Non-Equilibrium Transient Trajectory Shaping Control via Multiple Barrier Lyapunov Functions for a Class of Nonlinear Systems

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Abstract—This paper presents a non-equilibrium transient trajectory shaping (NETTS) control technique based on a set of Barrier Lyapunov Functions for single-input-single-output strict feedback nonlinear systems. The trajectory shapes of the system outputs or tracking errors during the course of convergence to the equilibrium points are important for some physical systems (e.g. some hybrid systems) where reference signals frequently jump. A smooth trajectory shaping control law consisting of a unidirectional switching mechanism and a control signal continuous approximation method is proposed to ascertain that the system tracking error transient trajectory travels within a shaped-boundary while approaching zero. A numerical example is utilized to show the performance of the proposed NETTS control.

I. INTRODUCTION

Stability has been the main focus for nonlinear system analyses and control designs. While closed loop stability is of paramount importance and/or sufficient for a large class of nonlinear control systems, for some physical (hybrid) systems, such as multi-mode combustion engines [13], where reference signals frequently and discontinuously change, the shapes of the tracking error non-equilibrium transient trajectories (after frequent controller switching or reference jumping) are also critical. It is often desirable that the nonlinear control laws not only can ensure stability but also guarantee the system non-equilibrium trajectories travel within specifically-shaped boundaries/regions before they arrive at the equilibrium points. Such a requirement is particularly meaningful for some nonlinear systems where the references frequently and abruptly jump and the system non-equilibrium trajectory shapes directly affect system performance and safety. For instance, during advanced multi-mode combustion engine transient operations, the desired engine in-cylinder conditions (ICCs) often instantaneously jump during combustion mode switching to avoid undesirable intermediate combustion [13][14]. The shapes of the ICC non-equilibrium transient trajectories after the (frequent) controller switching significantly affect the combustion events during the combustion mode transitions. Consequently, they greatly influence the engine fuel efficiency, emissions, drivability, and combustion noise. Inappropriate ICC non-equilibrium transient trajectory shapes may jeopardize the combustion stability and result in unacceptable engine behaviors such as excessive emissions and unstable combustion. In order to ascertain desirable and avoid adverse combustion behaviors during the combustion mode transition, it is necessary to ensure that the non-equilibrium ICC transient trajectories travel through certain regions in the in-cylinder condition state-space during the combustion mode transitions.

While the multiple Lyapunov Functions (MLF) technique can be used to analyze the overall stability of a large class of hybrid systems [3], the shape of the tracking error non-equilibrium transient trajectory after controller switching or reference jumping in hybrid systems relies on the individual control law for each of the subsystems of the overall hybrid system. The NETTS is different from the linear hybrid system bumpless transfer and the trajectory safety verification problems. The former concerns the system output tracking error boundedness (e.g. in the sense of $L_2$) after switching [15][16] and control input continuity at switching [4]. The latter addresses the issue of certifying (not designing control laws) that all system trajectories starting from a given initial set do not enter the unsafe regions [10][11]. The NETTS here can be interpreted as explicitly designing control laws that guarantee the nonlinear system tracking error trajectories (after the controller switching or reference jumping) to travel within certain shaped boundaries when they are converging to the origin.

Several existing methods have the capability of explicitly handling constraints for a class of nonlinear systems. Nonlinear model predictive control [1][8] and reference governor/management [2][5] (and the references therein) are the two well-known methods. However their generally-required on-line computational effort associated with the nonlinear optimization may raise a concern for some applications (e.g. engine control) where computational resources are very limited [17]. Recently, the backstepping technique was utilized to realize non-overshooting tracking for a class of nonlinear systems in the strict feedback form [6]. Very recently, a Barrier Lyapunov Function (BLF), originally proposed in [9], has been employed to address constant state and output constraints for nonlinear systems in the strict feedback form [9]. Combined with the backstepping technique [7], BLFs were used to derive bounded nonlinear control laws that can explicitly prevent the system constant constraints from being violated [9][12].

In this paper, a NETTS control method that employs Multiple Barrier Lyapunov Functions (MBLF) along with a corresponding switching control law is proposed to explicitly generate a smooth control signal that guarantees the tracking error non-equilibrium transient trajectory to travel within specifically-shaped boundaries for single-input-single-output strict feedback nonlinear systems. Such a transient trajectory shaping control method is particularly valuable for nonlinear systems that experience frequent reference jumping. The well-shaped transient trajectories not only can benefit the system performance and ensure safety, but also can increase
the switching bandwidth of the hybrid systems by reducing the minimum dwell-time between two adjacent switching.

The arrangement of this paper is as follows. In section II, single Barrier Lyapunov Function-based output constraint problem in [12] is briefly reviewed. A direct switching control design method based on a set of BLFs for trajectory shaping is introduced in section III. In section IV, a NETTS control law is proposed to overcome the drawbacks of the direct switching control approach. In section V, a numerical example is given to evaluate the effectiveness of the algorithms.

II. PROBLEM INTRODUCTION
A. The Systems under Consideration
The problem considered in this paper is shaping the transient non-equilibrium tracking control error of nonlinear SISO strict feedback systems. Without loss of generality, such systems can be described in the following form:
\[ \dot{x}_i = f_i(x_i) + g_i(x_i)\dot{x}_{i+1}, \quad i = 1, 2, \ldots, n-1, \]
\[ \dot{x}_n = f_n(x_n) + g_n(x_n)u, \]
with the system output being
\[ y = x_1. \]

Denote \( y_d \) the desired reference and \( z_1 = x_1 - y_d \) the tracking error. \( \dot{x}_i \) represents the state vector \([x_1, x_2, \ldots, x_i] \).

The following assumptions are made: \( y_d, y_d, \ldots, y_d^{(n)} \) are bounded. \( f_i, g_i \) are smooth functions in the definitional domain. There exists a constant \( g_0 \) such that:
\[ |g_i| \geq g_0 > 0. \]

B. Single Barrier Lyapunov Function Approach Review
In [12], a Barrier Lyapunov Function, originally proposed in [9], was used to design control law that can explicitly make the system output satisfy a constant output constraint, as illustrated in Proposition 1.

Proposition 1 (Single Barrier Lyapunov Function): For \( nth \)-order SISO system as (1)-(3), a backstepping control law via Lyapunov Function \( V = V_e \) in the form of
\[ V_1 = \frac{1}{2} \log \frac{k_x^2}{k_x^2-z_1^2}, \]
\[ V_i = V_{i-1} + \frac{1}{2} z_i^2, \quad i = 2, \ldots, n, \]
can ensure the system output tracking error remain in the constant boundary, i.e. if \( |z_1(0)| < k_b \), then \( |z_1| \) can not violate the bound \( k_b \) during the transient to the equilibrium point.

Here \( V_e \) is the Barrier Lyapunov Function and \( k_b > 0 \) is a given constant bound for the output tracking error.

The control law is in the following form:
\[ \alpha_1 = \frac{1}{g_0} (-f_1 - (k_b^2 - z_1^2)) k_1 z_1 + y_d, \]
\[ \alpha_2 = \frac{1}{g_2} (-f_2 + \alpha_1 - k_2 z_2 - \frac{g_2 y_1}{k_2^2 - z_1^2}), \]
\[ \alpha_i = \frac{1}{g_i} (-f_i + \alpha_{i-1} - k_i z_i - \frac{g_i y_{i-1}}{k_i^2 - z_{i-1}^2}), \quad i = 3, \ldots, n, \]
\[ u = \alpha_n, \]
where
\[ z_1 = x_1 - y_d, \]
\[ z_{i+1} = x_{i+1} - \alpha_i, \]
\[ k_i > 0, \quad i = 1, \ldots, n \] are given constants.

Proof: Backstepping. \( \alpha_{i-1} \) is given by
\[ \dot{z}_i = \sum_{l=1}^{i-1} \frac{\partial \alpha_{l-1}}{\partial x_l}(f_l + g_l x_{l+1}) + \sum_{l=0}^{i-1} \frac{\partial \alpha_{l-1}}{\partial y_l} (y_d') \]
for \( i = 2, \ldots, n. \)
\[ \dot{z}_1 = -(k_b^2 - z_1^2) k_1 z_1 + g_1 y_d, \]
\[ \dot{z}_2 = -k_2 z_2 - \frac{g_2 y_1}{k_2^2 - z_1^2} + g_2 z_2, \]
\[ \dot{z}_i = -k_i z_i - \frac{g_i y_{i-1}}{k_i^2 - z_{i-1}^2} + g_i z_{i+1}, \quad i = 3, \ldots, n - 1 \]
\[ \dot{z}_n = -k_b z_n - g_n y_{n-1} - \alpha_n. \]

Take time derivative of \( V_i \):
\[ \dot{V}_1 = \frac{z_2^2}{k_2^2 - z_1^2} = z_1 (f_1 + g_1 (y_1 + z_1)) \frac{y_d'}{y_d} \]
By choosing \( \alpha_i \) as (8),
\[ \dot{V}_i = -k_i \dot{z}_i^2 + \frac{g_i^2 \dot{z}_i^2}{k_i^2 - z_{i-1}^2} \]

Consider (12), then
\[ \dot{V}_2 = -k_2 \dot{z}_2^2 + \frac{g_2^2 \dot{z}_2^2}{k_2^2 - z_1^2} + \dot{z}_2 \dot{z}_2 = -k_1 \dot{z}_1^2 + \frac{g_1^2 \dot{z}_1^2}{k_1^2 - z_i^2} + \dot{z}_2 \dot{z}_2 - \frac{g_1 \dot{z}_1}{k_1^2 - z_i^2} + g_1 \dot{z}_3 \]
\[ \dot{V}_i \]
\[ \dot{V}_n = -\sum_{i=1}^{n} k_i \dot{z}_i^2, \quad i = 3, \ldots, n - 1 \]
\[ \dot{V}_n = -\sum_{i=1}^{n} k_i \dot{z}_i^2, \quad i = 3, \ldots, n - 1 \]
\[ \dot{V}_n = -\sum_{i=1}^{n} k_i \dot{z}_i^2, \quad i = 3, \ldots, n - 1 \]
\[ \dot{V} = \dot{V}_e \leq 0 \text{ and } \dot{V} < 0 \text{ for } \forall z_i \neq 0, \quad i = 1, \ldots, n \]

Thus, the asymptotic stability of the system equilibrium point is proved.

Remark: From (5) and (6), we can realize whereas \( |z_1| \) approaches to \( k_b \), Lyapunov Functions \( V_e \) and thus \( V \) approach to infinity. As the Lyapunov Function \( V \) cannot be greater than its initial value \( V_0, |z_1| \) should be bounded by \( k_b \).

III. DIRECT SWITCHING ALGORITHM FOR NETTS
Here we first introduce a method that uses multiple Barrier Lyapunov Functions and a direct switching control law to achieve the transient trajectory shaping purpose without considering the constraint of the control signal and the converging speed of the Lyapunov Function. Such an approach is refereed as direct trajectory shaping (DTS) method in the rest of this paper.

Theorem 1 (DTS): Given a desired strictly decreasing boundary set:
\[ K = \{ k_{b,1}, k_{b,2}, \ldots, k_{b,n} \} | k_{b,j} > k_{b,j+1} > 0, j = 1, \ldots, m - 1 \] and a buffer set \( \Delta = \{ \Delta_j | 0 < \Delta_i < \Delta_j, i = 1, \ldots, m \} \), if the condition \( |z_1(0)| \leq k_{b,1} - \Delta_1 \) is satisfied, then a series of control laws exists such that the tracking error has the following non-equilibrium trajectory-shaping property:
Once \( z_1 \) enters the region \( \{ |z_1| \leq k_{b,j} - \Delta_j \} \), \( j = 1, \ldots, m \), it will be bounded by \( k_{b,j} \) until the equilibrium point is achieved. Here, the \( k_{b,j} \) and \( \Delta_j (k_{b,j} > \Delta_j) \) are the constant boundary and buffer for the \( j \)th control law.

Proof: The value of \( z_1 \), when it enters the region \( \{ |z_1| \leq k_{b,j} - \Delta_j \} \), \( j = 1, \ldots, m \), is viewed as the initial condition in Proposition 1. Given a set of Lyapunov Functions as:
\[ V^j = \frac{1}{2} \log \frac{k_{b,j}^2}{k_{b,j}^2 - z_1^2} + \frac{1}{2} \sum_{l=2}^{m} z_l^2, \quad j = 1, \ldots, m \]

The corresponding control law \( u_j \) can be derived through backstepping method in Proposition 1, by choosing \( k_b \) ask \( k_{b,j} \), for \( \forall z_j \in \{ k_{b,j+1} - \Delta_{j+1} \leq |z_j| \leq k_{b,j} - \Delta_j \} \), \( j = 1, \ldots, m \) .

By unidirectional switching through the control laws \( u_j \) associated with the \( V^j \), the tracking error \( z_1 \) will be shaped
into a boundary specified by a series of strictly decreasing bound values in the set \( K \) during its transient to zero.

**Remark:** 1) \( \Delta \), \( j = 1, 2, \ldots, m \) is the buffer region for control design purpose, which will be explained in the following section; 2) the parameter \( k_i \) (the gain value in (7)-(10)) affects the Lyapunov Function converging speed according to Eq. (22); and 3) the control signal from the unidirectional switching control law designed in Theorem 1 is discontinuous during the switching, and thus it is difficult to be realized when actuators have rate limits.

**Notation:** \( i \) (\( i = 1, 2, \ldots, n \)) represents the \( i \)th element for an \( n \)th order system; \( j \) (\( j = 1, 2, \ldots, m \)) represents the \( j \)th boundary or \( j \)th switching (totally \( m \)) in trajectory shaping method.

In the following section, a smooth switching NETTS control algorithm is proposed to address three issues: 1) how to choose the parameter \( k_{ij} \) (the gain value gain \( k_i \) in the \( j \)th control law); 2) control signal continuity; and 3) the condition for choosing the buffer size.

**IV. SMOOTH SWITCHING ALGORITHM FOR NETTS**

In this section, a smooth and stable NETTS control approach using MBLF is proposed. Such a method is referred as the smooth trajectory shaping (STS) method in the rest of the paper.

**A. Choice of the Converging Parameters**

For the DTS method, at the moment of the \( j \)th control law switching, \( t_j \), the Lyapunov Function increases discontinuously when the boundary of output becomes smaller. Denote \( V^j \) the Lyapunov Function right before the \( j \)th switching. At the moment of the switching, \( t_j \), the new Lyapunov Function increases discontinuously due to the sudden change of the boundary value from \( k_{bj} \) to \( k_{bj+1} \) in the BLF (see Eqs. (5)(6)). So, at the time of switching,

\[
V^{j+1}(z(t_j)) > V^j(z(t_j)).
\]

According to (22), the decreasing speed of the new Lyapunov Function by the new control law, \( u_{j+1} \), is,

\[
\dot{V}^{j+1}(t_j) = -\sum_{i=1}^{n} k_i z_i^2(t_j),
\]

which is the same as the one provided by the prior control law, \( u_j \), for the \( V^j \) before the switching:

\[
\dot{V}^j(t_j) = -\sum_{i=1}^{n} k_i z_i^2(t_j).
\]

Thus, one can see that after the \( j \)th switching, for the new control law, \( u_{j+1} \), the Lyapunov Function \( V^{j+1} \) suddenly increased. Even though the output error \( z_i \) can be bounded into a smaller constraint \( k_{bj+1} \), the total Lyapunov Function will remain higher than that of the case without the \( j \)th switching for a certain period of time. Such a discontinuity may affect the convergences of the other states \( z_i, i = 2, \ldots, n \), and therefore influence the behavior of \( z_i \). Undesirable phenomenon such as “oscillation” or slow converging speed may occur (as shown in the simulation examples in section V). To avoid such abrupt change on Lyapunov Function during switching, the converging parameters can be modified.

Here, we give the condition for choosing \( k_{ij} \) after each switching. The \( i \) here refers to the parameters in the \( i \)th backstepping for the same control law and the \( j \) refers to the parameter in the \( j \)th control law (with \( k_{bj} \) being the bound).

**Proposition 2:** In the switching from the \( j \)th bound to \((j+1)\)th bound, i.e., the bound of the output changes from \( k_{bj} \) to \( k_{bj+1} \), to avoid the jump between Lyapunov Functions, \( k_{ij+1} \) can be chosen as:

\[
k_{ij+1} = k_{ij}/R,
\]

where

\[
R = \frac{\log \left( \frac{k_{bj}}{k_{bj+1} - (k_{bj} + k_{bj+1}) \alpha_{ij}} \right)}{2} > 0, \quad \alpha_{ij} = \frac{\sum_{i=2}^{n} z_i^2(t_j)}{2}.
\]

**Proof:** The Lyapunov Function before the \( j \)th switching is:

\[
V_j = \frac{1}{2} \log \frac{k_{bj}^2}{k_{bj} - z_1^2} + \frac{1}{2} \sum_{i=2}^{n} z_i^2,
\]

and after switching is

\[
V_{j+1} = \frac{1}{2} \log \frac{k_{bj+1}^2}{k_{bj+1} - z_1^2} + \frac{1}{2} \sum_{i=2}^{n} z_i^2.
\]

Denote:

\[
\tilde{V}_{j+1} = R V^{j+1} = \frac{1}{2} \log \frac{k_{bj+1}^2}{k_{bj+1} - z_1^2} + \frac{1}{2} \sum_{i=2}^{n} z_i^2.
\]

It is easy to verify that the Lyapunov Function is modified to be a continuous function between switching, i.e.,

\[
\tilde{V}^{j+1}_{j+1}(t_j) = \tilde{V}^{j+1}(t_j) = \dot{\tilde{V}}^{j+1}(t_j) = \frac{1}{2} \log \frac{k_{bj+1}^2}{k_{bj+1} - z_1^2} + \frac{1}{2} \sum_{i=2}^{n} z_i^2.
\]

By choosing \( k_{ij+1} = k_{ij}/R \) and conducting the same backstepping procedure as Proposition 1 based on \( \tilde{V}^{j+1} \) instead of \( V^{j+1} \), the control law \( \tilde{u}_{j+1} \) is derived and it satisfies

\[
\dot{\tilde{V}}^{j+1}(t_j) = -R \sum_{i=2}^{n} k_{ij+1} z_i^2(t_j) = -\sum_{i=2}^{n} k_{ij+1} z_i^2(t_j).
\]

Comparing with (27), we can realize that the converging speed remains the same level. By conducting the same parameter selection during every switching, the Lyapunov Function will decrease smoothly, once \( k_{ij} \) is chosen. To be noted, by proposition 2, \( k_{ij} \) (the \( k_i \) in the \( j \)th switching control law) is not the same as in subsection B, which is used for DTS.

**B. Continuity of Control Signal and Buffer Size**

In physical systems, the discontinuous control signals are usually difficult to be realized, and thus directly applying Theorem 1 may not be feasible. In this subsection, we deal with the control signal continuity problem during switching.

**Proposition 3 (Continuous Approximation):** During switching transient from bound \( k_{bj} \) to \( k_{bj+1} \), the control signal is continuous by the following approximation:

\[
k_{bj}(t) = k_{bj} + (k_{bj+1} - k_{bj}) \left[ 1 - \cos \left( \frac{\pi (t-t_j)}{T_j} \right) \right]^{n-1} > 0,
\]

where \( T_j \) is the switching period.
\[ \bar{k}_{ij}(t) = k_{ij} + (k_{ij+1} - k_{ij}) \left[ 1 - \cos \left( \frac{\pi (t - t_j)}{a_j} \right) \right]^{n-1} > 0, \quad (36) \]

for \( t \in [t_j, t_j + a_j] \). For \( t \in (t_j + a_j, t_{j+1}) \), \( k_{ij} \) and \( k_{ij+1} \) are chosen as positive constants according to Proposition 2. Here, \( a_j \) is the given transient time for the continuous approximation being active after each switching and its relation to the actuator rate limit is continuous.

**Proof:** As can be easily verified, the boundary function \( \bar{k}_{ij}(t) \geq k_{ij+1} \) (\( \bar{k}_{ij}(t) = k_{ij+1} \) only when \( t = t_j + a_j \)) and converging parameter \( k_{ij} \leq \bar{k}_{ij} \leq \bar{k}_{ij+1} \) are continuous and their \( n \)th derivatives exist after approximation \((n-1)\)th derivatives are continuous. Thus, according to (8)-(11), the approximated control signal is continuous.

To conserve that the stability and trajectory shaping results in Theorem 1 still hold during and after the continuous approximation period, Theorem 2 and 3 are given with the assumption that in system (1) both \( f_1 \) and \( g_1 \) are bounded for bounded \( x_1 \).

**Theorem 2 (Boundedness):** If \( |z_1(t)| < k_{b,j+1} - \frac{a_j}{2} < \bar{k}_{b,j}(t) \) holds during the period when the continuous approximation is active, i.e. \( t \in [t_j, t_j + a_j] \), and \( R(\Delta^{(1)}_{j+1})a_j > -0.5 \), \( R(\Delta^{(1)}_{j+1})a_j > -0.5 \)

\[ R(\Delta^{(1)}_{j+1}) = k_{b,j} - k_{j-1,j} \left[ \frac{g_1 q(\Delta^{(1)}_{j+1})}{2 k_{j,z_j}} \right]^2 \leq k_{j-1,j} - k_{j+1,j} \left[ \frac{g_1 q(\Delta^{(1)}_{j+1})}{2 k_{j,z_j}} \right]^2, \quad (37) \]

where,

\[ R(\Delta^{(1)}_{j+1}) = \frac{k_{b,j} - k_{j+1,j}}{2 k_{j,z_j}} \left[ \frac{\rho^{(1)}(z_j)}{2 k_{j,z_j}} \right]^2 \leq R(\Delta^{(1)}_{j+1}) = \frac{k_{b,j} - k_{j+1,j}}{2 k_{j,z_j}} \left[ \frac{\rho^{(1)}(z_j)}{2 k_{j,z_j}} \right]^2, \quad (38) \]

and \( R \) is the resultant positive constant in Proposition 2, then, \( z_1 \) is bounded with respect to the choice of \( \Delta^{(1)}_{j+1} \), the conditions of states \( x \) at the \( j \)th switching time \( t_j \), bound of tracking error trajectory \( k_{b,j} \), bounds of \( f_1 \) and \( g_1 \), and bounds of tracking reference \( y_d \) and its derivatives during the continuous approximation period \( t \in [t_j, t_j + a_j] \).

**Proof:** The Lyapunov Function after the \( j \)th switching is \( V^{(j+1)}(t) \)

\[ V^{(j+1)}(t) = R \left( \frac{1}{2} g_1 \rho^{(1)}(z_j) + \frac{1}{2} \sum_{i=2}^{n} \dot{z}_i \right)^2. \quad (40) \]

According to Proposition 3, the control law during the period of continuous approximation is:

\[ \alpha_1 = \frac{1}{g_1} (-f_1 - \bar{k}_{j,1}(t)z_1 + \dot{y}_d), \quad (41) \]

\[ \alpha_2 = \frac{1}{g_2} (-f_2 + \bar{k}_{j,2}(t)z_2 + g_1 z_1 - \frac{g_1 z_1}{2 k_{j,z_j}}), \quad (42) \]

\[ \alpha_i = \frac{1}{g_i} (-f_i + \bar{k}_{j,i}(t)z_i - \frac{g_1 z_1}{2 k_{j,z_j}}), \quad i = 3, \ldots, n, \quad (43) \]

\[ u = \alpha_i. \quad (44) \]

Applying a similar procedure as the proof in Proposition 1,

\[ \bar{V}^{(j+1)} = \frac{k_{b,j+1} - k_{j+1,j}}{2 k_{j,z_j}} \left[ \frac{\rho^{(1)}(z_j)}{2 k_{j,z_j}} \right]^2 \left[ \bar{k}_{j,1}(t)z_1 - \frac{1}{k_{b,j+1}-z_1} - \sum_{i=2}^{n} \bar{k}_{j,i}(t)z_i \right]^2. \quad (45) \]

Here we denote \( P = \frac{1}{k_{b,j+1}-z_1} - \frac{1}{k_{b,j} - z_1} \), which is zero at \( t = t_j + a_j \). Consider the first three terms in (45).

\[ \bar{V}^{(j+1)} \leq \frac{k_{b,j+1} - k_{j+1,j}}{2 k_{j,z_j}} \left[ \frac{\rho^{(1)}(z_j)}{2 k_{j,z_j}} \right]^2 \left[ \bar{k}_{j,1}(t)z_1 - \frac{1}{k_{b,j+1}-z_1} - \sum_{i=2}^{n} \bar{k}_{j,i}(t)z_i \right]^2. \quad (46) \]

Here we have

\[ \bar{k}_{j,1}(t) - k_{j,1} \left( \frac{\rho^{(1)}(z_j)}{2 k_{j,z_j}} \right)^2 \geq k_{j-1,j+1} \left( \frac{\rho^{(1)}(z_j)}{2 k_{j,z_j}} \right)^2 \geq k_{j-1,j+1} \left( \frac{\rho^{(1)}(z_j)}{2 k_{j,z_j}} \right)^2 = R(\Delta^{(1)}_{j+1}), \quad (47) \]

with

\[ |P| = \frac{1}{k_{b,j+1}-z_1} - \frac{1}{k_{b,j}-z_1} \leq \frac{k_{b,j+1}-k_{j+1,j}}{k_{b,j} - k_{j+1,j}}. \quad (48) \]

Thus, we have \( V^{(j+1)} \geq -R(k_{j-1,j+1} - \frac{\rho^{(1)}(z_j)}{2 k_{j,z_j}})^2 \sum_{i=3}^{n} \bar{k}_{j,i}z_i^2 \leq -R(\Delta^{(1)}_{j+1})z^2, \forall t \in [t_j, t_j + a_j] \). If \( R(\Delta^{(1)}_{j+1}) > 0 \), \( V^{(j+1)} \) is negative definite and thus \( V^{(j+1)}(t) \leq V^{(j+1)}(t_j), \forall t \in [t_j, t_j + a_j] \). According to (40), \( |z_2| \leq \sqrt{2V^{(j+1)}(t_j)/R} \). \( \forall t \in [t_j, t_j + a_j] \). If \( R(\Delta^{(1)}_{j+1}) \leq 0 \), \( \dot{V}^{(j+1)}(t_j - \frac{t_j + a_j}{2}) \leq R(\Delta^{(1)}_{j+1})z^2 dt \leq V^{(j+1)}(t_j) - \frac{R(\Delta^{(1)}_{j+1})z^2 max_a \dot{q}_j, \forall t \in [t_j, t_j + a_j]. \) According to (40), \( z^2 \leq 2z_{max} \leq 2V^{(j+1)}(t_j)/R \). Under the condition in (37), \( |z_1| \leq z_{2,\text{max}} \leq \sqrt{2V^{(j+1)}(t_j)/R \left[ 1 + 2R(\Delta^{(1)}_{j+1}) a_j \right]}, \forall t \in [t_j, t_j + a_j]. \) Here \( z_{2,\text{max}} \) is the maximal value of \( |z_2| \) and \( |z_1| \) is the maximal value of \( V^{(j+1)} \) during \( [t_j, t_j + a_j] \). Thus, by (41) and the assumption that \( y_d^{(1)}, x_1, g_1 \) and \( f_1 \) are bounded, \( x_1 \) is bounded. Then, \( z_2 \) is bounded due to \( z_2 = x_2 - \alpha_1 \). So \( x_1 \) is bounded according to Eq. (1) and \( \dot{z}_1 \) is bounded because of \( z_1 = x_1 = \dot{y}_d \). Here we denote \( |z_1| \leq M \) (\( M \) can be calculated from the bounds).

**Theorem 3 (Buffer Size Determination):** With the control signal continuous approximation as Proposition 3, the non-equilibrium transient trajectory shaping results in Theorem 1 still hold if the buffer size at the moment of switching satisfies the following condition:

\[ k_{b,j+1} > \Delta^{(1)}_{j+1} + \Delta^{(2)}_{j+1} > 0, \quad (49) \]

with \( \Delta^{(1)}_{j+1} \) satisfying the condition (37) and \( \Delta^{(2)}_{j+1} \) satisfying the following condition:

\[ \Delta^{(2)}_{j+1} > M \cdot a_j > 0. \quad (50) \]
Proof: Here we only need to show that condition (50) can ensure that $|z_1(t)| < k_{b,j+1} - \Delta_{j+1}^{(1)}$ holds during the period when the continuous approximation is active right after the switching, i.e. $t \in [t_j, t_j + a_j]$. Then, according to Theorems 1 and 2, Theorem 3 holds.

Suppose that even by choosing $\Delta_{j+1}^{(2)}$ and $a_j$ as (50), during $t \in [t_j, t_j + a_j]$, $|z_j|$ will increase beyond $k_{b,j+1} - \Delta_{j+1}^{(1)}$. As $|z_1(t_j)| = k_{b,j+1} - \Delta_{j+1}$ and $|z_1(t)|$ is continuous, there must exist a $t_p \in (t_j, t_j + a_j)$ such that $|z_1(t_p)| = k_{b,j+1} - \Delta_{j+1}^{(1)}$ and $|z_1(t)| < k_{b,j+1} - \Delta_{j+1}^{(1)}$, $\forall t \in [t_j, t_p)$. According to Theorem 2, $\forall t \in [t_j, t_p]$, $|\dot{z}_1(t)| \leq |k_{b,j+1} - \Delta_{j+1}^{(1)} + M \cdot a_j| < k_{b,j+1} - \Delta_{j+1}^{(1)}$. Contradiction. Therefore, such a $z_1(t_p)$ does not exist and $|z_1|$ cannot increase beyond $k_{b,j+1} - \Delta_{j+1}^{(1)}$, i.e., $|z_1(t)| < k_{b,j+1} - \Delta_{j+1}^{(1)}$ holds for $t \in [t_j, t_j + a_j]$.

Remark: With the control input continuous approximation during a period of $a_j$ after a switching, the $\Delta_{j+1}$ and $a_j$ determine the tradeoff between the demand on control signal rate of change and the slope steepness (determined by $k_{b,j+1}$ and $\Delta_{j+1}$) of the shaped boundary for the non-equilibrium transient tracking error trajectory.

V. NUMERICAL EXAMPLE

To illustrate the performance of the NETTS, we consider the same example as used in [12] except larger initial conditions:
$$\begin{align*}
\dot{x}_1 &= 0.1x_1^2 + x_2, \\
\dot{x}_2 &= 0.1x_1x_2 - 0.2x_1 + (1 + x_1^2)u, \\
y &= x_1,
\end{align*}$$
(51)
(52)
(53)

The tracking reference is:
$$y_d = 0.2 + 0.3\sin(t).$$
(54)

The initial conditions are:
$$x_1(0) = 0.75, x_2(0) = 1.2.$$  
(55)

A. Case 1: Quadratic Lyapunov Function Approach

Control law is designed by using a quadratic Lyapunov Function. As the backstepping control law design via quadratic Lyapunov Function is mature, we omit the detailed design procedure here. The converging parameters were chosen as $k_1 = k_2 = 2$. Fig. 1 shows the performance.

B. Case 2: Single Barrier Lyapunov Function Approach

A control algorithm was designed based on a Single Barrier Lyapunov Function (SBLF) as Proposition 1. The bound of output $k_b$ is set to be 0.6 and the converging parameters are $k_1 = k_2 = 2$. Fig. 2 shows the performance.

C. Case 3: Direct Trajectory Shaping Approach

In Case 3, a DTS control law is applied. The bounds were chosen as $k_{b,1} = 0.6$, $k_{b,2} = 0.15$, $k_{b,3} = 0.05$. The converging parameters are $k_1 = k_2 = 2$. As Fig. 3 shows, the tracking error of the system was kept within each bound during a respective period. However, after the last bound was applied, as analyzed before, the converging speed was slow and “chattering” happened. Furthermore, the control signal evolves discontinuously as shown is Fig. 3. As the control signal is related to the derivative of the output error, the oscillation of the output error made the control signal magnitude significantly large, as illustrated in Fig.3.

D. Case 4: Smooth Trajectory Shaping Approach

In this case, the smooth trajectory shaping control law is applied. The bounds were chosen as those in Case 3, i.e. $k_{b,1} = 0.6$, $k_{b,2} = 0.15$, $k_{b,3} = 0.05$. The initial converging parameters are $k_{1,1} = k_{2,1} = 2.1$. After each switching, these parameters were changed according to Propositions 2 & 3. As Fig. 4 illustrates, the chattering problem and control signal discontinuity were both eliminated. The demand on control signal rate of change was similar as the one for the SBL in Case 2.

E. Comparison of the Four Control Methods

The quadratic Lyapunov Function method (QLF) is the most common approach. In the example, however, it transgresses the bound of 0.6. To be mentioned, by choosing...
control gains appropriately [6], non-overshooting tracking response can be achieved by QLF in strict feedback system. Here we use the QLF with the same control gains as the other three cases. The SBLF can render the tracking error trajectory within the bound. However, both QLF and SBLF approaches cannot shape the transient trajectory. The DTS method provides the ability of constraining the tracking error in a desired region during a particular period. However, due to the low converging speed and control input discontinuity, it is not applicable in real control problem. The STS method exhibits superior performance over the other three methods. The phase planes and tracking errors in four cases are compared in Figs. 5 & 6.

![Fig. 4 Tracking error and control signal by the STS method.](image)

![Fig. 5 Comparison of the $z_1$-$z_2$ phase planes in four cases.](image)

![Fig. 6 Comparison of the tracking errors of the four control laws.](image)

VI. CONCLUSIONS AND FUTURE WORK

In this paper, a NETTS method based on a set of Barrier Lyapunov Functions is proposed to generate a continuous and smooth control signal which can ascertain that the non-equilibrium tracking error transient trajectory travels within a specifically-shaped boundary before it arrives at zero for SISO nonlinear systems in strict feedback form. The parameter selection, boundedness, stability, control input continuity, and the steepness of the shaped boundary are systematically addressed. A numerical example clearly demonstrated the advantages and effectiveness of the proposed method. In the future work, the control input constraints, system uncertainties, and the extensions to multi-input-multi-output nonlinear systems will be investigated further.

REFERENCES


