Stability analysis and state feedback control design of discrete-time systems with a backlash

Christophe Prieur, Ricardo C. L. F. Oliveira, Sophie Tarbouriech and Pedro L. D. Peres

Abstract—This paper considers the class of discrete-time nonlinear systems resulting from the connection of a linear system with a backlash operator. By conveniently exploiting the properties of the backlash, a class of candidate Lyapunov functions with quadratic terms and Lur’e type terms, derived from generalized sector conditions, is introduced. Using this class of Lyapunov functions, the stability of the time-shifted system is investigated. Additionally, the set of equilibrium points, which can be estimated, may be not reduced to the origin, since the backlash operator contains a dead-zone. Sufficient convex conditions, formulated in terms of semi-definite programming, are provided for the stability analysis and for the design of a linear stabilizing state-feedback controller. Numerical simulations illustrate the results and some computational issues.

I. INTRODUCTION

To overcome limitations of linear controllers in real process control, it may be fruitful to consider non-smooth nonlinearities, as hysteresis, backlash or dead-zone, in the synthesis problem. In this paper we focus on the analysis of stability and the control design for a linear system with a backlash in one of its inputs. A backlash operator (also called Krasnosel’skii-Pokrovskii hysteresis in [1]) is a slope-restricted nonlinearity with memory. Such a nonlinearity is present in mechanical systems and, if neglected during the control design or the stability analysis, may lead to important degradations of the closed-loop performance or even to the loss of stability (see, in particular, [2]). For control systems in the continuous-time domain, several approaches have been developed in the context of this type of non-smooth nonlinearity: see, for example, [3] and [4], [5], where an approach based on the inverse nonlinearity is applied. For continuous-time system with such a backlash, sufficient conditions based on semi-definite programming (SDP) have been provided in [6] (see also [7], [8] for a backlash and a saturation in the input).

The current paper proposes a constructive method to study both the stability analysis problem and the design of a stabilizing state-feedback linear controller for the nonlinear system. The class of nonlinear systems under consideration is a linear discrete-time system with a backlash in its input such as in [9], [10]. For the stability analysis case, it is assumed that the linear dynamics is Schur stable (as the Hurwitz stability has been assumed in [6], [8] for the continuous-time case) and, for the synthesis problem, that the linear system is stabilizable. As far as the stability analysis is concerned, the first main result (see Theorem 3.3 below) proposes some sufficient conditions to prove the convergence of the states to an equilibrium point. Moreover, the set of equilibrium points is estimated using the data of the backlash (this set may be not reduced to the origin, since the backlash operator contains a dead-zone). The sufficient conditions are written in terms of SDP constraints. A procedure to design a stabilizing state-feedback linear controller is also provided. To the best of the authors’ knowledge, this second main result (see Theorem 4.1 below) is the first contribution in terms of linear matrix inequality (LMI) conditions for the control design of a linear system with a backlash in the input.

The approach is based on quadratic Lyapunov functionals and on Lur’e type Lyapunov functions, including a generalized sector condition that uses the knowledge of the nonlinearity. In particular, without the term related to the sector condition, the corresponding LMI conditions have no solution (see the first point of Remark 3.2 below).

The paper is organized as follows. First, in Section II, we introduce the backlash operator and the system under consideration for the analysis problem. We also prove some properties on the backlash which will be instrumental to define the candidate Lyapunov functions. Then we solve the stability problem by studying the time-shifted system and by computing an estimation of the set of equilibrium. This is our first main result (see Section III). In Section IV we state numerically tractable sufficient conditions to compute a linear state feedback law. This is our second main result. In Section V we give some numerical simulations to illustrate our results. Section VI contains some concluding remarks and further research lines.

Notation. Given two vectors $x$ and $y \in \mathbb{R}^m$, we write $x \leq y$ if, for each $i \in \{1, \ldots, m\}$, we have $x_i \leq y_i$, where $x_i$ is the $i$-th component of $x$. $\text{diag}(A, B)$ denotes the diagonal matrices for which the diagonal elements are the matrices $A, B, I$ and $0$ denote respectively the identity matrix and the null matrix of appropriate dimensions. $|A|$ is the matrix constituted from the absolute value of each element of $A$, whereas $A_+$ and $A_-$ denote the two matrices constituted with non-negative entries such that $A = A^+ - A^-$ and $|A| = A^+ + A^-$. For two symmetric matrices, $A$ and $B$, $A > B$ (respectively $A \geq B$) means that $A - B$ is positive definite (respectively positive semi-definite). $A'$ denotes the transpose of $A$. The symbol $\star$ denotes the symmetric blocks in partitioned matrices.
Due to space limitation, the proofs are omitted.

II. DEFINITIONS AND PRELIMINARIES

Let us first introduce the nonlinear system under consideration for the stability analysis problem (as solved in Section III below), and prove some properties of the backlash operator.

Without any control, the nonlinear plant is described by:

\[
x(k+1) = Ax(k) + B\Phi[w](k), \quad k \geq 0
\]

\[
w(k) = Cx(k),
\]

where \( x \in \mathbb{R}^n \) is the state, the nonlinearity is \( \Phi \in \mathbb{R}^m \), the input of the nonlinearity is \( w \in \mathbb{R}^m \), and \( A, B, \) and \( C \) are real matrices of appropriate dimensions. This nonlinearity \( \Phi \) is assumed to be a backlash operator in discrete-time. More precisely, the backlash is defined componentwise by, for each \( w: \mathbb{N} \to \mathbb{R}^m \), for each \( i \in \{1, \ldots, m\} \), and for each \( k \in \mathbb{N} \),

\[
\Phi[w](k+1)_{(i)} = \begin{cases} 
\Phi[w](k)_{(i)} + \ell_{(i)}(w(k+1) - w(k)) & \text{if } w(k+1)_{(i)} \geq w(k)_{(i)} \\
\Phi[w](k)_{(i)} - \ell_{(i)}(w(k) - c_{(i)}) & \text{if } w(k+1)_{(i)} < w(k)_{(i)} \\
\Phi[w](k)_{(i)} & \text{otherwise}
\end{cases}
\]

(2)

where \( \ell \in \mathbb{R}^m \) has only positive entries, \( c_l \in \mathbb{R}^m \) and \( c_r \in \mathbb{R}^m \) are such that

\[
c_l \leq 0 \leq c_r.
\]

To ease the presentation, we will write \( \Phi(k) \) instead of \( \Phi[w](k) \). Denoting \( L = \text{diag}(\ell) \), we consider initial conditions such that

\[
L(Cx(0) - c_r) \leq \Phi(0) \leq L(Cx(0) - c_l).
\]

(3)

**Remark 2.1:** For any solution \( x: \mathbb{N} \to \mathbb{R}^n \) of (1) such that the initial condition satisfies (3), we have, for all \( k \geq 0 \),

\[
L(Cx(k) - c_r) \leq \Phi(k) \leq L(Cx(k) - c_l).
\]

(4)

The following lemma contains properties of the backlash operator which will be useful for the stability analysis.

**Lemma 2.2:** For any diagonal positive semi-definite matrices \( N_1 \) and \( N_2 \) in \( \mathbb{R}^{m \times m} \), we have, for all \( k \geq 0 \),

\[
(\Phi(k+1) - \Phi(k))'N_1(\Phi(k) - Lw(k)) \leq 0,
\]

(5)

and

\[
(\Phi(k+1) - \Phi(k))'N_2 \\
\times (\Phi(k+1) - \Phi(k) - L(w(k+1) - w(k))) \leq 0,
\]

(6)

for all solutions \( x(\cdot) \) of (1).

The conditions provided in Lemma 2.2 are also known in the literature as “sector conditions” and are an extension of the results given in [11].

III. STABILITY ANALYSIS

A. Stability analysis of the time-shifted system

Let us now solve the stability analysis problem for the nonlinear system (1). It is assumed that matrix \( A \) is Schur stable, i.e.

\[
x(k+1) = Ax(k), \quad k \geq 0
\]

is asymptotically stable. Since the backlash operator is defined in terms of its time-shift, it is useful to consider first the time-shifted version of (1). In this regard, let us consider the following auxiliary variables

\[
\tau(k) = x(k+1) - x(k), \quad (7)
\]

\[
\overline{\tau}(k) = \Phi(k+1) - \Phi(k).
\]

(8)

Using these variables, the aim is to investigate the stability of the system

\[
\tau(k+1) = A\tau(k) + B\overline{\tau}(k).
\]

(9)

The stability analysis conditions are given in the next theorem.

**Theorem 3.1:** If there exist a symmetric positive definite matrix \( P \in \mathbb{R}^{n \times n} \), a diagonal positive semi-definite matrix \( N_1 \in \mathbb{R}^{m \times m} \) and a diagonal positive definite matrix \( N_2 \in \mathbb{R}^{m \times m} \) satisfying

\[
N_1(I + LC(A - I)^{-1}B)' = (N_1(I + LC(A - I)^{-1}B))' \geq 0,
\]

(10)

\[
\left[ A'PA - P \quad A'PB + (N_1LC(A - I)^{-1})' + (N_2LC)' \right]
\]

\[
\times \left[ B'PB - 2N_2 + N_1(I + LC(A - I)^{-1}B) \right] < 0,
\]

(11)

then system (9) is asymptotically stable for all initial conditions satisfying (3).

In the proof of Theorem 3.1, the following candidate Lyapunov function is introduced, for each \( \tau : \mathbb{N} \to \mathbb{R}^m \), \( \overline{\tau} : \mathbb{N} \to \mathbb{R}^m \), and \( k \in \mathbb{N} \),

\[
v(\tau, \overline{\tau}, k)
\]

\[
= \tau(k)'P\tau(k) + \Phi(k)'S\Phi(k)
\]

\[
- 2 \sum_{j=0}^{k-1} \overline{\tau}(j)'N_1(\Phi(j) - LC(A - I)^{-1}(\tau(j) - B\Phi(j)))
\]

\[
- 2 \sum_{j=0}^{k-1} \overline{\tau}(j)'N_2(\overline{\tau}(j) - LC\tau(j))
\]

(12)

where \( P \) and \( S \) are symmetric positive definite matrices, \( N_1 \) is a diagonal positive semi-definite matrix and \( N_2 \) is a diagonal positive definite matrix of appropriate dimensions.

**Remark 3.2:** Some comments about this result and its proof are in order.

- The Lyapunov function (12) used in the proof has pure quadratic terms and also Lur’e type terms. These latter terms use the generalized sector conditions computed in Lemma 2.2 and properties of the backlash. Without these properties, we would have only a quadratic Lyapunov function candidate (let \( N_1 = N_2 = 0 \) in (12)) and, following the steps of the proof of Theorem 3.1, we would
obtain the LMI (11) with $N_1 = N_2 = 0$. Clearly, this inequality does not admit a feasible solution $P$ (consider the symmetric positive semi-definite term $B^PB$ in the diagonal). As a consequence, the knowledge of the sector conditions is essential for the stability analysis result.

Let us note that the conditions (10) and (11) do not depend on $c_r$ and $c_l$. This is due to the fact that, in Theorem 3.1, we are studying the stability of the time-shifted system (9), and the variables $\pi$ and $\Phi$ do not depend neither on $c_l$ nor on $c_r$. Thus, when the sufficient conditions (10) and (11) hold, the "speed" variable $\pi$ converges to zero (whatever the values of $c_r$ and $c_l$ are), as stated in Theorem 3.1. On the other hand, under these conditions, the position variable $x$ converges to a point $x(\infty)$ which depends on $c_r$ and $c_l$ (see Theorem 3.3 below).

Let us give some comments on the expression of the Lyapunov function (12). First let us note that the sums are for all index $j$ from $0$ to $j = k - 1$. Considering the index $k$ into that sum introduces the new variable $\pi(k + 2)$ in the computation of $\Delta v$. Moreover, this Lyapunov function is close in spirit to the Lyapunov function for continuous-time systems with a backlash as introduced in [6]. However, the proof of the negativity of its time-shift is different to the one for the continuous-time case, since to cancel the product between $\Phi$ and $\Phi$, we need to use part of the square of $\Phi$ (see the term (2.2) in the LMI (11)).

### B. Computation of the equilibrium-set

We follow the lines given in [12] by adapting to the discrete-time case. An equilibrium point should satisfy $x(\infty) = Ax(\infty) + B\Phi(\infty)$. This yields

$$x(\infty) = (I - A)^{-1}B\Phi(\infty).$$

(13)

Substituting (13) in (4), one has

$$-L_{cl} \leq (I - LC(I - A)^{-1}B)\Phi(\infty) \leq -L_{cl}.$$  

(14)

Let

$$R = (I - LC(I - A)^{-1}B)^{-1},$$

$$R = R_+ - R_-,$$

where $R_+$ and $R_-$ are real matrices with positive entries. Now note that (14) implies

$$-R_+L_{cl} \leq R_+(I - LC(I - A)^{-1}B)\Phi(\infty) \leq -R_+L_{cl} ,$$

and

$$-R_-L_{cl} \leq R_-(I - LC(I - A)^{-1}B)\Phi(\infty) \leq -R_-L_{cl} .$$

Combining both equations and using the definition of $R$, one has

$$-R_+L_{cl} + R_-L_{cl} \leq \Phi(\infty) \leq -R_+L_{cl} + R_-L_{cl} .$$

(15)

This is an estimation of the output of the backlash at the equilibrium.

Now, to estimate $x(\infty)$, we combine (13) and (15), by multiplying (15) by $(I - A)^{-1}B_+^R$ and $(I - A)^{-1}B_-^R$, and by subtracting the results. We get

$$((I - A)^{-1}B_+^R)(-R_1L_{cl} + R_2L_{cl})$$

$$-((I - A)^{-1}B_-^R)(-R_1L_{cl} + R_2L_{cl})$$

$$\leq x(\infty) \leq ((I - A)^{-1}B_+^R)(-R_1L_{cl} + R_2L_{cl})$$

$$-((I - A)^{-1}B_-^R)(-R_1L_{cl} + R_2L_{cl}) .$$

(16)

Relation (16) defines an estimation of the region containing the equilibrium points $x(\infty)$.

Combining Theorem 3.1 with the previous computation, we get our first main result about the stability analysis:

**Theorem 3.3:** If there exist symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, a diagonal positive semi-definite matrix $N_1 \in \mathbb{R}^{m \times m}$ and a diagonal positive definite matrix $N_2 \in \mathbb{R}^{m \times m}$ satisfying (10) and (11), then, for all initial conditions satisfying (3), the solutions of (1) converge to a point $x(\infty)$ satisfying the estimation (16).

Computing a numerical solution $(P, N_1, N_2)$ for the semi-positive definiteness constraints (10) and (11) can be difficult due to the equality constraint (10). Note that some LMI solvers (like the LMI Control Toolbox [13]) do not handle equality constraints. For SDP solvers that can manage equality constraints (like Sedumi [14]), the solution $(P, N_1, N_2)$ that is numerically computed is such that $N_1$ is symmetric positive definite (and not only semi-definite) and the LMI (10) is positive (and not only non-negative). See Section V-C below for exhaustive numerical simulations illustrating this fact.

From a numerical point of view, to analyze the stability of (1), it is then interesting to firstly consider the particular case in which we let $N_1 = 0$ in Theorem 3.3, and secondly to consider the case where we assume that $N_1$ is a diagonal positive definite matrix$^2$. Therefore it is interesting to firstly find a solution $(P, N_2)$ to the following problem.

**Problem 3.4:** Compute a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a diagonal positive definite matrix $N_2 \in \mathbb{R}^{m \times m}$ solution to

$$\begin{bmatrix}
A^TPA - P & A^TPB + (N_2L_{cl})'
\end{bmatrix} < 0 .$$

(17)

Secondly, if there does not exist a solution $(P, N_2)$ to problem (17), one has to find a solution $(P, N_1, N_2)$ to the following problem.

**Problem 3.5:** Compute a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and two diagonal positive definite matrices $N_1$ and $N_2 \in \mathbb{R}^{m \times m}$ solution to (11) and to

$$N_1(I + LC(A - I)^{-1}B) = (N_1(I + LC(A - I)^{-1}B))' > 0 .$$

(18)

Problem 3.4 is also very interesting for the design of a linear stabilizing state-feedback law (in contrast to Problem 3.5 where we have to tackle the inverse of the dynamic matrix $A$). This is the aim of next section.

$^2$In the case $m > 1$, the diagonal elements of matrix $N_1$ could be either positive or zero. To ease the presentation, only the separate cases $N_1 = 0$ and $N_1 > 0$ are addressed.
IV. CONTROL DESIGN

In this section we consider the synthesis problem for the nonlinear system. To do that we consider (1) with a control
\( u(k) \), yielding
\[
\begin{align*}
  x(k+1) &= Ax(k) + Bu(k) + B\Phi(k), \quad k \geq 0 \\
  w(k) &= Cx(k),
\end{align*}
\]
where \( B, C \in \mathbb{R}^{n \times p} \). The class of controllers is that of one linear state-feedback law (some generalizations may be possible, see the conclusion): \( u = Kx \), where \( K \in \mathbb{R}^{p \times n} \). Let us assume that the pair \((A, B)\) is stabilizable. We are now in position to state our second main result about the control design problem:

**Theorem 4.1:** If there exist a symmetric positive definite matrix \( W \in \mathbb{R}^{n \times n} \), a diagonal positive definite matrix \( N \in \mathbb{R}^{n \times m} \) and a matrix \( Z \in \mathbb{R}^{p \times n} \) such that
\[
\begin{bmatrix}
  -W & WA' + Z'B'C - WC'L' \\
  * & -W \\
  * & * \\
  * & -2N
\end{bmatrix} < 0
\]
then, letting \( K = ZW^{-1} \), for all initial conditions satisfying (3), the solutions of system (19) in closed-loop with \( u = Kx \) converge to a point \( x(\infty) \) satisfying the estimation (16).

V. NUMERICAL SIMULATIONS

A. Example 1 – Stability analysis

This experiment is concerned with the stability analysis of a single-input mechanical system describing the displacement \( x \) of a mass. The aim is to investigate the effect of an hysteretic phenomena related to the friction between the mass and the ground [15], [16]. The system is governed by the differential equation:
\[
  m\ddot{x}(t) + c\dot{x}(t) + \Phi[x](t) = u(t)
\]
where \( m \) is the mass, \( c \) is the damping constant and \( u(t) \) is the control force given by the PID controller:
\[
u(t) = -k_p x(t) - k_d \dot{x}(t) - k_i \int x(\tau) d\tau .
\]
The backlash operator \( \Phi \) corresponds to the friction and is, in this case, a function of the displacement. To apply the stability conditions developed in this paper, it is necessary to discretize the differential equation. By adopting the sampling time \( T_s = 0.05 \) seconds, using a zero-order interpolation (implemented in Matlab through the routine \texttt{c2d}) and choosing the values \( m = 1, c = 2, k_p = 10, k_d = 8, k_i = 4, c_r = -c_l = 0.5 \) and \( L = 1 \), a state-space representation (19) is obtained, with
\[
\begin{bmatrix}
  0.9999 & 0.0477 & 0.0011 \\
  -0.0043 & 0.9109 & 0.0392 \\
  -0.1652 & 0.2308 & 0.6758 \\
\end{bmatrix},
\begin{bmatrix}
  -0.0011 \\
  -0.0392 \\
  0.3242
\end{bmatrix},
\begin{bmatrix}
  0.0264 & 0.0264 & 0.05982 \\
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\end{bmatrix}
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  0.0264 & 0.5982 & 0.3242 \\
  0.0264 & 0.5982 & 0.3242
\end{bmatrix}
\]

Theorem 3.1 provides a feasible solution, assuring the stability of the system. The solution \((P, N_1, N_2)\) is given by
\[
\begin{bmatrix}
  2.5206 & 0.6852 & 0.4504 \\
  0.6852 & 3.5804 & 0.2433 \\
  0.4504 & 0.2433 & 0.9797
\end{bmatrix},
\begin{bmatrix}
  0.0133, \\
  0.6624
\end{bmatrix},
\begin{bmatrix}
  0.0133, \\
  0.6624
\end{bmatrix}
\]

For illustrative purposes, some simulations of the discretized system are performed with six different initial conditions for \( x(0) \) and for \( \Phi(0) = 0 \). The trajectories are depicted in Figure 1. Using the results of Theorem 3.3, it is possible to establish the following estimate for the equilibrium of the system:
\[
\begin{bmatrix}
  0 \\
  0 \\
  -0.5
\end{bmatrix} \leq x(\infty) \leq \begin{bmatrix}
  0 \\
  0 \\
  0.5
\end{bmatrix},
\]

which is corroborated by the simulations presented (see the thick black line depicted in Figure 1).

**Fig. 1.** Simulation of the discretized mechanical system given in Example 1 for six different initial conditions.

B. Example 2 – Control design

Consider system (1) with the state-space matrices
\[
\begin{bmatrix}
  -1 & -2 \\
  -1 & -1
\end{bmatrix}, \quad \begin{bmatrix}
  1 \\
  1
\end{bmatrix}, \quad \begin{bmatrix}
  -1 & 1 \\
  1 & -1
\end{bmatrix},
\]
\[
\begin{bmatrix}
  0 & 0 \ \\
  0 & 0
\end{bmatrix}
\]

For illustrative purposes, a time simulation has been performed for four different initial conditions for \( x(0) \) and for \( \Phi(0) \) = 0. The phase portrait is depicted in Figure 2. As can
be seen, the state converges to the interior of the black box, that is an estimative of the equilibrium set, given by
\[
-0.6 \leq x(\infty) \leq 0.6,
\]
on obtained through inequalities (16). Additionally, the evolution of \(x_1(k), x_2(k), \Phi(k)_{(1)}\) and \(\Phi(k)_{(2)}\) as a function of time \(k\) are shown in Figure 3 for the initial condition \(x(0) = [-2 -2]^T\).

**C. Computational issues about Problems 3.4 and 3.5**

To investigate the numerical behavior of Problem 3.4 (where \(N_1 = 0\)) and Problem 3.5 (where \(N_1 > 0\)), we performed exhaustive numerical simulations through randomly generated systems of the form (1) with the dimensions \(n = 2, \ldots, 5\) and \(m = n - 1, \ldots, 1\). For each pair \((n, m)\), one hundred systems were generated and the feasible solutions are shown in Table I.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(m)</th>
<th>Problem 3.4</th>
<th>Problem 3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>94</td>
<td>94</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>90</td>
<td>90</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>58</td>
<td>35</td>
</tr>
<tr>
<td>1</td>
<td>92</td>
<td>92</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>21</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>44</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>57</td>
<td>37</td>
</tr>
<tr>
<td>1</td>
<td>85</td>
<td>86</td>
<td></td>
</tr>
</tbody>
</table>

Two distinct situations can be observed from the results. Firstly, for single-input systems \((m = 1)\), where the additional variable \(N_1\) is a scalar, the numerical behavior for both Problems is quite similar: when \(m = 1\), for each system considered, we note that Problems 3.4 and 3.5 are both feasible or both infeasible. The only exception is in the case \((n = 5, m = 1)\) where the degree of freedom given by \(N_1\) guaranteed one more feasible solution.

Secondly, when \(m > 1\), the situation is very different, with a great advantage for Problem 3.4.

The conclusion of these simulations is that, except for a few systems, if Problem 3.4 does not have any solution, then Problem 3.5 does not have any solution as well. In other words, it is often useless to consider Problem 3.5. This motivated us to consider only Problem 3.4 (as done in Section V-B) for the control design.

**VI. CONCLUSION AND PERSPECTIVES**

The class of nonlinear systems resulting from a linear system with a backlash in its input was investigated in this paper. The time is discretized and the nonlinearity is defined by a slope-restricted operator with a memory. Using appropriate properties of the backlash nonlinearity, firstly, the stability analysis problem is addressed and, secondly, the design of a linear control law is investigated. The sufficient conditions to solve these two problems are written in terms of numerically tractable conditions, since they use LMI constraints.

As soon as the stability analysis problem is concerned, our main result contains an inequality constraint and an equality one (this is Theorem 3.3). This sufficient condition yielded two different numerical procedures (Problems 3.4 and 3.5). We checked by numerical simulations that, in general, it is sufficient to consider Problem 3.4 and, except for a few number of systems, it is not necessary to consider Problem 3.5. This motivated us to consider only Problem 3.4 for the control design.

Some LMI conditions have been already derived for the stability analysis of a nonlinear system with a backlash (consider e.g. [6], [8] for a continuous-time dynamics). However, to the best of our knowledge, the result of Theorem 4.1 is the
first one in the literature giving LMI conditions to design a control for a nonlinear system with a backlash in one input. The class of controllers, considered in this paper, is the one of linear state feedback laws. Some works are in progress to stabilize the system by means of other control strategies, as output or nonlinear feedback control laws (as proposed, in a different context, in [17]).

Another possible further research line is the study of systems with two nonlinearities which are assumed to be nested (a backlash and a saturation e.g.). For this problem it may be useful to combine the results of the present paper with techniques of [18] dealing with nested saturations.

REFERENCES