A Ball and Curved Offset Beam Experiment

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Abstract—The straight beam in the ball and beam control experiment was replaced by a curved beam mounted away from its center of rotation. The resulting system is much harder to understand, model and control than the ball and straight beam. Nonetheless, the apparatus was constructed, a model was derived using Lagrangian mechanics, and a controller was designed using a linearized model and the LQR. This controller was implemented and successfully stabilized the ball on the curved offset beam.

I. INTRODUCTION

The ball and straight beam experiment is used commonly in teaching labs as an example of a system that is relatively easy to understand and model but challenging to control. As a result there are many publications dealing with different implementations of the apparatus and different controller designs for it. There are two typical constructions of the system. The first places the motor shaft at the center of the beam. The second has the pivot at one end of the beam and the motor drives the other end up or down. The dynamics of the two versions are slightly different but the complexity is approximately the same. The location of the ball (φ in Fig.1) is typically sensed by using a conductive ball that shorts resistive wires/strips or conductive plastics. To determine the position and angle of the beam, a potentiometer is commonly used. Recently, optical tracking techniques for both the ball and the beam have been used with some success.

The mathematical model of the system derived for the pivot at the left of the beam as shown in Fig. 1, is as follows [4]:

\[
0 = \left( m \ddot{\phi} + mg \sin(\theta) \right) - m \phi \ddot{\theta}^2
\]

\[
Q = \left( m \phi^2 + I_b \right) \ddot{\theta} + 2m \ddot{\phi} \dot{\theta} + (mg \phi + Mgr \cos(\theta)) \cos(\theta)
\]

where \( m \) is the mass of the ball, \( M \) is the mass of the beam, \( g \) is the acceleration due to gravity, \( I_b \) is the inertia of the beam about the pivot, \( r \) is the distance from the pivot to the center of the beam, \( \theta \) is the beam angle, \( \phi \) is the ball position, and \( Q \) is the torque applied by the motor. The \( Mgr \cos(\theta) \) term disappears if the pivot is at the center of the beam. Note that the rotation of the ball has been omitted as its effect is small relative to those included.

In contrast, this paper describes the design, construction, and control of a related system that is harder to understand, model, and control but still very reasonable to use in a teaching laboratory. The beam in this new system is part of a convex downward circle as shown in Fig. 2. The pivot point—equivalently, the motor shaft—is not on the beam but is offset to a point below the beam as shown in the figure. These changes make the control problem harder and more interesting. Deriving the mathematical model of this physical system is, as is shown in this paper, harder than for the straight beam but not too hard. Controlling the linearized version of the curved offset beam and ball is complicated by the appearance of at least one right half plane pole—two in our implementation. We believe these added difficulties enhance the educational value of the new experiment.

II. BACKGROUND

The earliest reference to the ball and beam experiment that we have found is due to P. E. Wellstead [1]. Wellstead acknowledges in that article that the ball and beam experiment “is widely used in universities throughout Europe; it was first introduced to the author by Johan Wieslander of the Division of Automatic Control, Lund Institute of Technology.” Wellstead has also published a paper on a ball and hoop experiment [2]. Normally, the ball rolls inside the hoop but Wellstead does mention the possibility of placing the ball on the outside of the hoop. However, his illustration of this would be completely uncontrollable as will be explained shortly.

A standard version of the ball and beam experiment simplifies the control problem by making the ball much lighter than the beam and by making the beam dynamics very fast compared to those of the ball. If one then linearizes the system about a beam angle of zero and a ball position at the center of the beam, the result is that the dynamics of the system separate into beam dynamics driving the ball dynamics. The ball dynamics have transfer function \( \hat{G}(s) = \frac{\phi(s)}{\dot{\phi}(s)} = \frac{1}{s^2} \) where \( \phi(s) \) is the Laplace transform of the ball’s...
position and $\theta(s)$ is the Laplace transform of the beam angle. A similar implementation of the ball and curved beam (with no offset) results in a ball subsystem that has two real poles, one of which is in the right half plane. This will be discussed in some detail later in this paper.

There are at least hundreds of references to the ball and beam experiment and its control in the literature. It is typical to analyze the stability of the system based on the linearized model and controller. However, in [5], standard PD controllers, both serial and parallel, were used and the stability of the nonlinear system was analyzed using the Lyapunov method. Nonlinear control techniques have also been used for the system. For instance, in [4], an approximate input-output linearization method with state feedback was investigated and compared to Jacobian linearization. In [3], the lambda method was implemented onto the ball and beam system and the matching equations derived. The resulting controller was less stable than a simple linear control law but the result is ambiguous because the opposite was the case in the inverted pendulum experiment.

III. FORMULATION OF THE PROBLEM

A. Derivation

Consider the coordinate system shown in Fig. 2, where the beam is an arc of a circle of radius $r$ and $d$ is the distance between the center of the motor shaft and the center of the beam circle. The $x$, $y$ positions and velocities of the ball are

$$x = r \sin(\phi - \theta) + d \sin(\theta)$$
$$y = r \cos(\phi - \theta) - d \cos(\theta)$$
$$\dot{x} = r \cos(\phi - \theta) (\dot{\phi} - \dot{\theta}) + d \cos(\theta) \dot{\theta}$$
$$\dot{y} = -r \sin(\phi - \theta) (\dot{\phi} - \dot{\theta}) + d \sin(\theta) \dot{\theta}$$

The kinetic energy in the system (ignoring the small amount of energy in the rotation of the ball), $T$, is approximately

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_\theta \dot{\theta}^2$$

The ball contributes a potential energy

$$V = mgy$$

For simplicity, the beam’s center of mass is adjusted to coincide with the motor shaft. This is not essential as any offset in the beam’s center of mass can be easily compensated for in the controller. Substituting equations (6) through (7) into $L = T - V$ yields the Lagrangian

$$L = \frac{m r^2}{2} (\dot{\phi} - \dot{\theta})^2 + \frac{m d^2}{2} \dot{\theta}^2 - m r d \cos(\phi) (\dot{\phi} - \dot{\theta}) \dot{\theta}$$
$$+ \frac{I_\theta}{2} \dot{\theta}^2 - m g d \cos(\phi - \theta) - m g d \cos(\theta)$$

Substituting the Lagrangian into the Euler-Lagrange equations,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i$$

for the radial force $Q_\rho$ resulting equation solved for the radial force $Q_\rho$ and $Q_\phi$ are the generalized forces (torques) corresponding to the two generalized coordinates $\theta$ and $\phi$, and solving for $\phi$ and $\dot{\phi}$ produces the following equations for the dynamics of the system ($Q_\phi = 0$)

$$\ddot{\theta} = \frac{1}{m d^2 + I_b - m d^2 \cos(\phi)^2} [m r d \sin(\phi) (\dot{\phi} - \dot{\theta})^2 - m g d \sin(\theta) - m g d \cos(\phi) \sin(\phi - \theta) - m d^2 \cos(\phi) \sin(\theta) \dot{\theta}^2 + Q_\theta]$$

$$\ddot{\phi} = \frac{-1}{r (m d^2 + I_b - m d^2 \cos(\phi)^2)} [m r g d \sin(\theta) - m g d^2 \cos(\theta) \sin(\phi) - m r^2 d \sin(\phi) (\dot{\phi} - \dot{\theta})^2 + m r d^2 \cos(\phi) \sin(\phi) (\dot{\phi} - \dot{\theta})^2 + m r g d \cos(\phi - \theta) + m r d^2 \cos(\phi) \sin(\phi) \dot{\theta}^2 - m d^3 \sin(\phi) \dot{\theta}^2 - I_{bg} \sin(\phi - \theta) - I_\theta d \sin(\phi) \dot{\theta}^2 - (r - d \cos(\phi)) Q_\theta]$$

B. Force of Interaction

This derivation assumed that the ball will always remain in contact with the beam. However, it is clear that there are many situations in which the ball will leave the beam. In this case, the equations derived for the dynamics of the ball and beam would no longer be appropriate. To address this, the force required to keep the ball in contact with the beam was calculated by first substituting the time dependent variable $\rho$ for the parameter $r$, thereby giving the system another degree of freedom. Then, the additional Lagrangian $\frac{d}{dt} \frac{\partial L}{\partial \dot{\rho}} - \frac{\partial L}{\partial \rho} = Q_\rho$ was derived, the constraint $\rho = r$ was enforced, and the resulting equation solved for the radial force $Q_\rho$ required to satisfy the constraint (see [6]). The equations for $\theta$ and $\phi$ are unaffected by this substitution. These steps yield the following result

$$Q_\rho = m d \cos(\phi) \dot{\theta}^2 + m d \sin(\phi) \dot{\theta}$$
$$- m r (\dot{\phi} - \dot{\theta})^2 + m g \cos(\phi - \theta)$$

$Q_\rho \geq 0$ implies that the ball is in contact with the beam. $Q_\rho < 0$ is physically impossible as the beam cannot pull on the ball so this condition implies that the ball leaves the beam. Thus, any simulation of the ball and beam is invalid once $Q_\rho < 0$ unless some additional provision is
made to describe the ball and beam moving separately. By ensuring that the force of constraint during an experiment or simulation never becomes negative, we can insure that the mechanics derived in the previous section is valid.

C. Dimensionless Parameters

To simplify the dynamics of the model and to broaden our results, we use dimensionless parameters. The initial model has seven parameters: $m$, $I_b$, $d$, $r$, $g$, $Q_0$ and $t$. We scale to a dimensionless time using $\tau = \sqrt{\frac{I}{g}} t$, $\frac{d\theta}{dt} = \sqrt{\frac{I}{mg}} \frac{d\theta}{d\tau}$, and $\dot{Q}_b(t) = \frac{g}{\tau^2} Q_0(t)$. From now on, $\dot{Q}_b$, $\dot{\theta}$, $\dot{\phi}$, and $\dot{\phi}$ will designate variables in the $\tau$ domain. We are also able to reduce the number of parameters to four by using the following substitutions: $I = \frac{I_b}{md^2}$, $Q = \frac{Q_0}{md\tau}$, $R = \frac{r}{d}$, $\hat{\gamma} = \sqrt{\frac{I}{g}}$ and $\tau$ is the independent variable. This reduces the model to

$$\dot{\theta} = \frac{1}{I + \sin^2(\phi)} \left[ R \sin(\phi)(\dot{\theta} - \phi) - R \sin(\theta) + Q \right]
- R \cos(\phi) \sin(\phi - \theta) - \cos(\phi) \sin(\phi) \dot{\theta}^2]$$ (12)

$$\dot{\phi} = \frac{1}{I + \sin^2(\phi)} \left[ \sin(\phi) \cos(\phi) (\phi - R)(2\theta - \phi)
+ \sin(\phi) \cos(\phi) + \sin(\phi) - (I - R \sin(\phi))
- R \sin(\theta) - 2 \cos(\phi) \sin(\phi) \dot{\theta}^2
+ \dot{\theta}^2 \sin(\phi) \left( R + \frac{1}{R} \right) + Q \left( 1 - \frac{1}{R} \cos(\phi) \right) \right]$$ (13)

$$\dot{Q}_\rho = \cos(\phi) \dot{\theta}^2 + \sin(\phi) \dot{\theta} - R (\dot{\phi} - \dot{\theta})^2
+ R \cos(\phi - \theta)$$ (14)

where $\dot{Q}_\rho = Q_\rho \frac{\tau}{md}$. 

D. Linearized Model

In contrast to the ball and straight beam which only has equilibrium points for which $\theta = 0$, the ball and curved beam has equilibrium points for many values of $\theta$. In fact, setting $Q_e = R \sin(\theta_e)$, $\phi_e = \theta_e$, and $\dot{\phi}_e = \dot{\theta}_e = 0$ establishes an equilibrium point for each value of $\theta$ as long as the beam is a full semicircle.

For simplicity, we choose to study the model linearized about $\theta_e = \phi_e = 0$. The linear model is then

$$\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\phi} \\ \dot{\phi} \end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{R}{T} - 1 & 1 + \frac{1}{T} & 0 & 0 \\
0 & 0 & \frac{1}{T} & (1 - \frac{1}{R})
\end{bmatrix}
\begin{bmatrix} \theta \\ \phi \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
\frac{1}{T}
\end{bmatrix}
Q$$ (15)

To analyze the linear system, we can examine the controllability and observability matrices. The controllability matrix

$$\mathcal{C} = \begin{bmatrix}
B & AB & A^2 B & A^3 B
\end{bmatrix}
= \begin{bmatrix}
0 & \frac{1}{T} & \frac{1}{T^2} & \frac{1}{T^3} \\
0 & \frac{R}{TR} - 1 & 0 & \frac{R^2}{T^2R} - 1 \\
0 & 0 & \frac{1}{T^2R} & \frac{R}{T^3R} - 1 \\
0 & 0 & 0 & \frac{R^2}{T^3R} - 1
\end{bmatrix}$$ (16)

always has full rank provided $R$ and $I$ are non-zero and finite and provided $I \neq 2R^2 - R$. Note that $T \rightarrow \infty$ when $d \rightarrow 0$, i.e., when the pivot is placed at the center of the circle. In this case, the highest point of the beam is always directly above the pivot, regardless of $\theta$. Thus, the ball must climb in order to reach the equilibrium point and this is not possible. This proves that placing the ball on the outside of the hoop, as proposed by Wellstead, results in an uncontrollable system.

The observability matrix when measuring only the position of the ball, the $\phi$ variable, is

$$\mathcal{O} = \begin{bmatrix}
C & CA & CA^2 & CA^3
\end{bmatrix}^T$$

$$= \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\frac{R}{T} - 1 & 1 + \frac{1}{T} & 0 & 0 \\
0 & 0 & \frac{R}{T} - 1 & 1 + \frac{1}{T}
\end{bmatrix}$$ (17)

It is somewhat surprising that $\mathcal{O}$ has full rank provided $R \neq I$. When $\theta$ is also measured (something that is easy to do) the system is obviously still observable.

The model linearized about zero (see Eqn. (15)) has two additional noteworthy properties. Firstly, the parameter $\gamma$ does not appear in Eqn. (15). Thus, the shape of the response of the system is the same for all values of $\gamma$. Changing $r$ while holding $R$ and $I$ constant only affects the time scale of the responses.

When, in addition, $R = 1$ (equivalently, the pivot is in the curved beam), Eqn. (15) decomposes into the following two equations.

$$\dot{\theta} = -\frac{1}{T} \phi + \frac{1}{T} Q$$ (18)

$$\dot{\phi} = (1 + \frac{1}{T}) \phi - (1 - \frac{1}{T}) \theta$$ (19)

This shows that the system decomposes into two sub-systems in series although there is an internal feedback loop. More precisely, Eqn. (19) describes a ball subsystem with input $\theta$ and output $\phi$. This subsystem has poles at $1 \pm \sqrt{1(1 + \frac{1}{T})}$. Eqn. (18) describes a beam subsystem with two inputs, a feedback of $\phi$ and an exogenous input $Q$, and one output, $\theta$. This demonstrates that the offset has the effect of coupling $Q$ into the ball equation, thereby complicating the control problem.

E. Motor Model

A DC motor was incorporated into the model for the simulations and experiments using the following equation

$$Q = \alpha V - \frac{\beta \dot{\theta}}{T}$$ (20)

where $\alpha$, and $\beta$ are the motor parameters and the $\dot{\theta}$ arises from the conversion to dimensionless "time" $\tau$. The motor includes a saturation at an input voltage $V_{sat}$. The output torque of this motor model is the torque input of the ball and beam model. The input voltage to the motor is the new control variable in the system.
The system linearized about the state vector equal to zero then becomes

\[
\begin{bmatrix}
\dot{\theta} \\
\phi \\
\dot{\phi}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{R}{\tau} - 1 & 1 + \frac{1}{\tau} & -\beta/\tau & 0
\end{bmatrix}
\begin{bmatrix}
\theta \\
\phi \\
\dot{\theta} \\
\dot{\phi}
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
\frac{Q}{\tau}(1 - \frac{1}{\tau})
\end{bmatrix} V
\]

(21)

with the new input variable, \(V\), corresponding to the voltage used to drive the motor.

**F. Controller**

There are at least three control design strategies that could be applied to the linearized version of the ball and curved beam system. The first two are essentially classical. Since the linearized system is controllable and observable when the ball angle alone is measured, one can try any classical procedure for the single-input single-output system with input \(V\) and output \(\phi\). A measure of how challenging this can be is easily obtained. Writing the transfer function \(G(s) = \frac{\phi(s)}{V(s)}\) when \(R = 1.5\), \(I = 10\), and \(\hat{\tau} = 3.6\), one obtains (These values are explained in Section (IV))

\[
G(s) = \frac{-2s^2 - 3.5}{s^4 - 2.2s^3 - 1.2s^2 + 2.6s - 12}
\]

(22)

This transfer function has poles at 2.2, 1, .05, and −1.1. It has zeros at ±j13.

When \(R = 1\) and both \(\phi\) and \(\theta\) are measured, a classical two-loop control strategy is effective provided one has a good enough motor. To understand the procedure, first examine the block diagram in Fig. (3).

![Fig. 3. The ball and curved beam system when R=1, including a DC motor](#)

Suppose that \(I = 2.5\). This is a reasonable value for our apparatus when \(R = 1\). It should be clear from Fig. (3) that larger \(I\) makes the control problem easier as long as the motor is powerful enough. Assume that we have a very good motor and motor driver, i.e., suppose that \(\alpha = 10\) and that \(\beta\hat{\tau}/I = 10\). The resulting system can be stabilized by the following two step procedure.

First, close an inner loop using feedback of \(\theta\). One can implement this by setting

\[
V = -10\theta + V_1
\]

(23)

where \(V_1\) is the new control input

Second, implement a lead compensator in the outer loop feedback of \(\phi\). Thus,

\[
V = -10\theta + 100\left(\frac{1 - 29s + 1}{1 - 29s + 1}\right)\phi
\]

(24)

This controller will stabilize the ball and curved beam system linearized about zero, as can easily be verified. The ideas behind the design are classical. They are an adaptation to the curved beam of ideas expressed by Wellstead in [1]. The inner (\(\theta\)) loop makes the beam dynamics wide band and fast compared to that of the ball. Because the beam dynamics consist of poles at 0 and \(-\beta\hat{\tau}/I\) a pure gain (10 in our case) suffices. The intrinsic feedback from the ball position acts as a negligibly small perturbation to these fast beam dynamics.

The outer (\(\phi\)) loop is then a second-order system with poles at \(\pm\sqrt{1 + \frac{1}{\tau}} = \pm1.29\). A lead compensator with its zero approximately equal to the stable pole and its pole conveniently further to the left (we use \(-5 \times 1.29\)) will then stabilize this system as long as the feedback gain is large enough (−100 produces a transient response with time constant \(\leq 5\)). This controller does require a very good motor and a motor input that does not saturate until the input voltage gets fairly large. In our case, an initial displacement of .3 radians would need a motor plus motor driver capable of ±70V without saturating. There are tradeoffs. A smaller gain reduces the control magnitude but slows the transient response. Because of the right half plane pole, any classical controller will need relatively high gain.

We have successfully tested this controller in simulation. We cannot test it on the real system because our motor is not powerful enough. The weaker motor we use makes the control problem more challenging.

The approach we took to designing a successful controller for the physical system was based on the linear quadratic regulator (LQR). This is also a controller design based on the linearized system. Assuming full state feedback, the controller is designed to minimize a performance measure

\[
J = \int_0^\infty (x^T Q x + u^2) dt
\]

where the weights are \(Q = 10I_{4 \times 4}\). These weights ensure that the controller moves fast enough to keep the ball in the linear region, which is the primary concern for making this idea work well.

Because the full state is not available, the LQR needs to be augmented by some estimator of the two unobserved states \(\dot{\theta}\) and \(\dot{\phi}\). We have chosen to estimate these states by differentiating and then low pass filtering \(\dot{\theta}\) and \(\dot{\phi}\). This works well enough and is easily explained to undergraduates.

Finally, there are many nonlinear design approaches that one might take. We are eager to try them.

**G. Apparatus**

A version of the physical system was designed and built. The major issues were to create a curved and offset beam
and to develop a sensor for the ball's position. The straight beam usually includes a linear resistor which is shorted by the conducting ball as a way to sense the ball's position. It was decided to emulate this model if possible.

A 30° piece of thin-walled, hollow stainless steel rod was curved in a hand-cranked rolling machine so that an arc with a radius of 0.94 meters was made. This tube was then cut in half and both pieces were trimmed down to have the same arc length. A block of Delrin was machined into a 2" X 1" X 2 rectangle and two 3/8" holes were then drilled 1" apart in the blocks to accommodate and hold the ends of the tubes. Delrin acts as an insulator between the two rails.

One piece of the tubing was slowly rotated as 34 gauge, vinyl coated magnet wire was wrapped around its length. The wire was allowed to wind next to itself as its protective coating keeps the wire from shorting to itself and the stainless steel rod. The overall resistance of this linear resistor was about 34 ohms. Once the wire was wrapped firmly around the tube, it was secured on both ends and longer lengths of wire were soldered to the loose ends. Both the wrapped and unwrapped tubes were then pressed into the Delrin blocks. The area where the ball makes contact with the wire on the beam was sanded down with 600 grit sandpaper to remove the protective coating on the magnet wire. This allowed a one inch diameter copper ball to act as the wiper arm on the linear potentiometer wound on the curved beam. The current return path through the other aluminum rail has negligible resistance. Applying a voltage of 1.2 volts DC across the beam completed the creation of both the curved beam and the ball position sensor.

Next, a simple plywood form was cut to hold the blocks into place and screwed into the back of the blocks. Multiple holes were drilled .04m apart along the vertical center line of the wooden to allow for the attachment of this assembly onto a Quanser SRV-02 Plant Motor set. We attached inputs "Analog 0" and "Analog 1" from the Quanser PCI DAC board to the ball sensor and the angle sensor (θ sensor) built into the SRV-02, respectively. The ground leads of both the "Analog 0" (ball position) and the 1.2 volt DC power supply are tied together. In our first trial of this experiment we had the "sensing rail" tied to ground through a 1 k Ohm resistor. This produced a serious noise problem whenever the ball lost contact with the wire wrapped rail. To reduce this noise problem we removed the 1kOhm resistor and allowed the ball position sensor rail voltage to simply discharge through the Quanser PCI DAC board if the ball lost contact with the wire wrapped rail. The input impedance of the Quanser PCI DAC board is high so the discharge rate of the ball position sensor voltage is slow. As a result, this effectively acts as a low-pass filter, thereby partially solving our noise problem.

For this implementation of the apparatus, the beam has an overall arc length of .3 m, a radius, $r = .94 m$, and the ball has a mass of approximately $m = .05 kg$. The pivot point could be varied from within the curved beam to below the beam at .04m intervals.

The motor used was the SRV-02 rotary servo supplied by Quanser Consulting. This motor has a torque constant, $K_m = 0.00767 Nm/amp$, a gear ratio, $K_g = 70$, $R_m = 2.6$ and a maximum voltage, $V_{sat} = 5.0 V$. These values result in $\alpha = .21$ and $\beta = .11$.

Measurement of the beam angle, $\theta$, was accomplished using the servo's own internal sensor, which has an accuracy of ±0.05 degrees. Measurement of the ball's position on the beam $\phi$ was accomplished using the sensor described above. The two velocities were estimated by simple differentiation (while more accurate methods exist, this method is simple and sufficiently accurate for our experiments). The appearance of the device is shown in Fig. 4.

The controller was implemented digitally using MATLAB/Simulink to create the high level code for the controller. The Digital to Analog conversion of the computed control signal was done by the Quanser MultiQ-PCI DAC. The Analog to Digital conversion of the sensor data was performed by the Quanser MultiQ-PCI ADC. The Simulink code was converted to real time C-code by means of Quanser’ WinCon. The sampling interval was .001secs.

Before each day’s operation, the copper ball was polished to remove the oxidation.

1) Improvement: It is apparent that using a soft copper wire for our experiment will work in the short run but this will not last for extended periods of use. The magnet wire started to deform under use and caused the resistance along the length of the tube to become nonlinear. A solution to this was found by using a bare 32 gauge nichrome wire wrapped simultaneously alongside a 30 gauge spacer wire over a stainless steel tube covered in shrink tubing. Once the two wires were wrapped around the stainless steel tubing, the spacer wire was carefully removed and the remaining nichrome wire was coated with an epoxy to hold it into place. This gives us a very durable surface for the copper ball to ride on that will maintain its linear resistive properties for a long time.

Fig. 4. The apparatus. The Quanser SRV-02 can be seen at the rear. Note the curved beam and the wooden beam holder. The wire wound section of the beam is the rail furthest from the camera.
IV. RESULTS

A. Simulations

To predict the performance of the system with the linear controller, we simulated the non-linear system with the controller in Matlab and Simulink. The dimensions of our simulated beam were as follows: $r = 0.75m$, $I_{beam} = 0.05kg/m^2$, $m_{ball} = 0.02kg$, with the pivot point at $d = 0.5m$. This set up is equivalent to the following dimensionless parameters: $R = 1.5$, $I = \frac{0.05}{0.75}r = 10$, and $\hat{r} = \sqrt{\frac{0.8}{0.75}} = 3.6$. In this model, we assumed that there was full access to the state.

Using the parameter values above, an LQR controller was designed. The weighting matrix $Q$ was chosen equal to $10I_{4 \times 4}$. The resulting feedback gain matrix was

$$K = \begin{bmatrix} 25.5 & -39.9 & 17.1 & -37.23 \end{bmatrix}$$

(26)

This control was applied to the nonlinear simulation. The results are shown in Figs. (5) and (6).

![Ball and Beam Simulation with $R = 0.94/0.90$ and initial $\phi$ of 0.3 radians](image)

Fig. 5. The simulation results with $R = 1.5$ and $\phi(0) = 0.3$ radians

Note that the force of interaction plotted in Fig. (5) shows that the ball would leave the beam at around $\tau = 12$. This is clearly unsatisfactory so a smaller initial $\phi$ was tried. This result is shown in Fig. (6).

![Ball and Beam Simulation with $R = 0.94/0.90$ and initial $\phi$ of 0.2 radians](image)

Fig. 6. The simulation results with $R = 1.5$ and $\phi(0) = 0.2$ radians

Thus, the simulations predicted that the ball would be controllable with an initial offset of up to $\phi = 0.2$ radians. This is satisfactory although it would be desirable to increase the domain of attraction of the closed-loop system. Although the LQR controller could probably be tweaked to produce improvement it would be much more interesting to attack the problem directly by means of a nonlinear controller design.

B. Laboratory Experiments

We applied the LQR controller to the actual apparatus and were successfully able to stabilize the ball on the beam at two different pivot locations, $R = 0.94/0.90$ and $R = 0.94/0.86$. The dimensions of our real beam were as follows: $r = 0.94m$, $I_{beam} = 0.05kg/m^2$, $m_{ball} = 0.02kg$, with the pivot point at $d = 0.9m$ or $d = 0.86m$. This set up is equivalent to the following dimensionless parameters: $R = 1.04$ or $R = 1.1$, $I = 3.09$ or $I = 3.38$, and $\hat{r} = 3.2$.

In both cases we used the control feedback gain matrix

$$K = \begin{bmatrix} 10.2 & -31 & 17 & -37 \end{bmatrix}$$

(27)

The controller stabilizes the ball angle for both values of R, as can be seen in Figs. (7) and (8). In both cases the transient response is somewhat underdamped. In both cases, the control input saturates in the early stages of the transient response, as can be seen in Figs. (9) and (10). This indicates that more work is needed in order to find the best choice for $Q$. It is encouraging that the same controller stabilized the system for both values of $R$.

The large amount of noise in the control signals in Figs. (9) and (10) is due to the approximate differentiation of $\theta$ and $\phi$. This noise has little or no effect because it is at a frequency that is high compared to the bandwidth of the closed-loop system. In fact, we have since changed the low-pass filter from $\frac{1}{\tau + 1}$ to $\frac{1}{\tau + 1}$ and this has largely eliminated the noise with no noticeable effect on the closed-loop system.

![Ball and Beam Experiment with $R = 0.94/0.90$](image)

Fig. 7. The experimental results with $R = 1.04$. The ball and beam angles are shown.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

In this paper, we have added complexity to the original ball and beam problem by creating a curved beam and by offsetting the location of the pivot point. We analyzed the feasibility of simple control techniques, classical two-loop and LQR, on a linearized model and tested the LQR-based controller in both Matlab simulations and laboratory experiments. The construction of the apparatus required the
use of several techniques different from the ball and straight beam to ensure sensor and model accuracy.

B. Future Works

There is much more to be done with the ball and curved offset beam experiment. First, the performance of the controllers based on linearization should be more fully explored. The largest possible domain of attraction, the best transient response, and stabilization to equilibrium points other than at $\theta = 0$ are all of interest.

Second, nonlinear control methods need to be explored. The key issue is to maximize the domain of attraction of the origin. The major difficulty that must be overcome is the tendency of the ball to leave the beam. Curvature and offset both exacerbate this problem.

Third, other curvatures can be explored. The techniques used here can also be used to place a position sensor on an arbitrarily curved beam. It would be interesting to try, for example, a quartic or catenary curve for the beam.

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