x_i|) \leq V_i(x_i) \leq \tau_i(|x_i|), \quad x_i \in \mathbb{R}^{N_i}
$$

holds along the trajectories $x_i(t)$ for all $x_j \in \mathbb{R}^{N_i}, j \neq i$ and all $r \in \mathbb{R}^M$, where $\sigma_i, \tau_i \equiv 0, i = 1, 2, ..., n$.

The inequality (4) is called a dissipation inequality. This assumption means that each subsystem $\Sigma_i$ with the inputs $x_j$, $j \neq i$ and $r$ is integral input-to-state stable (iISS), and that $V_i$ is an iISS Lyapunov function for the disconnected subsystem $\Sigma_i$ [3]. Under a stronger assumption $\alpha_i \in \mathcal{K}_\infty$, the subsystem $\Sigma_i$ is input-to-state stable (ISS), and the function $V_i$ is an ISS Lyapunov function [17]. By definition, the set of iISS systems is essentially larger than the set of ISS systems. Note that the function $V_i$ is said to be an iISS Lyapunov function even when $\alpha_i \in \mathcal{P}$ [3]. Nevertheless, to allow for feedback loops in $\Sigma$, this paper assumes $\alpha_i \in \mathcal{K}$ which is a strict subset of $\mathcal{P}$. It has been proved that a feedback interconnection of iISS systems defined with dissipation inequalities (4) is iISS only if $\alpha_i \in \mathcal{K}$ holds for all $i$ [9]. The goal of this paper is to construct an iISS Lyapunov function $V(x)$ of the network $\Sigma$ with respect to input $r$ and state $x$, and to find a condition under which the construction is possible.

Theorem 1: Suppose that there exist continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2, ..., n$, such that

$$
\lambda_i(s) > 0, \quad \forall s \in (0, \infty), \quad i = 1, 2, ..., n
$$

holds for some $\delta_1, \delta_2, ..., \delta_n \in \mathcal{K}_\infty$. Then, the system $\Sigma$ is iISS with respect to input $r$ and state $x$. If

$$
\alpha_i \in \mathcal{K}_\infty, \quad i = 1, 2, ..., n
$$

are satisfied additionally, the network $\Sigma$ is ISS. Furthermore, an iISS (ISS) Lyapunov function is

$$
V(x) = \sum_{i=1}^{n} \int_{0}^{V_i(x_i)} \lambda_i(s) ds.
$$

In the case of globally asymptotically stability of $x = 0$ for $r(t) \equiv 0$, we can remove (7) from the above theorem.

In this paper, $\Sigma$ is said to be a cyclic system if
holds for $\sigma_i \in \mathcal{K}$, $i = 1, 2, ..., n$. The cyclic system is shown in Fig.1. The system $\Sigma$ is always cyclic when $n = 2$. In order to deal with cyclic systems, this paper adopts a cyclic notation for the indices $\{1, 2, ..., n\}$ of the $n$ subsystems. For instance, the indeces $i+1$ and $i+2$ indicate 1 and 2, respectively, if $i = n$. In a similar manner, the indeces $i-1$ and $i-2$ indicate $n$ and $n-1$, respectively, if $i = 1$.

The existing iSS small-gain results[6], [12], [13] deal with the case of $n = 2$ in Theorem 1. This paper aims at providing a solution $\{\lambda_i(s)\}$, $i = 1, 2, ..., n$, to the problem posed in Theorem 1 for general $n \geq 2$, and derive their formulas amenable to (12) and more general interconnections.

**IV. A SOLUTION FOR CYCLIC SYSTEM**

This section considers the problem of (5)-(8) and (10) for the cyclic system (12) with $n \geq 2$, and provides a solution $\Lambda(x)$ which directly gives us a Lyapunov function $V(x)$ of the network $\Sigma$ as in (11). Note that this section employs the circulating indices of $\{1, 2, ..., n\}$, e.g. $\alpha_{n+1} = \alpha_1$ and $\sigma_0 = \sigma_n$.

The following lemma defines $\{\lambda_1(s), \lambda_2(s), ..., \lambda_n(s)\}$ and shows that they conform to (5), (6), (7) and (10).

**Lemma 1:** Consider $\alpha_i, \sigma_i \in \mathcal{K}$ and $\hat{\alpha}_i, \hat{\sigma}_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$. Assume that

\( \lim_{s \to \infty} \alpha_{i+1}(s) = \infty \lor \lim_{s \to \infty} \sigma_{i+1}(s) = \infty \), $i = 1, 2, ..., n$

holds. Let $\nu_i, i = 1, 2, ..., n$, be non-decreasing continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying

\[
\begin{align*}
0 < \nu_i(s) &< \infty, \quad s \in (0, \infty), \ i = 1, 2, ..., n \\
\lim_{s \to \infty} \nu_i(s) &< \infty, \quad i = 1, 2, ..., n
\end{align*}
\]

Define

\[
\lambda_i(s) = \frac{\hat{\lambda}_i(s)}{\nu_i(s)}, \quad i = 1, 2, ..., n
\]

\[
\hat{\lambda}_i(s) = (\tau_{i+1} - 1) \left( 1 + \frac{\alpha_i(\hat{\alpha}_i^{-1}(s))}{\tau_i} \right)^{n-1} \prod_{q=1}^{n} \left[ p_{i-q,i}(s) \right]^{n-1} \psi + 1
\]

where

\[
0 \leq \psi_i, \quad \psi_i < \tau_i, \quad i = 1, 2, ..., n
\]

\[
Y_i = \begin{cases} 
\infty & \text{if } \lim_{s \to \infty} \alpha_i(s) > \lim_{s \to \infty} \tau_i \sigma_i(s) \\
\lim_{s \to \infty} \frac{1}{\sigma_i} \alpha_i(\hat{\alpha}_i^{-1}(s)) & \text{otherwise}
\end{cases}
\]

\[
\theta_i(s) = \frac{\hat{\sigma}_i}{\sigma_i} \frac{1}{\alpha_i} \tau_i \sigma_i(s), \quad s \in [0, \nu_i]
\]

\[
\zeta_i(s) = \begin{cases} 
\infty & \text{if } \lim_{s \to \infty} \tau_i \sigma_i(s) < \lim_{s \to \infty} \alpha_i(\hat{\alpha}_i^{-1}(s)) \\
\frac{1}{\sigma_i} \frac{1}{\alpha_i} \hat{\sigma}_i^{-1} & \text{otherwise}
\end{cases}
\]

\[
L_{i+n-1, i+n} = Y_{i+n-1} = \lim_{s \to \infty} \zeta_{i+n-1}(s)
\]

\[
L_{i+n-1, i+n+1} = \zeta_{i+n-1} \hat{\sigma}_{i+n-1} \circ \cdots \circ \hat{\sigma}_2 \circ \hat{\sigma}_1(Y_{i+1})
\]

\[
\theta_i(s) = \left( \frac{\hat{\alpha}_i^{-1}}{\alpha_i} \right) \frac{1}{\tau_i} \sigma_i(\hat{\alpha}_i^{-1}(s))
\]

\[
\hat{\theta}_i(s) = \left( \frac{\hat{\alpha}_i^{-1}}{\alpha_i} \right) \frac{1}{\tau_i} \sigma_i(\hat{\alpha}_i^{-1}(s))
\]

Then, $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ are non-decreasing continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ and satisfies

\[
0 < \lambda_i(s) < \infty, \quad s \in (0, \infty), \ i = 1, 2, ..., n
\]

\[
\lim_{s \to \infty} \lambda_i(s) = \lim_{s \to \infty} \lambda_i(s) < \infty, \ i = 1, 2, ..., n.
\]

The following is the main result which demonstrates that the functions $\lambda_i$ given in Lemma 1 solve the problem posed by Theorem 1.

**Theorem 2:** Consider $\alpha_i, \sigma_i \in \mathcal{K}$, $\hat{\alpha}_i, \hat{\sigma}_i \in \mathcal{K}_\infty$ and non-decreasing continuous functions $\nu_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2, ..., n$. Assume that (11), (14) and (19) are satisfied. If there exist $c_i > 1, i = 1, 2, ..., n$ such that

\[
\alpha_i^{-1} \circ \hat{\sigma}_i^{-1} \circ \alpha_i^{-1} \circ c_i \sigma_i \circ \alpha_i^{-1} \circ \hat{\sigma}_i^{-1} \circ \alpha_i^{-1} \circ c_2 \sigma_2 \circ \cdots \circ \alpha_i^{-1} \circ \hat{\sigma}_i^{-1} \circ \alpha_i^{-1} \circ c_n \sigma_n
\]

\[
\alpha_n^{-1} \circ \hat{\sigma}_n^{-1} \circ \alpha_n^{-1} \circ c_n \sigma_n(s) \leq s, \forall s \in \mathbb{R}_+
\]

holds, the functions $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ given in (15) and (16) with $\tau_i$ and $\psi_i \geq 0$ satisfying

\[
0 < \psi_i \leq \tau_i + 1, \quad i = 1, 2, ..., n
\]

achieve (5)-(8) and (10) with $\delta(s) = \delta_2(s) = \cdots = \delta_n(s) = bs$ for some $b > 0$.

Clearly, the choice $\nu_1 = \nu_2 = \cdots = \nu_n = \nu$ with an arbitrary positive constant $\nu$ fulfills (13), (14) and (19). Thus, Theorem 2 is ready for establishing the iSS property of the cyclic system (12). However, when we consider networks consisting of multiple loops, having flexibility of $\nu_i$'s as functions is useful as shown in Section V. The following two theorems show that there exist and we can construct non-decreasing continuous functions $\nu_i$ satisfying (13), (14) and (19).

**Theorem 3:** Consider $\alpha_i, \sigma_i \in \mathcal{K}$ and $\hat{\alpha}_i, \hat{\sigma}_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$. If there exist $c_i > 1, i = 1, 2, ..., n$ such that (20) holds, there exist $k \in \{1, 2, ..., n\}, \tau_1, \tau_2, ..., \tau_n > 1$ and $\hat{\alpha}_k, \hat{\alpha}_{k+1}, ..., \hat{\alpha}_{k+n-3}, \hat{\sigma}_k, \hat{\sigma}_{k+1}, ..., \hat{\sigma}_{k+n-3} \in \mathcal{K}$ such that

\[
\begin{align*}
\alpha_k^{-1} \circ \hat{\sigma}_k \circ \alpha_k^{-1} \circ \tau_k \hat{\sigma}_k \circ \cdots \\
\alpha_{k+n-3} \circ \hat{\sigma}_{k+n-3} \circ \alpha_{k+n-3} \circ \tau_{k+n-3} \circ \hat{\sigma}_{k+n-3} \\
\alpha^{-1} \circ \hat{\sigma}_{k+n-2} \circ \alpha_{k+n-2} \circ \tau_{k+n-2} \circ \hat{\sigma}_{k+n-2} \\
\alpha_{k+n-1} \circ \hat{\sigma}_{k+n-1} \circ \alpha_{k+n-1} \circ \tau_{k+n-1} \circ \hat{\sigma}_{k+n-1}
\end{align*}
\]

\[
\leq s, \forall s \in \mathbb{R}_+
\]

\[
\tau_l \leq c_i, \quad l = 1, 2, ..., n
\]

\[
\hat{\alpha}_i(s) \leq \alpha_i(s), \quad \forall s \in \mathbb{R}_+, \ i = k, k+1, ..., k+n-3
\]

\[
\hat{\sigma}_i(s) \geq \sigma_i(s), \quad \forall s \in \mathbb{R}_+, \ i = k, k+1, ..., k+n-3
\]

\[
\hat{\sigma}_k^{-1} \in \mathcal{K}
\]

\[
\hat{\theta}_k^{-1} \circ \alpha_{k+n-3} \circ \hat{\theta}_k^{-1} \in \mathcal{K}
\]

\[
\hat{\theta}_k^{-1} \circ \alpha_{k+n-3} \circ \hat{\theta}_k^{-1} \circ \cdots \circ \hat{\theta}_k^{-1} \circ \in \mathcal{K}
\]
where\
\[
\hat{\theta}_l^{-1} = \hat{\sigma}_l^{-1} \circ \frac{1}{\tau_i} \hat{\alpha}_l \circ \tau_l^{-1}, \quad l = k, k+1, \ldots, k+n-3.
\]

**Theorem 4:** Let \( k \in \{1, 2, \ldots, n\} \). Assume that \( \alpha_i, \sigma_i, \hat{\alpha}_i, \hat{\sigma}_i \in \mathbb{K}, \) \( \alpha_i, \sigma_i \in \mathbb{K}_\infty \) and \( \tau_i > 1 \) hold for \( i = 1, 2, \ldots, n \) and \( l = k, k+1, \ldots, k+n-3 \). Suppose (23)-(30) and
\[
\lim_{s \to \infty} \alpha_{k+n-1}(s) = \infty \quad \vee \quad \lim_{s \to \infty} \sigma_{k+n-2}(s) < \infty
\]
are satisfied. Then, the functions
\[
\beta_k(s) = \frac{1}{\tau_{k+n-2}} \alpha_{k+n-2} \circ \tau_{k+n-2}^{-1} \circ \alpha_{k+n-2} \circ \hat{\theta}_{k+n-3}^{-1} \\
\vdots \\
\beta_{k+n-3}(s) = \frac{1}{\tau_{k+n-2}} \alpha_{k+n-2} \circ \tau_{k+n-2}^{-1} \circ \alpha_{k+n-2} \circ \hat{\theta}_{k+n-3}^{-1} \circ \cdots \circ \alpha_{k+2} \circ \hat{\theta}_{k+1}^{-1} \circ \alpha_{k+1} \circ \hat{\theta}_{k}^{-1}(s)
\]
are of class \( \mathbb{K} \) and satisfy
\[
\{ \lim_{s \to \infty} \alpha_{k+i}(s) < \infty \Rightarrow \lim_{s \to \infty} \beta_{k+i}(s) < \infty \},
\]
i = 0, 1, \ldots, n - 1 (32)
\[
\beta_{k+i}(\theta_{k+i}(s)) \leq \beta_{k+i+1}(\theta_{k+i+1}(s)), \quad s \in [0, \gamma_{k+i}]
\]
i = 0, 1, \ldots, n - 1. (33)

Furthermore, the functions
\[
\nu_{k+i}(s) = \nu(\beta_{k+i}(s)), \quad i = 0, 1, \ldots, n - 1
\]
are non-decreasing continuous functions and satisfy (13), (14) and (19) if \( \nu \) is a non-decreasing continuous function from \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \) satisfying
\[
0 < \nu(s) < \infty, \quad s \in (0, \infty).
\]

Remark 1: When \( \alpha_i \in \mathbb{K}_\infty \) holds for all \( i = 1, 2, \ldots, n \), the inequality (20) with \( c_i > 1, i = 1, 2, \ldots, n \) reduces to the application of the generalized ISS small-gain condition developed in [5], [16] to the cyclic system (12). When \( \alpha_1, \ldots, \alpha_n \in \mathbb{K}_\infty \) holds, we can assume \( \sigma_1, \ldots, \sigma_n \in \mathbb{K}_\infty \) without loss of generality since we can always replace \( \sigma_1, \ldots, \sigma_n \in \mathbb{K} \) by new \( \hat{\sigma}_1, \ldots, \hat{\sigma}_n \in \mathbb{K}_\infty \) satisfying \( \sigma_i(s) \leq \hat{\sigma}_i(s), \forall s \in \mathbb{R}_+ \) and maintaining (20). Since all the \( L_n \)'s become \( \infty \), the functions \( \lambda_i \) in (15) and (16) are less complicated in the ISS case. The assumption (H1) is always fulfilled by \( \alpha_1, \ldots, \alpha_n \in \mathbb{K}_\infty \).

Remark 2: The cyclic iISS small-gain condition presented in the form of (20) is asymmetric in contrast to the ISS case where (20) become symmetric. Indeed, the existence of \( c_1, \ldots, c_n > 1 \) fulfilling (20) requires
\[
\{ \lim_{s \to \infty} \alpha_i(s) = \infty \vee \lim_{s \to \infty} \alpha_i(s) > \lim_{s \to \infty} \sigma_i(s) \}, \quad i = n (36)
\]
while (36) is not necessary for \( i \neq 4 \). This property (36) implicitly requires that \( \Sigma_n \) be ISS [17]. In the same way, the expression (20) with \( c_1, \ldots, c_n > 1 \) requires
\[
\lim_{s \to \infty} \alpha_i(s) = \infty \vee \lim_{s \to \infty} \alpha_i(s) > \lim_{s \to \infty} \sigma_i(s) \rightarrow \infty, \quad 1 \leq i \leq n
\]
are satisfied. Then, the functions
\[
\beta_k(s) = \frac{1}{\tau_{k+n-2}} \alpha_{k+n-2} \circ \tau_{k+n-2}^{-1} \circ \alpha_{k+n-2} \circ \hat{\theta}_{k+n-3}^{-1} \\
\vdots \\
\beta_{k+n-3}(s) = \frac{1}{\tau_{k+n-2}} \alpha_{k+n-2} \circ \tau_{k+n-2}^{-1} \circ \alpha_{k+n-2} \circ \hat{\theta}_{k+n-3}^{-1} \circ \cdots \circ \alpha_{k+2} \circ \hat{\theta}_{k+1}^{-1} \circ \alpha_{k+1} \circ \hat{\theta}_{k}^{-1}(s)
\]
are of class \( \mathbb{K} \) and satisfy
\[
\{ \lim_{s \to \infty} \alpha_{k+i}(s) < \infty \Rightarrow \lim_{s \to \infty} \beta_{k+i}(s) < \infty \},
\]
i = 0, 1, \ldots, n - 1 (32)
\[
\beta_{k+i}(\theta_{k+i}(s)) \leq \beta_{k+i+1}(\theta_{k+i+1}(s)), \quad s \in [0, \gamma_{k+i}]
\]
i = 0, 1, \ldots, n - 1. (33)

Furthermore, the functions
\[
\nu_{k+i}(s) = \nu(\beta_{k+i}(s)), \quad i = 0, 1, \ldots, n - 1
\]
are non-decreasing continuous functions and satisfy (13), (14) and (19) if \( \nu \) is a non-decreasing continuous function from \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \) satisfying
\[
0 < \nu(s) < \infty, \quad s \in (0, \infty).
\]

This paper refers to the inequality (20) with \( c_i > 1, i = 1, 2, \ldots, n \) as the cyclic iISS small-gain condition. It generalizes the iISS small-gain condition originally developed for the two subsystems case [13] to deal with the cyclic system (12) of arbitrary \( n \). Setting \( n = 2 \), the small-gain condition (20) provided by Theorem 2 reduces to the two subsystems result in [13]. However, the formula of \( \lambda_i \)'s given in this paper is different from the one given in [13]. The solution to the problem posed in Theorem 1 is not unique. Every pair of \( \lambda_1 \) and \( \lambda_2 \) developed for the two subsystems case in [6], [12], [13] did not share a common form. In order to render the construction of Lyapunov functions amenable to complicated interconnections of systems, this paper unifies the equations of all \( \lambda_i \)'s into a single formula (15)-(16).
Assumption 2: Each pair of loops existing in $\Sigma$ shares at most one common subsystem $\Sigma_i$.

An example fulfilling this assumption is shown in Fig. 2 which contains three loops $\{\Sigma_1, \Sigma_2, \Sigma_3\}$, $\{\Sigma_1, \Sigma_2, \Sigma_3\}$ and $\{\Sigma_3, \Sigma_6\}$. This network configuration is represented by

$$\Gamma(s) = \begin{bmatrix}
0 & \sigma_{12} & 0 & \sigma_{14} & 0 & 0 \\
0 & 0 & \sigma_{23} & 0 & 0 & 0 \\
\sigma_{31} & 0 & 0 & 0 & 0 & \sigma_{36} \\
0 & 0 & 0 & 0 & \sigma_{45} & 0 \\
\sigma_{51} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma_{63} & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\alpha_1^{-1}(s_1) \\
\alpha_2^{-1}(s_2) \\
\alpha_3^{-1}(s_3) \\
\alpha_4^{-1}(s_4) \\
\alpha_5^{-1}(s_5) \\
\alpha_6^{-1}(s_6)
\end{bmatrix}. \quad (38)

For any network satisfying Assumption 2, we can verify that the network $\Sigma$ is iISS with respect to input $r$ and state $x$ if the small-gain condition of the form (20) and (H1) holds for all the loops in $\Sigma$. The iISS property of $\Sigma$ is established by the function $V(x)$ given in (11) with $\lambda_i$ given by (15)-(16) applied to the individual loops. The key is to obtain a common $\lambda_i(s)$ from different loops containing $\Sigma_i$ by making use of the flexibility of $\nu$ in (15). As in the simple cyclic network (12), the function $V(x)$ becomes an ISS Lyapunov function of $\Sigma$ if $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, \ldots, n$.

The rest of this section shows how to compute $\lambda_i$’s for the example of Fig. 2. A set of small-gain conditions which can establish the iISS property of $\Sigma$ in Fig. 2 is

$$\begin{array}{l}
\omega_1 \circ \alpha_1 \circ \alpha_2^{-1} \circ \sigma_{14} \circ \sigma_{23} \circ \sigma_{36} \circ \sigma_{45} \\
\omega_2 \circ \alpha_5 \circ \alpha_6 \circ \sigma_{51} \leq s, \quad \forall s \in \mathbb{R}_+ \\
\alpha_1^{-1} \circ \alpha_1 \circ \alpha_3^{-1} \circ \sigma_{12} \circ \sigma_{23} \circ \sigma_{36} \circ \sigma_{45} \\
\alpha_3^{-1} \circ \alpha_3 \circ \alpha_5^{-1} \circ \sigma_{31} \leq s, \quad \forall s \in \mathbb{R}_+ \\
\alpha_5^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \sigma_{56} \circ \sigma_{63} \leq s, \quad \forall s \in \mathbb{R}_+
\end{array} \quad (39) - (41)

where $d_1$ and $d_3$ are allowed to be any real number in $(0, 1)$. Let (15) be represented as

$$\lambda_i(s) = \tilde{\lambda}_{i,1}(s) \nu_1(\beta_{i,1}(s)) \quad (42)
$$

which is (15) applied to the $l$-th loop in $\Sigma$. In the $l = 1$ case, we use (42) for $i \in \{1, 4, 5\}$. In the same manner, we use (42) for $i \in \{1, 2, 3\}$ in the $l = 2$ case, and $i \in \{3, 6\}$ in the $l = 3$ case. Solve

$$\begin{array}{l}
\tilde{\lambda}_{1,1}(s) = \nu_2(\beta_{1,1}(s)), \quad s \in \mathbb{R}_+ \\
\tilde{\lambda}_{2,1}(s) = \nu_1(\beta_{2,1}(s)), \quad s \in \mathbb{R}_+
\end{array} \quad (43) - (44)
$$

for non-decreasing function $\nu_1, \nu_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Using the function $\nu_2$ obtained, we solve

$$\begin{array}{l}
\tilde{\lambda}_{2,3}(s) = \nu_2(\beta_{2,3}(s)), \quad s \in \mathbb{R}_+ \\
\tilde{\lambda}_{3,3}(s) = \nu_2(\beta_{3,3}(s)), \quad s \in \mathbb{R}_+
\end{array} \quad (45) - (46)
$$

for non-decreasing function $\nu_3, \nu_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. We also compute a non-decreasing function $\tilde{\mu}_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ achieving

$$\begin{array}{l}
\tilde{\mu}_2(\beta_{2,1}(s)) = \mu_1(\beta_{1,1}(s)), \quad s \in \mathbb{R}_+
\end{array} \quad (47)
$$

Then, the functions

$$\begin{array}{l}
\lambda_6(s) = \tilde{\lambda}_{3,6}(s) \nu_3(\beta_{3,6}(s)) \\
\lambda_3(s) = \tilde{\lambda}_{3,3}(s) \nu_3(\beta_{3,3}(s)) \\
\lambda_2(s) = \tilde{\lambda}_{2,2}(s) \nu_2(\beta_{2,2}(s)) \mu_2(\beta_{2,2}(s)) \\
\lambda_1(s) = \tilde{\lambda}_{2,1}(s) \nu_2(\beta_{2,1}(s)) \mu_1(\beta_{2,1}(s)) \\
\lambda_4(s) = \tilde{\lambda}_{1,4}(s) \nu_1(\beta_{1,4}(s)) \mu_1(\beta_{1,4}(s)) \\
\lambda_5(s) = \tilde{\lambda}_{1,5}(s) \nu_1(\beta_{1,5}(s)) \mu_1(\beta_{1,5}(s))
\end{array} \quad (42)
$$

are in the form of (42) and achieve the matching

$$\begin{array}{l}
\lambda_1(s) = \tilde{\lambda}_{1,1}(s) \nu_1(\beta_{1,1}(s)) = \tilde{\lambda}_{2,1}(s) \nu_2(\beta_{2,1}(s)) \\
\lambda_3(s) = \tilde{\lambda}_{2,3}(s) \nu_2(\beta_{2,3}(s)) = \tilde{\lambda}_{3,3}(s) \nu_3(\beta_{3,3}(s))
\end{array} \quad (43)
$$

for some non-decreasing functions $\nu_1, \nu_2, \nu_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. The feasibility of this recursive procedure is guaranteed by Assumption 2. It is straightforwardly proved based on (32) that the existence of the solution $\nu_1, \nu_2, \nu_3, \mu_1$ to (43)-(47) is guaranteed if either $\sigma_2, \ldots, \sigma_6 \in \mathcal{K}_\infty$ or $\alpha_2, \ldots, \alpha_6 \in \mathcal{K} \setminus \mathcal{K}_\infty$ holds. The existence of $\nu_1, \nu_2, \nu_3, \mu_1, \mu_2, \mu_1$ in the other cases can be verified with the help of Lemma 2.

VI. CONCLUDING REMARKS

The results of this paper indicate that a cyclic interconnection of multiple iISS subsystems is eligible to be stable if one subsystem is ISS with respect to its feedback input. The interconnection is guaranteed to be stable if the gain of the ISS subsystem is small enough to fulfill the cyclic iISS small-gain condition. This paper has focused on the explicit construction of a continuously differentiable iISS Lyapunov function verifying this fact. Covering iISS subsystems by a cyclic small-gain condition is also new in the literature. The Lyapunov function derived in this paper is more useful than the ones in [6, 12, 13] since it is composed of homogenous scaling functions $\lambda_i$. Unifying the $\lambda_i$’s into a single formula not only enables us to solve the cyclic problem involving more than two subsystems, but also aims at the convenience in dealing with networks in more general structures. Although an illustration is given in Section V, the treatment of general networks needs to be discussed in a separate paper by removing Assumption 2. Providing a less conservative stability test by getting rid of the partitioning parameters $d_i$ in (39)-(41) is also a topic of future study. Finally, it is mentioned that discussion on potential conservativeness of the small-gain criteria is worthy of investigation[10].

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APPENDIX

A. A Technical Lemma

Lemma 2: Consider $\beta \in \mathcal{K}$ and a non-decreasing continuous function $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\{B :=$
lim_{s \to \infty} \beta(s) < \infty \Rightarrow \lim_{s \to \infty} \lambda(s) < \infty$. Then, $\nu \circ \beta(s) = \lambda(s)$ holds for all $s \in \mathbb{R}_+$ if a continuous function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\nu(s) = \lambda \circ \beta^{-1}(s)$ for all $s \in [0, B)$.

### B. Sketch of the Proof of Theorem 2

Define $b_i > 0$, $b > 0$ and functions $\lambda_{0i}$ by

$$b_i = \frac{\tau_i}{\epsilon_i(\tau_i-1)+1}, \quad i = 1, 2, \ldots, n, \quad b = \min_{i \in \{1, 2, \ldots, n\}} b_i$$

$$\lambda_{0i}(s) = \begin{cases} \lambda_1 \circ \theta_i(s), & s \in [0, Y_i) \\ \lim_{s \to \infty} \lambda_i(s), & s \in [Y_i, \infty) \end{cases}, \quad i = 1, 2, \ldots, n.$$  

Due to (18), the $\lambda_{0i}(s)$'s are finite for each $s \in [0, \infty)$. The choice $\delta_i(s) = bs$ implies that (8) holds if $\lambda_1, \ldots, \lambda_n$ satisfy

$$\lambda_{0i}(s) \sigma_i(s) \leq \epsilon_i + 1 - \frac{\tau_{i+1} - 1}{\tau_{i+1}} \lambda_i(\Omega_i^{-1}(s)) \alpha_{i+1}(\Omega_i^{-1}(\bar{\Omega}_i^{-1}(s))) \quad (48)$$

for all $s \in \mathbb{R}_+$, $i = 1, 2, \ldots, n$. By definition, we have

$$L_{i+n,n+i+2} \leq \cdots \leq L_{i+n,n+i+1} \leq L_{i+n,n+i} = Y_{i+n} = Y_i.$$  

Using (15)-(16), we obtain

$$\lambda_{0i}(s) \sigma_i(s) = (\tau_{i+1} - 1) \cdot \left[ \sigma_{i+1}(\Omega_{i+2}^{-1}(\theta_{i+2}(\cdots \Omega_i^{-1}(\theta_i(s)))) \right]^{\psi+1} \quad \cdots$$

$$\lambda_{0i}(s) \sigma_i(s) = (\tau_{i+1} - 1) \cdot \left[ \lim_{e \to L_{i+n,n+i+1}} \sigma_{i+1}(\Omega_{i+2}^{-1}(\theta_{i+2}(\cdots \Omega_i^{-1}(\theta_i(e)))) \right]^{\psi+1} \quad \cdots$$

$$\lambda_{0i}(s) \sigma_i(s) = \begin{cases} \lim_{e \to \infty} \lambda_i(e), \quad s \in [Y_i, \infty) \end{cases} \quad (50)$$

From (19) we obtain

$$\left( \begin{array}{c} \sigma_{i+1} \circ \Omega_{i+2}^{-1} \circ \theta_{i+2} \circ \cdots \circ \Omega_i^{-1} \circ \theta_i(s) \\ \nu_i \alpha_i(s) \end{array} \right) \quad \left( \begin{array}{c} \psi+1 \\ \psi+1 \\ \psi+1 \end{array} \right)$$

$$\leq \left( \begin{array}{c} \tau_{i+1} \psi+1 \\ \tau_{i+1} \psi+1 \\ \tau_{i+1} \psi+1 \end{array} \right)$$

With the help of (19), (21) and (22), we arrive at (48).

### REFERENCES


