A SOS-based robust fault detection method for polynomial nonlinear systems

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Abstract—A novel nonlinear Luenberger-like filter design method is discussed for fault detection of nonlinear polynomial systems subject to unknown inputs. The disturbance rejection/fault sensitivity conditions are characterized by exploiting Algebraic Geometry methods. By means of the Positivstellensatz Theorem, a Sum of Squares (SOS) formulation of the design problem is obtained and an adaptive threshold is proposed to reduce the effect of false alarms. The effectiveness of the filter is illustrated via a numerical example taken from the literature.

I. INTRODUCTION

The demand for increased productivity leads to more challenging operation conditions for many modern engineering systems. These requirements increase the possibility of faults and/or malfunctions, which are characterized by anomalous and unpredictable changes in system dynamics. Advanced supervision and Fault Detection (FD) devices can help to improve plant reliability, efficiency and self-reconfiguration capabilities by early detection and accommodation of system failures.

The issue of fault detection has been addressed by many authors in several books and survey articles where different design methodologies have been exploited (model based approach, parameter estimation, generalized likelihood ratio etc.). See [5], [9] and references therein for comprehensive and up-to-date tutorials.

Amongst all existing methodologies, the design and analysis of FD model-based paradigms based on analytical redundancy approach has received a significant attention in the last two decades. The appeal of this methodology relies on more powerful information processing techniques which are compelled to generate the required redundancy factors. This class of FD algorithms exploits in fact model information to generate additional signals, referred to as residuals, that are conceived to be compared during the on-line operations with the corresponding measured quantities and generate fault alarms when a large discrepancy arises. This approach hinges upon two components: the first, an unknown input estimator whose role is to simultaneously decouple the residuals from the exogenous disturbances (robustness issue) and increase the sensitivity of the residuals w.r.t. faults. The second, a decision logic whose role is to possibly discriminate and isolate a single fault amongst many potential candidates [5], [17].

Robust Fault detection (FD) problems for linear systems have been extensively studied over the past three decades, and a lot of powerful methods have been developed. For a more thorough review, the reader is referred to [17] and references therein. However, the use of linear approaches gives rise to unavoidable level of conservativeness if the system to be monitored is strongly nonlinear and many working points are covered during operations. Because many industrial systems are nonlinear in nature, the development of nonlinear FD methods plays a significant role in practical applications. Recently, observer-based approaches [2], [4], [11], [27], [7], adaptive-residual-threshold approach [9], fuzzy-model-based [3] and neural-network-based methods [25] have been proposed. Most of the above-mentioned methods are based on a decoupling framework, where the model uncertainty and all possible faults can be decoupled through an appropriate state coordinate transformation and residual generation technique. However, the model uncertainty is often unstructured, which makes it difficult to achieve exact decoupling between faults and modelling errors.

Recently, the development of sum-of-squares (SOS) programming techniques and the Positivstellensatz Theorem from real algebraic geometry makes it possible to state nonlinear control analysis and synthesis problems for polynomial systems as SOS programs which are computationally tractable [18], [23], [24]. In particular, SOS decomposition problems have been found to be solvable by means of semidefinite programming techniques whose computational complexity is polynomial in the problem size [26], [19], [16]. As one of its major merits, the SOS-based approach provides less conservative results than most alternative available methods. In the literature, first attempts to exploit polynomial methodologies for Fault Detection and Diagnosis problems are very recent, see [8], [12]. In the first contribution, observer design for diagnostic purpose is studied by means of homogeneous Lyapunov functions and generalized Krasovskii-type Lyapunov functions. Also, Lyapunov functions which are nonquadratic with respect to the control system output are taken into consideration. In the second contribution, fault-tolerant control (FTC) problems for nonlinear systems, with guaranteed cost or $H_\infty$ performance objective in the presence of actuator faults are analyzed. There, the sum of squares (SOS) method is used to solve this kind of optimization problems in a reliable and efficient manner. One key difference between the proposed design and some existing FTC control methods is that the controller is built through algorithmic construction of the Lyapunov function.

Moving from these considerations, here we propose a
novel FD design procedure for nonlinear systems whose vector field is a multivariate polynomial in the state plant components. The proposed filter is designed so as to jointly robustly decouple the residuals from the disturbances and conversely enhance the sensitivity of the residual vector to faults. The residual generator consists of a “Luenberger-like” unknown nonlinear input observer (UNIO) [11] whose gain is also a multivariate polynomial in the state estimated variables. The disturbance/reference input rejection requirements and the fault sensitivity prescriptions are recast as SOS decomposition conditions and the UNIO design problem is proved to be a bilinear program (BMI) in the objective variables and solvable by means of local optimization algorithms.

A decision logic unit based on an adaptive threshold function is also proposed. The residuals are evaluated by means of a “rms-norm” time-based function and compared to a non-conservative threshold, which is derived according to SOS inequalities technicalities.

A simulation example taken from the literature (nonlinear model of an induction motor) is finally presented in order to illustrate the benefits and the effectiveness of the proposed SOS-based FD scheme.

II. PROBLEM STATEMENT

Let us consider the following non-linear plant

\[
\begin{align*}
\dot{x}(t) &= a(x(t)) + b(x(t)) u(t) + e(x(t)) f(t) + g(x(t)) d(t) \\
y(t) &= h(x(t))
\end{align*}
\]  

(1)

where

- \(x(t) \in \mathbb{R}^n\) denotes the state, \(u(t) \in \mathbb{R}^m\) the reference input and \(y(t) \in \mathbb{R}^p\) the measured output;
- \(f(t) \in \mathbb{R}^{n_f}\) denotes the fault signal, \(d(t) \in \mathbb{R}^{n_d}\) the exogenous disturbance. Both signals and the reference input as well, are supposed to be finite energy signals with given radius norms belonging to the following proper subsets of \(L_2\),

\[
\begin{align*}
\Omega_f &\triangleq \left\{ f(\cdot) \bigg| \exists \varepsilon_f > 0 \text{ s.t. } \int_0^\infty \| f(t) \|_2^2 \, dt \leq \varepsilon_f \right\} \\
\Omega_d &\triangleq \left\{ d(\cdot) \bigg| \exists \varepsilon_d > 0 \text{ s.t. } \int_0^\infty \| d(t) \|_2^2 \, dt \leq \varepsilon_d \right\} \\
\Omega_u &\triangleq \left\{ u(\cdot) \bigg| \exists \varepsilon_u > 0 \text{ s.t. } \int_0^\infty \| u(t) \|_2^2 \, dt \leq \varepsilon_u \right\}
\end{align*}
\]

- \(a(x) \in \mathbb{R}^{n_f}[x], b(x) \in \mathbb{R}^{n \times n_f}[x], e(x) \in \mathbb{R}^{n \times n_f}[x], g(x) \in \mathbb{R}^{n \times n_d}[x] \) and \(h(x) \in \mathbb{R}^p[x]\) are array matrices of multivariate polynomials, respectively.

In what follows we will suppose that the plant is exponentially stable and we will restrict our attention to process/actuators faults/disturbances. This is not a serious limitation because sensor faults/disturbances can be easily treated as process/actuators faults/disturbances as indicated in [22].

Based on the system representation (1), the objective is to design a diagnostic device capable to efficiently detect system dynamic deviations from the normal operating conditions due to faults. Here, we restrict our attention to “Luenberger-like” residual generators having the following expression (see, for example, [6])

\[
\begin{align*}
\dot{x}(t) &= a(\dot{x}(t)) + b(\dot{x}(t)) u(t) + \\
L(\dot{x}) &\left( y(t) - h(\dot{x}(t)) \right)
\end{align*}
\]

(2)

where \(L(\dot{x}) \in \mathbb{R}^{n \times p}[x]\) denotes the observer gain which is supposed to be a matrix of multivariate polynomials in the state estimate. As standard in robust FD problems, the designed filter needs to be sensitive to failures, viz. capable to distinguish failures from other unknown disturbances and reference inputs. We can now formally state the FD problem as follows

**Problem N-FD** - Given the nonlinear plant (1), compute a residual generator (2) which \(\forall t \geq 0\), achieves the following objectives

\[
\begin{align*}
\int_0^t \| r_d(\sigma) \|_2^2 \, d\sigma &\leq \gamma_d^2 \int_0^t \| d(\sigma) \|_2^2 \, d\sigma \\
\int_0^t \| r_d(\sigma) \|_2^2 \, d\sigma &\leq \gamma_u^2 \int_0^t \| u(\sigma) \|_2^2 \, d\sigma \\
\int_0^t \| r_f(\sigma) \|_2^2 \, d\sigma &\geq \beta^2 \int_0^t \| f(\sigma) \|_2^2 \, d\sigma
\end{align*}
\]

where \(\gamma_d, \gamma_u, \beta\) are suitable energy gains quantifying the performance of the residual generator and \(r_d, r_f\) the residuals obtained by cascading the observer output (eq. (2)) with the following two filters (see Fig. 1)

\[
\begin{align*}
\dot{x}_d(t) &= A_d x_d(t) + B_0 r(t) \\
\dot{r}_d(t) &= C_d x_d(t) \\
\dot{x}_f(t) &= A_f x_f(t) + B_f r(t) \\
\dot{r}_f(t) &= C_f x_f(t) + D_f r(t)
\end{align*}
\]

(6) (7)

having respectively a low-pass and a high-pass structure.

Note that the conditions (3) and (4) translate into the robust decoupling of residuals w.r.t. exogenous disturbances and reference inputs over a prescribed frequency window whereas condition (5) means that the residuals need to be sensitive w.r.t. to faults over a set of frequencies which, for solvability reasons, does not overlap with the disturbance decoupling frequencies set. Frequency weighting is important from practical points of view if the disturbances and faults have known non-overlapping spectra, e.g. the disturbances and reference inputs are slow signals whereas the faults needs to be promptly detected.

Now, by defining the augmented state

\[
\xi^T \triangleq [x^T \  \dot{x}^T \  x_d^T \  \dot{x}_d^T \  x_f^T \  \dot{x}_f^T] \in \mathbb{R}^{4n}
\]

it is possible to characterize the computation of the observer gain \(L(\dot{x})\) according to the setup shown in Fig. 1 whose state
where $\nu_d, \nu_u, \nu_f \geq 0$ are a-priori fixed scalarization terms and $\dot{V}$ denotes the total time derivative operator

$$\dot{V} = \frac{\partial V}{\partial \xi} (A(\xi, L) + B(\xi) u + \mathcal{E}(\xi) f + \mathcal{G}(\xi) d)$$

In the sequel, the conditions (10)-(12) will be rephrased by means of the SOS machinery. The use of a SOS approach will involve a certain unavoidable level of conservativeness, but on the other hand it has the appealing advantage of rephrasing the proposed nonlinear problem within a SDP paradigm. We have the following result:

**Proposition 1** - A pair $(V(\xi, f, u, d), L(x))$ satisfying the Problem SOS-N-FD prescriptions can be derived if there exist variables $s_i \in \Sigma_{n_c+n_d+n_f+m}$, $i = 1, \ldots, 12$, such that

$$V - l_1 \in \Sigma_{n_c+n_d+n_f+m}$$

where $l_1(\xi, f, u, d)$ is a chosen positive definite polynomial.

**Proof** - Let us consider eqs. (10)-(12) which can be translated into the following implications in the state variables $(\xi, f, u, d)$

$$\{\|u(t)\|_2 \leq \varepsilon_u\} \cap \{\|d(t)\|_2 \leq \varepsilon_d\} \cap \{\|f(t)\|_2 \leq \varepsilon_f\} \cap \{V \leq 1\} \subseteq \{\gamma_d^2 \|d\|^2_2 - \|r_d\|^2_2 - \dot{V} \geq 0\}$$

$$\{\|u(t)\|_2 \leq \varepsilon_u\} \cap \{\|u(t)\|_2 \leq \varepsilon_u\} \cap \{\|f(t)\|_2 \leq \varepsilon_f\} \cap \{V \leq 1\} \subseteq \{\gamma_u^2 \|u\|^2_2 - \|r_u\|^2_2 - \dot{V} \geq 0\}$$

$$\{\|u(t)\|_2 \leq \varepsilon_u\} \cap \{\|d(t)\|_2 \leq \varepsilon_d\} \cap \{\|f(t)\|_2 \leq \varepsilon_f\} \cap \{V \leq 1\} \subseteq \{\|f\|^2_2 - \beta^2 \|f\|^2_2 - \dot{V} \geq 0\}$$

By jointly exploiting the “Positivstellensatz” (P-satz) Theorem [16] and the S-procedure, the SOS inequalities (13) follow.

Note that the involved decision variables and polynomials do not enter linearly in the constraints and a BMI optimization problem results from (13). By jointly exploiting the yalmip matrix inequality parser and the package PENBMI [14], [15], it is possible to easily write the SOS conditions and achieve a local optimal solution.
III. THRESHOLD COMPUTATION

A key aspect of the proposed nonlinear FD paradigm is the evaluation of the generated residuals by means of a decision logic whose task is to decide if a fault has occurred at some time instance. The common design objective is to make this decision occurring with a predetermined and possibly low false-alarm error rate.

The easiest way of detecting a fault is to fix a threshold above which the residual function is considered to specify a fault occurrence in spite of any possible disturbances/noise realization. In most fault diagnostic systems, this approach is used to provide inputs to the supervisory level of the system.

Moving from these considerations, we will choose an adaptive threshold function for fault detection

\[ J_r(t) > J_{th}(t) \implies \text{with faults} \quad \text{alarm at instant } t, \]

\[ J_r(t) < J_{th}(t) \implies \text{no faults} \quad (17) \]

is considered where \( J_r(t) \) denotes a proper residual evaluation function. Let us consider now the residual expression

\[ r(t) = h(x(t)) - h(\hat{x}(t)) \quad (18) \]

Amongst all possible residual evaluation measures, a convenient choice is to consider the following time-based 'rms' function

\[ J_r(t) \triangleq \sqrt{\frac{1}{T} \int_0^T \|r(t)\|^2 dt} \quad (19) \]

To derive the threshold \( J_{th}(t) \) we will consider the fault-free case condition and the differential equations governing the state and state estimation evolutions

\[
\begin{align*}
\dot{x}(t) &= a(x(t)) + b(x(t)) u(t) + g(x(t)) d(t) \\
\dot{\hat{x}}(t) &= a(\hat{x}(t)) + b(\hat{x}(t)) u(t) + L(\hat{x})(y(t) - h(\hat{x}(t)) \\
r_0(t) &= h(x(t)) - h(\hat{x}(t))
\end{align*}
\]

where the observer gain \( L(\hat{x}) \) has been previously computed by solving Problem SOS-N-FD, eq. (13). Then, we will regard the model described in eq. (20) as a nonlinear system driven by the input \([u^T d^T]^T\). Therefore, a way to obtain a computable threshold consists in characterizing the induced \( L_2 \to L_2 \) gain, \( \gamma_{tot} > 0 \), such that

\[
\|r_0(t)\|_2 \leq \gamma_{tot} \left\| \begin{bmatrix} u(t) \\ d(t) \end{bmatrix} \right\|_2 \quad (21)
\]

This gain can be obtained by exploiting SOS arguments similar as those used in [13]. In fact, the inequality (21) can be turned into the following set containments

\[
\{ x \in \mathbb{R}^n, \dot{z} \in \mathbb{R}^n, u \in \mathbb{R}^m, d \in \mathbb{R}^n \mid \|x\|_2 \leq \varepsilon_x, \|d\|_2 \leq \varepsilon_d \} \cap \\
\{ x \in \mathbb{R}^n, \dot{z} \in \mathbb{R}^n, u \in \mathbb{R}^m, d \in \mathbb{R}^n \mid V(\xi)_{f=0} \leq 1 \} \leq \\
\{ x \in \mathbb{R}^n, \dot{z} \in \mathbb{R}^n, u \in \mathbb{R}^m, d \in \mathbb{R}^n \mid \|x\|_2 \leq \gamma_{tot} \left\| \begin{bmatrix} u \\ d \end{bmatrix} \right\|_2 \leq 0 \} \quad (22)
\]

where \( H \in \Sigma_{2n} \) is a decision variable and \( V \) has been obtained from Problem SOS-N-FD. Note that with the notation \( H \), we characterize the total time derivative of \( H(\cdot) \) w.r.t. the differential flow (20). A tight bound on \( \gamma_{tot} \) may be achieved by employing a generalized S-procedure argument and by solving the following SOS optimization procedure

\[
\min_{H \in \Sigma_{2n}} \gamma_{tot} \text{ s.t.} \\
\begin{align*}
H &= l \in \Sigma_{2n} \\
&-\left( H + \|r_0\|_2 - \gamma_{tot} \left\| \begin{bmatrix} u \\ d \end{bmatrix} \right\|_2 \right) \\
&- p_1(\varepsilon_o - \|u\|_2) - p_2(\varepsilon_d - \|d\|_2) - p_3(1 - v_f = 0) \in \Sigma_{2n+m+n_d} \quad (23)
\end{align*}
\]

with \( p_1, p_2, p_3 \in \Sigma_{2n+m+n_d} \) and \( l \in \Sigma_{2n} \) is an a-priori chosen positive polynomial. A a result, we obtain the following threshold

\[
J_{th}(t) = \sqrt{\frac{1}{T} \int_0^T \epsilon_{tot}^2 \left( \|u(\sigma)\|^2 + \varepsilon_d^2 \right) d\sigma} \quad (24)
\]

IV. NUMERICAL EXAMPLE

Consider the following polynomial \((d, q)\) model of the induction motor taken from [21]. The state variables are the rotor angular speed \( \Omega \) and the \((d, q)\) projections of the stator current and rotor flux: \( i_{sd}, i_{sq}, \varphi_{rd} \) and \( \varphi_{rq} \). The reference inputs are the \((d, q)\) projections of the stator voltage: \( v_{sd}, v_{sq} \) and the exogenous disturbance is represented by the load torque \( C_r \) which is unknown but bounded. The plant outputs are the only state variables that can be measured, \( L_1, \sigma \) and \( \tau \). Under usual hypotheses the model equations are

\[
\begin{align*}
\dot{x}_1 &= \frac{p M}{L_r} (x_5 x_2 - x_4 x_3) - \frac{C_r}{J} x_1 - \frac{C_r}{J} x_3 \\
\dot{x}_2 &= \frac{M}{L_{rd}} x_4 - \frac{K}{L_r} x_2 - p x_3 x_1 \\
\dot{x}_3 &= \frac{M}{L_{rd}} x_5 - \frac{K}{L_r} x_3 + p x_2 x_1 \\
\dot{x}_4 &= \gamma x_4 + \frac{M}{L_{rd}} \left( \frac{R_{sd}}{L_{rd}} x_2 + p x_1 x_3 \right) + \frac{1}{L_{rd} \sigma} v_{sd} \\
\dot{x}_5 &= \gamma x_5 + \frac{M}{L_{rd} L_{sq}} \left( \frac{R_{sq}}{L_{rd}} x_3 - p x_2 x_3 \right) + \frac{1}{L_{rd} \sigma} v_{sq}
\end{align*}
\]

where \( R_r, L_r, R_s, L_s \) are denoting the rotor/stator resistance/inductance, \( M \) is the rotor/stator mutual inductance, \( p \) is the number of pole pairs, \( K \) is the damping coefficient, \( J \) denotes the moment of inertia and

\[
\sigma = 1 - \frac{M^2}{L_r L_s}, \quad \gamma = \left( \frac{1}{\sigma} \right) \left( \frac{R_r}{L_s} (1 - \sigma) \frac{R_r}{L_r} \right)
\]

the plant output is \( y = (x_1, x_4, x_5)^T \) and numerical nominal values of the plant parameters are

\[
R_s = 10 \Omega, \quad L_s = 0.38 \ H, \quad R_r = 3.5 \Omega, \quad L_r = 0.3 \ H, \quad p = 2 \ M = 0.3 \ H, \quad J = 0.02 \ kgm^2, \quad K = 0.04 \ Nm \ s \ rad^{-1}
\]

We will consider anomalies on the plant coming from the rotor resistance and the faulty event will represent a deviation of such a parameter from its nominal value. As a consequence, it is possible to simply rewrite the electrical motor model (25) in the form (1), with the polynomials
involved in the differential flow given by

\[
a(x) = \begin{bmatrix}
p M \\
J L r \\
\end{bmatrix}
\]

and derived by means of Algebraic Geometric arguments. It can be proved that the obtained inequalities

We are interested to design a nonlinear filter capable to track fault signals in the frequency range \( \Omega = [0,1] \text{rad/s} \) and featuring a \(-40 \text{dB/dec}\) roll-off at higher frequencies. The design knobs are here summarized:

- Candidate Lyapunov function degree \( \partial(V) = \partial(H) = 2 \)
- and observer gain degree \( \partial(L) = 0; \)
- Scalar optimization weights, \( \nu_d = \nu_u = \nu_f = 1; \)
- \( L_2 \) ball radii, \( \varepsilon_d = 2, \varepsilon_u = \varepsilon_f = 1. \)

The SOS optimization procedure (9)-(12) has been implemented by exploiting the yalmip parser and the PENBMI solver. The computed observer gain and the disturbance rejection/fault sensitivity levels are as follows

\[
L = \begin{bmatrix}
344.837 \\
13.05 \\
19.873 \\
630.734 \\
-1118.105 \\
\end{bmatrix}, \gamma_d = 0.287, \gamma_u = 0.241, \beta = 0.853
\]

In Fig. 2, good disturbance decoupling properties of the filter in the absence of fault and reference inputs can be observed, the disturbance being a zero mean white noise input with variance equal to 2. We have considered the incipient fault profile depicted in Fig. 3. An anomalous event on the Rotor Resistance is observed starting at time \( t = 3 \text{ sec.} \). Note that at time \( t = 7 \text{ sec.} \) such a quantity reaches its largest deviation (50% of the nominal value) and the fault is recovered at time \( t = 18 \text{ sec.} \). The threshold gain has been computed by solving the SOS optimization problem (23) and \( \gamma_{tot} \) has been found equal to 0.325. In Fig. 4, the residual response (Upper sub-figure) \( r(t) \) and the evaluation function \( J_r \), together with the threshold \( J_{th} \) (Lower sub-figure), are depicted. The disturbance profile is identical to the fault-free simulation case (Fig. 2) and the reference inputs are two unitary amplitude steps. A quite satisfactory behavior for the designed residual generator can be observed. The anomalous behavior on the Rotor Resistance is in fact detected at time \( t = 11 \text{ sec.} \) and the alarm ends at time \( t = 25 \text{ sec.} \)

V. Conclusions

A solution to the Fault Detection problem for nonlinear polynomial plants has been proposed via a Luenberger-like unknown input observer.

The fault sensitivity constraints and disturbance/reference input rejection requirements have been converted into SOS optimization conditions expressed in terms of the filter design parameters and derived by means of Algebraic Geometric arguments. It can be proved that the obtained inequalities
are equivalent to SDP procedures and easily affordable from a computational point of view.

Real and false alarms have been discriminated by means of a detection logic based on a standard time-windowed adaptive threshold, again numerically derived by exploiting SOS arguments. A numerical example, taken from the literature, has been used to show the effectiveness of the proposed approach. The results testify good filter detection capabilities.

REFERENCES


