Global Finite-Time Stabilization of a Class of Upper-Triangular Systems

Shihong Ding, Chunjiang Qian, and Shihua Li

Abstract—In this paper, we consider the problem of global finite-time stabilization for a class of upper-triangular systems. First, we use the generalized adding a power integrator technique to design a homogeneous controller which locally finite-time stabilizes the upper-triangular systems. Then, we integrate a series of nested saturation functions with the homogeneous controller and adjust the saturation level to ensure global attractivity. The combination of two steps yields the global finite-time stability of the considered upper-triangular systems.

I. INTRODUCTION

In this paper, we consider the global finite-time stabilization for a class of upper-triangular systems

\[ \dot{x}_i = d_{i+1}(x)x_{i+1} + f_i(x_{i+1}, \ldots, x_n, u), \quad i = 1, \ldots, n-1, \]
\[ \dot{x}_n = d_n(x)u + f_n(u) \]

(1.1)

where \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n, u \in \mathbb{R} \) are system states and control input, respectively. The nonlinear functions \( d_i(x), \quad i = 1, \ldots, n \) and \( f_i(x_{i+1}, \ldots, x_n, u), \quad i = 1, \ldots, n \) are \( \mathcal{C}^0 \) functions which are not exactly known.

Compared to the massive results achieved for lower-triangular systems, in the literature there are fewer results on the global stabilization of upper-triangular systems. Among the existing results in handling the upper-triangular systems, much of the attention has been focused on the stabilization of a chain of linear integrators perturbed by an upper-triangular nonlinear vector field. In this case, there are two major successful tools including the nested saturation method introduced in [14], [15] and later expanded in many works such as [6], [5], [17], [2], and the integrator forwarding technique [7], [13], [12], which is based on the Lyapunov’s direct design.

Note that the aforementioned results only achieve global asymptotic stabilization. Since finite-time stable systems have better convergence and disturbance rejection properties [1], the research on finite-time stabilization of upper-triangular systems is of theoretical and practical interest. This paper aims to solve the global finite-time stabilization problem for the upper-triangular systems (1.1). To this end, we will combine the top-down and bottom-up design procedure introduced by [16] with a generalized adding a power integrator recently developed in [8], [9]. First, using the generalized adding a power integrator method [8], [9], we will design a homogeneous control law from top to bottom for system (1.1) without considering the perturbation terms \( f_i(\cdot) \)'s. Then we will show this homogeneous stabilizing law also renders the original upper-triangular system (1.1) locally stable in a given region. Second, we will impose a series of nested saturations to the homogeneous controller. Under the saturated control law, from the bottom to top, we can also prove that the closed-loop systems will converge to the given region in finite time by adjusting the saturation level, where the controller is no longer saturated and ensures the finite-time stability.

II. LOCAL STABILIZATION OF SYSTEM (1.1)

We first introduce the following assumptions:

Assumption 2.1: In a neighborhood of the origin, the following holds

\[ |f_i(t)| \leq \rho_i(x_{i+1}^{q_i-1} + \cdots + x_n^{q_n-1} + |u|^{q_{n+1}}) \]

(2.1)

for positives constants \( \rho \) and \( q_{ij} \) satisfying

\[ q_{ij} > r_{i+1}/r_j > 0, \quad 1 \leq i \leq n, \quad i + 1 \leq j \leq n + 1. \]

(2.2)

with a constant \(-\frac{1}{n+1} < \tau < 0\).

Assumption 2.2: There exist \( \bar{d}_i > 0 \) and \( \bar{d}_j > 0 \) such that

\[ 0 < \bar{d}_i \leq d_i(x) \leq \bar{d}_i, \quad i = 1, \ldots, n. \]

(2.4)

We first propose a method to local stabilize system (1.1).

Theorem 2.1: Under Assumptions 2.1 and 2.2, there exist a constant \( \beta_1^* = n/d_1 \) and functions \( \beta_i^* : \mathbb{R}^{i-1} \to \mathbb{R}, \quad i = 2, \ldots, n \), such that for any constants \( \beta_i \) satisfy \( \beta_1 \geq \beta_1^*, \quad \beta_2 \geq \beta_2^*(\beta_1), \quad \ldots, \quad \beta_n \geq \beta_n^*(\beta_1, \ldots, \beta_{n-1}) \), the following control law

\[ u = -\beta_n(x_n^{1/r_n} - x_n^{1/r_n})^{r_{n+1}} \]

(2.5)

with \( x_1^* = 0, \quad x_i^* = -\beta_{i-1}(x_{i-1}^{1/r_{i-1}} - x_{i-1}^{1/r_{i-1}})^{r_i}, \quad i = 2, \ldots, n \), locally finite-time stabilizes system (1.1).

Proof. For simplicity, we define \( X_i = (x_1, \ldots, x_i), \quad i = 1, \ldots, n \), which will be used throughout this paper. At first, the controller (2.5) will be constructed to globally stabilize the following system

\[ \dot{x}_i = d_i(x)x_{i+1}, \quad i = 1, \ldots, n-1, \quad \dot{x}_n = d_n(x)u. \]

(2.6)

The process to construct the control law for system (2.6) is similar to the one introduced in [9] except for the flexibility in selecting controller gains \( \beta_i \)'s proposed in this paper.
Step 1: For a Lyapunov function $V_1(x_1) = \int_0^{x_1} (s_1/r_1 - 0)^2 - r_2 ds$, the derivative of $V_1$ along system (2.6) is
\[ \dot{V}_1(x_1) = d_1(x) x_1^{2-r_2}/r_1 x_2^* + d_1(x) x_1^{2-r_2}/r_1 (x_2 - x_2^*) \] (2.7)
where $x_2^*$ is a virtual control law. We select the virtual controller $x_2^*$ as $x_2^* = -\beta_1 \xi_1^2$, with $\xi_1 = x_1^{1/r_1}$, $\beta \geq \beta_1 = n/d_1$. By (2.7), we have
\[ \dot{V}_1 \leq -n \xi_1^2 + d_1(x) \xi_1^{2-r_2}(x_2 - x_2^*). \] (2.8)

Inductive step: Suppose that at step $i - 1$, there exist a constant $\beta_i = n/d_1$ and functions $\beta_k^*: \mathbb{R}^{k-1} \to \mathbb{R}$, $k = 2, \ldots, i - 1$, such that for any constants $\beta_k$'s satisfying $\beta_1 \geq \beta_1^*$, $\beta_2 \geq \beta_2^*(\beta_1)$, $\ldots$, $\beta_{i-1} \geq \beta_{i-1}^*(\beta_1, \ldots, \beta_{i-2})$, the following holds
\[ V_{i-1}(X_{i-1}) \leq -(n - i + 2)(\xi_1^2 + \cdots + \xi_i^2) + d_{i-1}(x) \xi_{i-1}^{2-r_{i-1}^*}(x_i - x_i^*), \] (2.9)
where
\[ V_{i-1}(X_{i-1}) = \sum_{k=1}^{i-1} \int_{x_k^*}^{x_k} s_1^{1/r_k} - x_k^{1/r_k} 2 - r_k ds, \]
\[ x_1^* = 0, x_k^* = -\beta_{k-1} \xi_{k-1}^2, k = 2, \ldots, i, \]
\[ \xi_k = x_k^{1/r_k} - x_k^{1/r_k}, k = 1, \ldots, i. \] (10)
We claim that (2.9) will also hold at step $i$. To complete the induction argument at the $i$-th step, we consider the Lyapunov function
\[ V_i(X_i) = V_{i-1}(X_{i-1}) + W_i(X_i), \] (11)
with
\[ W_i(X_i) = \sum_{k=1}^{i-1} \int_{x_k^*}^{x_k} s_1^{1/r_k} - x_k^{1/r_k} 2 - r_k \] ds. The derivative of $V_i(X_i)$ along system (2.6) is
\[ \dot{V}_i(X_i) \leq -(n - i + 2)(\xi_1^2 + \cdots + \xi_i^2) + \sum_{k=1}^{i-1} \frac{\partial W_i(X_i)}{\partial x_k} \dot{x}_k \]
\[ + d_{i-1}(x) \xi_{i-1}^{2-r_{i-1}}(x_i - x_i^*) \]
\[ + d_i(x) \xi_i^{2-r_i}(x_{i+1}^* + d_i(x) \xi_i^{2-r_i}(x_{i+1}^* - x_{i+1})), \] (12)
for a virtual controller $x_{i+1}^*$ to be determined later. Next we estimate each term in the right hand side of (12).

Note that $0 < r_i \leq 1$, by Young’s inequality
\[ d_{i-1}(x) \xi_{i-1}^{2-r_{i-1}}(x_i - x_i^*) \leq \overline{d}_{i-1} 2^{2-r_i}|\xi_{i-1}^{2-r_{i-1}}| |\xi_i^r| \]
\[ \leq \frac{1}{2} \xi_{i-1}^2 + c_i \xi_i^2, \] (13)
where $c_i$ is a positive constant.

To facilitate the proof, we give the following proposition whose proof can be found in [9].

**Proposition 2.1:** There exists a gain $\gamma_i(\beta_1, \ldots, \beta_{i-1}) > 0$ whose value is determined by $\beta_1, \ldots, \beta_{i-1}$ such that
\[ \sum_{k=1}^{i-1} \frac{\partial W_i(X_i)}{\partial x_k} \leq \frac{1}{2} (\xi_1^2 + \xi_2^2 + \cdots + \xi_i^2) + \gamma_i(\beta_1, \ldots, \beta_{i-1}) \xi_i^2. \]
Substituting (2.13) and Propositions 2.1 into (12) yields
\[ \dot{V}_i(X_i) \leq -(n - i + 1)(\xi_1^2 + \cdots + \xi_i^2) + (c_i + \gamma_i(\beta_1, \ldots, \beta_{i-1}) \xi_i^2 + d_i(x) \xi_i^{2-r_i}(x_{i+1}^* + d_i(x) \xi_i^{2-r_i}(x_{i+1}^* - x_{i+1})). \]

Choosing the gain $\beta_i \geq \beta_i^*(\beta_1, \ldots, \beta_{i-1}) := ([\xi_i + \gamma_i(\beta_1, \ldots, \beta_{i-1}) + n - i + 1]/d_i^2$, the controller $x_{i+1}^* = -\beta_i \xi_{i+1}^2$ yields $V_i(X_i) \leq -(n - i + 1)(\xi_1^2 + \cdots + \xi_i^2) + d_i(x) \xi_i^{2-r_i}(x_{i+1} - x_{i+1}^*)$. This completes the inductive proof.

From the inductive proof above, at step $n$, we can find series of gains $\beta_1^*, \beta_2^*(\beta_1), \ldots, \beta_n^*(\beta_1, \ldots, \beta_{n-1})$ such that for any $\beta_i$'s satisfying $\beta_1 \geq \beta_1^*$, $\beta_2 \geq \beta_2^*(\beta_1)$, $\ldots$, $\beta_n \geq \beta_n^*(\beta_1, \ldots, \beta_{n-1})$, the control law (2.5) yields
\[ V_n(X_n)(2.6)-(2.5) \leq -(\xi_1^2 + \cdots + \xi_n^2) \] (14)
where $V_n(X_n) = \sum_{k=1}^{n} \int_{x_k^*}^{x_k} s_1^{1/r_k} - x_k^{1/r_k} 2 - r_k ds$.

Next we show that under Assumptions 2.1-2.2 system (1.1)-(2.5) is finite-time stable. By (14), the derivative of $V_n(X_n)$ along system (1.1) under the control law (2.5) is
\[ \dot{V}_n(X_n)(1.1)-(2.5) \leq -(\xi_1^2 + \cdots + \xi_n^2) + \omega_n(X_n)f_1(\xi) \]
\[ + \cdots + \omega_n(X_n) f_n(\xi), \] (15)
where $\omega_n(X_n) = \partial V_n(X_n)/\partial x_i$, $i = 1, \ldots, n$.

According to homogeneous definition 1, it is easy to verify $x_k^*(\xi_1, \ldots, \xi_n)$ and $V_k^*(\xi_1, \ldots, \xi_n)$ are $c^{2-r_k}V_k(X_k)$. It implies by homogeneous system theory the following relations
\[ \xi_i(\xi_1, \ldots, \xi_n, x_i) = c \xi_i(X_i) \]
\[ \omega(\xi_1, \ldots, \xi_n) = c^{2-r_k} \omega(X_i) \] (16)

With $u = x_{n+1}$, according to Assumption 2.1, one obtains
\[ |f_i(x_{i+1}, \ldots, x_n, u)| \leq \rho \sum_{j=1}^{n+1} |x_j|^{r_{n+1} - r_{i+1}} \sum_{j=1}^{n+1} |x_j|^{q_{n+1} - r_{n+1}} \]
Noting that $u = x_{n+1}$, we denote
\[ H_1(X_n) = \xi_1^2 + \cdots + \xi_n^2 \]
\[ H_2(X_n) = \omega_1(X_n)(|x_1|^2 + \cdots + |x_n|^{r_{n+1}}) \]
\[ + \cdots + \omega_n(X_n)|x_{n+1}| \]

It is easy to verify from (16) that $H_1(X_n), H_2(X_n)$ and $V_n^2(X_n)$ are homogeneous of degree $2$ with respect to the dilation $(r_1, \ldots, r_n)$. According to Lemma 2.3 in [11], it can be concluded that there exists a positive constants $\overline{c}$ and $c$ such that $H_2(X_n) \leq \varepsilon V_n^2(X_n)$ and $H_1(X_n) \geq cV_n^2(X_n)$. With the help of this relation, (15) becomes
\[ V_n(1.1)-(2.5) \leq -cV_n^2(X_n) + \overline{c} \rho V_n^2(X_n) \]
\[ + \sum_{i=1}^{n} (|x_{i+1}|^{q_{i+1} - r_{i+1}} + \cdots + |x_{n+1}|^{q_{n+1} - r_{n+1}}) \]
(17)

Note that by (2.2) we have $q_{ij} > r_{i+1}$. Therefore we can find a region $\Omega = \{X_n|V_n(X_n) \leq \lambda_0\}$ with $\lambda_0 > 0$ such that for all $X_n \in \Omega$,
\[ \sum_{i=1}^{n} (|x_{i+1}|^{q_{i+1} - r_{i+1}} + \cdots + |x_{n+1}|^{q_{n+1} - r_{n+1}}) \leq c/(2\overline{c}). \]

For a more precise definition of dilation and homogeneity, please refer to [10], [4].
By (2.17), it can be clearly seen that for all \( X_n \in \Omega \),
\[
\dot{V}(X_n(t)) \leq -cV(X_n(t)/2).
\] (2.18)
It follows from Theorem 4.2 in [1] that system (1.1) is locally finite-time stable.

III. GLOBAL STABILIZATION OF (1.1)

We show that, under Assumptions 2.1-2.2, the global stabilization problem for system (1.1) can be solved by the following control law
\[
u = u_n(X_n(t)) = -\beta_n \sigma^{r+1}(x_n^{1/r_n} - u_n^{1/r_n}(X_n(t)) \tag{3.1}
\]
where \( u_i(X_i(t)) = -\beta_i \sigma^{r+1}(x_i^{1/r_i} - u_i^{1/r_i}(X_i(t))) \), \( i = 1, \ldots, n - 1, u_0 = 0 \),
\[
\sigma(x) = \begin{cases} \varepsilon \text{ sign}(x), & \text{for } |x| > \varepsilon \\ x, & \text{for } |x| \leq \varepsilon 
\end{cases}
\]
for a small constant \( \varepsilon > 0 \) to be determined later, and the gains \( \beta_i \)'s are selected as
\[
\beta_1 > \max \{ \beta^*_1, 2^{2^i}(1 + 2\tilde{d}_1)/d_1 \},
\]
\[
\beta_i > \max \{ \beta^*_i, 2^{2^i + 4}(1 + \tilde{d}_{i-1})/d_{i-1} + 2d_i \} / d_i
\]
with \( i = 2, \ldots, n \) and
\[
\alpha_1(\beta_1) = \beta_1^{1/r_1} (\tilde{d}_1(1 + \beta_1) + 1),
\]
\[
\alpha_j(\beta_1, \ldots, \beta_j) = \frac{\beta_j^{1/r_j + 1}}{r_j} (1 + \beta_{j-1})^{1/r_j - 1} (\tilde{d}_j(1 + \beta_j) + 1) + \beta_j^{1/r_j + 1} \alpha_{j-1}(\beta_1, \ldots, \beta_{j-1}),
\]
\( j = 2, \ldots, n - 1 \). (3.2)

We begin our proof by introducing an important lemma whose proof can be found in [3].

Lemma 3.1: For system (1.1), under control law (3.1), for every \( i = 1, \ldots, n - 1 \), there exist a constant \( 0 < \epsilon_1 < 1 \) and functions \( \alpha_i(\beta_1, \ldots, \beta_i) \) defined as (3.3) such that for any \( 0 < \epsilon < \epsilon_1 \), the following inequalities hold
\[
(a) \ |f_i(x_{i+1}, \ldots, x_n, u)| \leq \epsilon^{r_i+1}
\]
\[
(b) \ |u_i^{1/r_i+1}(X_i(t)) - u_i^{1/r_i+1}(X_i(t))| \leq \epsilon^{r_i+1}(1 + \beta_{i-1}),
\]
\( j = i + 1, \ldots, n \). (3.3)

With the help of Lemma 3.1, we are ready to prove the following main result of the paper.

Theorem 3.1: Under Assumptions 2.1-2.2, there is a constant \( \epsilon \in (0, \epsilon_1] \) such that control law (3.1) will globally stabilize the upper-triangular system (1.1).

Proof. The proof is divided into two steps: First, we show that the control law with coefficients \( \beta_i \)'s preset in (3.2) ensures that all states will converge to a small region determined by the saturation level \( \epsilon \). Then, we adjust the saturation level to guarantee that the states will enter the domain of attraction as defined in Theorem 2.1.

An inductive method will be used to show that the states will enter a small region one by one and stay there forever. In the following proof, the parameter \( \epsilon \) is set to be \( 0 < \epsilon \leq \epsilon_1 \).

Initial step. In this step we will prove there exists a time instance \( t_1 \) such that for \( t \geq t_1 \)
\[
X_n(t) \in Q_n = \left\{ X_n : |x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t))| < \epsilon \right\}.
\]

We first use a contradiction argument to prove that there exists a time instance \( t_1 \) such that \( x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t)) \leq \frac{\delta}{2} \). Otherwise, it can be assumed that for all \( t \geq 0 \) we have \( |x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t))| > \frac{\delta}{2} \).

We first consider the case when
\[
x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t)) > \frac{\epsilon}{2}.
\] (3.6)
Under (3.6), there exists \( \epsilon \in (0, \epsilon_1] \) such that for all \( t \geq 0 \),
\[
x_n(t) = -\beta_n \sigma_n^{r+1}(x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t))) + f_n(u) \leq -\beta_n \sigma_n^{r+1}(1/2)^{n+1} \epsilon^{r+1} + \epsilon^{r+1} - \mu_n \epsilon^{r+1}
\]
with
\[
\mu_n = \beta_n \sigma_n^{r+1}(1/2)^{n+1} - 1 > 0
\] (3.7)
determined by (3.2). It follows that for \( t \geq 0 \), \( x_n(t) < x_n(0) - \mu_n \epsilon^{r+1} t \). This, together with (3.6) and the fact
\[
|u_n^{1/r_n}(X_n(t))| \leq \beta_n \epsilon \]
shows for \( t \geq 0 \)
\[
\epsilon \leq x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t)) \leq (x_n(0) - \mu_n \epsilon^{r+1} t^{1/r_n} + \beta_n \epsilon^{r+1} \epsilon.
\] (3.8)
As time goes to infinity, (3.8) leads to a contradiction disavowing (3.6). Similarly, we can show the case \( x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t)) \leq -\frac{\epsilon}{2} \), \( \forall t > 0 \), is also impossible. In conclusion, there must exist a time instance \( t_1 \) such that
\[
|x_n^{1/r_n}(t_1) - u_n^{1/r_n}(X_n(t_1))| \leq \epsilon/2.
\]
Next, by using a contradiction argument again, we will prove that the following holds
\[
|\dot{x}^{1/r_n}(t) - u_n^{1/r_n}(X_n(t))| \leq \epsilon, \quad \text{for } t \geq t_1.
\] (3.9)
Suppose (3.9) is not true, which means there is at least one time instance \( t_1 \) when \( |\dot{x}^{1/r_n}(t_1) - u_n^{1/r_n}(X_n(t_1))| = \epsilon \).

Specifically, there are \( t_1 < \infty \) and \( t_1 < \infty \) such that either
\[
x_n^{1/r_n}(t_1) - u_n^{1/r_n}(X_n(t_1)) = \epsilon/2
\] (3.10)
\[
x_n^{1/r_n}(t_1) - u_n^{1/r_n}(X_n(t_1)) = \epsilon
\] (3.11)
\[
\frac{\epsilon}{2} \leq x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t)) \leq \epsilon, \quad t \in [t_1, t_1]
\] (3.12)
in the positive region, or \( x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t)) = -\epsilon/2 \), \( x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t)) = -\epsilon \), \( -\epsilon \leq x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t)) \leq -\epsilon/2 \), \( t \in [t_1, t_1] \) as the negative case.

Next we will first prove that the positive case (3.10)-(3.11)-(3.12) is impossible. To this end, by (3.12) and the fact that \( \dot{x}_n(t) = -d_n(x) \beta_n \sigma_n^{r+1}(x_n^{1/r_n}(t) - u_n^{1/r_n}(X_n(t))) + f_n(u) \), there exists \( \epsilon \in (0, \epsilon_1] \) such that
\[
\dot{x}_n(t) = -\mu_n \epsilon^{r+1}, \quad t \in [t_1, t_1]
\] (3.13)
where $\mu_n$ is defined as (3.7). It follows from (3.13) that
\[
\mu_n e^{\epsilon n + 1} (t_1^* - t_1^*) \geq x_n(t_1^*) - x_n(t_1^*). \tag{3.14}
\]
By (3.10) and the fact that $|u_{n-1}^{1/r_n}(X_{n-1})| \leq \beta_n^{-1} \epsilon$, the following holds since $r_n < 1$
\[
x_n(t_1^*) \leq (1 + \beta_{n-1}) e^{\epsilon n} \leq (1 + \beta_{n-1}) e^{\epsilon n}. \tag{3.15}
\]
Similarly, by (3.11), we have
\[
x_n(t_1^*) \geq -(1 + \beta_{n-1}) e^{\epsilon n} \geq -(1 + \beta_{n-1}) e^{\epsilon n}. \tag{3.16}
\]
Together with (3.15) and (3.16), (3.14) leads to the following estimate
\[
t_1^* - t_1^* \leq \frac{x_n(t_1^*) - x_n(t_1^*)}{\mu_n e^{\epsilon n + 1}} \leq \frac{2}{\mu_n} (1 + \beta_{n-1}) e^{-\tau}. \tag{3.17}
\]
Furthermore, by (3.14), we have $x_n(t_1^*) \leq x_n(t_1^*)$ which implies
\[
x_n^{1/r_n}(t_1^*) - u_{n-1}^{1/r_n}(X_{n-1}(t_1^*)) \leq x_n^{1/r_n}(t_1^*) - u_{n-1}^{1/r_n}(X_{n-1}(t_1^*)) + u_{n-1}^{1/r_n}(X_{n-1}(t_1^*)) - u_{n-1}^{1/r_n}(X_{n-1}(t_1^*)). \tag{3.18}
\]
Substituting (3.10) and (3.11) into (3.18) we have
\[
\epsilon/2 \leq \left| u_{n-1}^{1/r_n}(X_{n-1}(t_1^*)) - u_{n-1}^{1/r_n}(X_{n-1}(t_1^*)) \right|. \tag{3.19}
\]
Using (3.12) and the fact that $|u_{n-1}^{1/r_n}(X_{n-1})| \leq \beta_n^{-1} \epsilon$, we have
\[
|x_n(t)| \leq (1 + \beta_{n-1}) e^{\epsilon n}, \, t \in [t_1^*, t_1^*] \tag{3.20}
\]
which enables us to use Lemma 3.1 to estimate (3.19) as
\[
\epsilon/2 \leq \left| u_{n-1}^{1/r_n}(X_{n-1}(t_1^*)) - u_{n-1}^{1/r_n}(X_{n-1}(t_1^*)) \right| \leq \alpha_{n-1}(\beta_1, \ldots, \beta_n) e^{\epsilon^{1+\tau} (t_1^* - t_1^*)}. \tag{3.21}
\]
Substituting (3.17) into (3.21) yields
\[
\epsilon/2 \leq \frac{\beta_n}{\mu_n} (1 + \beta_{n-1})(\beta_1, \ldots, \beta_n) e. \tag{3.22}
\]
On the other hand, by the definition of $\mu_n$ and the choice of $\epsilon$ (3.2) we have
\[
\mu_n = \frac{d_n(1/2)^{r_{n+1}} \alpha_{n-1}}{\alpha_{n-1}(\beta_1, \ldots, \beta_n) e} \tag{3.23}
\]
Combining (3.22) and (3.23) leads to
\[
\epsilon/2 \leq \frac{\beta_n}{\mu_n} (1 + \beta_{n-1})(\beta_1, \ldots, \beta_n) e < \epsilon/2
\]
which obviously is a contradiction. Therefore the case of (3.10)-(3.11)-(3.12) will never happen. Similarly, it can be shown, using an almost same argument as the positive case, that $x_n^{1/r_n}(t) - u_{n-1}^{1/r_n}(X_{n-1}(t))$ will never cross $-\epsilon$. Hence for $t \geq t_1$ we have
\[
|x_n^{1/r_n}(t) - u_{n-1}^{1/r_n}(X_{n-1}(t))| < \epsilon
\]
which implies for $t \geq t_1$, we have
\[
X_n(t) \in Q_n = \left\{ X_n : |x_n^{1/r_n}(t) - u_{n-1}^{1/r_n}(X_{n-1}(t))| < \epsilon \right\}.
\]

**Inductive step.** We suppose that at step $i - 1$, there exist $0 \leq t_1 \leq \cdots \leq t_{i-1}$, such that for $t \geq t_{i-1}, j = n - i + 2, \ldots, n$
\[
X_j(t) \in Q_j(X_j) = \left\{ X_j : |x_j^{1/r_j}(t) - u_{n-i}^{1/r_j}(X_{n-i}(t))| < \epsilon \right\}. \tag{3.24}
\]
Next we will prove the above relation will also hold at step $i$. Similar to the initial step, first we will prove that there exists a time instance $t_i \geq t_{i-1}$ such that
\[
|x_n^{1/r_{n-i+1}}(t_i) - u_{n-i}^{1/r_{n-i+1}}(X_{n-i}(t_i))| \leq \epsilon. \tag{3.25}
\]
If there is no such $t_i$, it can be assumed that for $t \geq t_{i-1}$,
\[
|x_n^{1/r_{n-i+1}}(t) - u_{n-i}^{1/r_{n-i+1}}(X_{n-i}(t))| > \epsilon. \tag{3.26}
\]
We first consider the case when
\[
x_n^{1/r_{n-i+1}}(t) - u_{n-i}^{1/r_{n-i+1}}(X_{n-i}(t)) > \epsilon, \, t \geq t_{i-1},
\]
which consequently leads to the estimate of the controller
\[
u_{n-i+1}(X_{n-i+1}(t)) = -\beta_n^{i+1} \sigma^{r_{n-i+2}}(x_n^{1/r_{n-i+1}}(t) - u_{n-i}^{1/r_{n-i+1}}(X_{n-i}(t))) \leq -\beta_n^{i+1} (\epsilon/2)^{r_{n-i+2}}. \tag{3.27}
\]
With the help of (3.27), we have
\[
x_{n-i+1}(t) = d_{n-i+1}(x) u_{n-i+1}(\cdot) + f_{n-i+1}(\cdot)
\]
\[
+ d_{n-i+1}(x) \left[ x_{n-i+2}(t) - u_{n-i+1}(\cdot) \right] \leq -\beta_n^{i+1} \sigma^{r_{n-i+2}}(x_{n-i+2}(t) - u_{n-i+1}(\cdot)) + d_{n-i+1}(x) \left[ x_{n-i+2}(t) - u_{n-i+1}(\cdot) \right]. \tag{3.28}
\]
Next, we estimate the terms in (3.28). First, by (3.24), we have $|x_j| < (1 + \beta_j) \epsilon_j$, $j = n - i + 2, \ldots, n$. Therefore by Lemma 3.1,
\[
|x_{n-i+1}(x_{n-i+2}, \ldots, x_n)| \leq \epsilon^{r_{n-i+2}}, \, t \geq t_{i-1}. \tag{3.29}
\]
Noting that $0 < r_{n-i+2} < 1$, it can be concluded that
\[
|x_{n-i+2} - u_{n-i+1}| \leq 2^{r_{n-i+2}} x_{n-i+2}^{1/r_{n-i+2}} - u_{n-i+1}^{1/r_{n-i+2}} \leq 2 \epsilon^{r_{n-i+2}}. \tag{3.30}
\]
Substituting (3.29) and (3.30) into (3.28) yields that for $t \geq t_{i-1}$,
\[
x_{n-i+1}(t) \leq -\mu_{n-i+1} \epsilon^{r_{n-i+2}} \tag{3.31}
\]
with
\[
\mu_{n-i+1} = \beta_n^{i+1} d_{n-i+1}(1/2)^{r_{n-i+2}} - 1 - 2 d_{n-i+1}, \tag{3.32}
\]
which is a positive constant. As a matter of fact, by the choice of $\beta_i$’s in (3.2), it can be verified
\[
\mu_{n-i+1} \geq 2^{r_{n-i+2}} 4(1 + d_{n-i+1}(1/2)^{r_{n-i+2}} + 2 d_{n-i+1}) \times \left( \frac{1}{2} d_{n-i+1}^{1/2} \right)^{r_{n-i+2}} - 1 - 2 d_{n-i+1}
\]
\[
> 4(1 + \beta_{n-i}) \alpha_{n-i}(\beta_1, \ldots, \beta_n-i) > 0.
\]
Integrating (3.31) in \([t_{i-1}, t]\) yields
\[
x_{n-i+1}(t) \leq x_{n-i+1}(t_{i-1}) - \mu_{n-i+1} e^{r_{n-i+2}} (t - t_{i-1}).
\] (3.34)

By (3.34) and the fact that \(|u_{n-i}(X_{n-i})| \leq \beta_{n-i} e^{r_{n-i+1}}\), we obtain from (3.26) that
\[
\frac{\epsilon}{2} < x_{n-i+1}(t) - u_{n-i}(X_{n-i}(t)) \leq (x_{n-i+1}(t_{i-1}) - \mu_{n-i+1} e^{r_{n-i+2}} (t - t_{i-1})) \frac{1}{r_{n-i+1}}
\]
which implies that \(\frac{\epsilon}{2} < -\infty\) as \(t\) goes to infinity. This contradiction shows that the assumption (3.26) will never hold. For the case \(x_{n-i+1}(t) - u_{n-i}(X_{n-i}(t)) < -\frac{\epsilon}{2}\), the proof is similar and is omitted here for the sake of space. Hence it can be concluded that there exists \(t_i\) such that
\[
|x_{n-i+1}(t_i) - u_{n-i}(X_{n-i}(t_i))| \leq \epsilon/2.
\]

Following the same line of the proof of the initial step, next we use a contradiction argument to prove that for \(t \geq t_i\),
\[
|x_{n-i+1}(t) - u_{n-i}(X_{n-i}(t))| \leq \epsilon.
\]
Otherwise, we can assume that there exist \(t' < +\infty\) and \(t'' < +\infty\) such that we have either the positive case as
\[
x_{n-i+1}(t') - u_{n-i}(X_{n-i}(t')) = \epsilon/2
\] (3.35)
\[
x_{n-i+1}(t'') - u_{n-i}(X_{n-i}(t'')) = \epsilon
\] (3.36)
\[
\epsilon/2 \leq x_{n-i+1}(t) - u_{n-i}(X_{n-i}(t)) \leq \epsilon, \quad t \in [t', t''],
\] (3.37)
or the negative case as
\[
x_{n-i+1}(t') - u_{n-i}(X_{n-i}(t')) = -\epsilon/2,
\]
\[
x_{n-i+1}(t'') - u_{n-i}(X_{n-i}(t'')) = -\epsilon
\]
\[
\epsilon/2 \leq x_{n-i+1}(t) - u_{n-i}(X_{n-i}(t)) \leq \epsilon, \quad t \in [t', t''].
\]
Next, we focus on the positive case (3.35)-(3.36)-(3.37) to show neither case would happen. First, note that condition (3.37) implies that (3.27) holds. This, together with (3.29)-(3.30) guaranteed by the inductive assumption, implies that (3.31) holds as well for \(t \in [t', t'']\). Integrating (3.31) yields
\[
\mu_{n-i+1} e^{r_{n-i+2}} (t'' - t') \leq x_{n-i+1}(t'') - x_{n-i+1}(t').
\] (3.38)

By (3.35) and the fact that \(|u_{n-i}(X_{n-i})| \leq \beta_{n-i} e^{r_{n-i+1}}\), we obtain
\[
x_{n-i+1}(t') \leq (1 + \beta_{n-i}) e^{r_{n-i+1}}.
\] (3.39)

Similarly, by (3.36), we have
\[
x_{n-i+1}(t'') \geq -(1 + \beta_{n-i}) e^{r_{n-i+1}}.
\] (3.40)

Substituting (3.39) and (3.40) into (3.38) leads to
\[
t'' - t' \leq \frac{2}{\mu_{n-i+1}} (1 + \beta_{n-i}) e^{r_{n-i+1}}
\]
\[
= \frac{2}{\mu_{n-i+1}} (1 + \beta_{n-i}) e^{-\tau}.
\] (3.41)

Note that (3.31) implies \(x_{n-i+1}(t_i) \leq x_{n-i+1}(t_i')\). Hence, we have
\[
x_{n-i+1}(t_i) - u_{n-i}(X_{n-i}(t_i)) \leq x_{n-i+1}(t_i') - u_{n-i}(X_{n-i}(t_i')) + \frac{1}{\mu_{n-i+1}} (X_{n-i}(t_i') - u_{n-i}(X_{n-i}(t_i'))
\]
which leads to the following inequality by using (3.35)-(3.36),
\[
\epsilon/2 \leq u_{n-i}(X_{n-i}(t_i')) - u_{n-i}(X_{n-i}(t_i)).
\] (3.42)

On the other hand, by (3.37) and the fact that \(|u_{n-i}(X_{n-i})| \leq \beta_{i} e^{r_{n-i+1}}\), it can be concluded that for \(t \in [t_i', t_i''\],
\[
|x_{n-i+1}(t)| \leq (1 + \beta_{n-i}) e^{r_{n-i+1}}
\]
which, together with (3.24), implies that for \(t \in [t_i', t_i''\],
\[
|x_j(t)| \leq (1 + \beta_{j-1}) e^{r_j}, \quad j = n - i + 1, \cdots, n.
\] (3.43)

Note that we have assumed \(0 < \epsilon \leq \epsilon_i < 1\). Hence with the help of (3.43), we obtain from Lemma 3.1 that
\[
|u_{n-i}(X_{n-i}(t_i')) - u_{n-i}(X_{n-i}(t_i))| \leq \alpha_{n-i}(\beta_1, \cdots, \beta_{n-i}) e^{1+\tau} (t_i'' - t_i').\]
\[
(3.44)
\]

Substituting (3.41) and (3.44) into (3.42) yields
\[
\epsilon/2 \leq \frac{2}{\mu_{n-i+1}} (1 + \beta_{n-i}) e^{1+\tau} (t_i'' - t_i').
\]
\[
(3.45)
\]

Note that in (3.33), we have verified \(\frac{2}{\mu_{n-i+1}} (1 + \beta_{n-i}) e^{1+\tau} (t_i'' - t_i') < \frac{\epsilon}{2}\) which in turn implies the following contradiction
\[
\epsilon/2 \leq \frac{2}{\mu_{n-i+1}} (1 + \beta_{n-i}) e^{1+\tau} (t_i'' - t_i').
\]
\[
(3.46)
\]

Therefore the positive case (3.35)-(3.36)-(3.37) will never happen. For the negative case \(-\epsilon < x_{n-i+1}(t) - u_{n-i}(X_{n-i}(t)) < -\epsilon/2\), the proof is similar to the positive case and thus is omitted here. Hence for \(t \geq t_i\) we can conclude that
\[
|x_{n-i+1}(t) - u_{n-i}(X_{n-i}(t))| < \epsilon
\]
which implies for \(t \geq t_i\), we have \(X_{n-i+1}(t) \in Q_{n-i+1} = \{X_{n-i+1} : |x_{n-i+1}(t) - u_{n-i}(X_{n-i}(t))| < \epsilon\}\). This, together with (3.24), shows that for \(t \geq t_i\), \(j = n - i + 1, \cdots, n\),
\[
X_j(t) \in Q_j = \{X_j : |x_j(t) - u_{j-1}(X_{j-1}(t))| < \epsilon\}
\]
(3.45)

This completes the inductive proof.

**Final step.** Proceeding by this way, we can obtain there exists a time instance \(t_{n-1}\) such that for \(t \geq t_{n-1}\) we have
\[
X_j(t) \in Q_j = \{X_j : |x_j(t) - u_{j-1}(X_{j-1}(t))| < \epsilon\}
\]
(3.45)

with \(j = 2, \cdots, n\), which implies for \(t \geq t_{n-1}\),
\[
|x_j(t)| \leq (1 + \beta_{j-1}) e^{r_j}, \quad j = 2, \cdots, n.
\]
Therefore by Lemma 3.1, one obtains for $t \geq t_{n-1}$
\[ |f_1(x_2, \cdots, x_n, u)| \leq \epsilon r^2. \] (3.46)

In addition, we have for $t \geq t_{n-1}$,
\[ |x_2(t) - u_1(x_1(t))| \leq 2^{1-r_2}x_2^{1/2}(t) - u_1^{1/2}(x_1(t)) |r_2^2 \leq 2\epsilon r^2. \] (3.47)

With the help of the above relations, it can be concluded from
\[ \dot{x}_1 = d_1(x)u_1(x_1) + f_1(x_2, \cdots, x_n, u) + d_1(x)(x_2 - u_1(x_1)) \] (3.48)

that for $t \geq t_{n-1}$
\[ \dot{x}_1(t) < -\mu_1 \epsilon r^2 < 0, \quad \text{for } x_1^{1/r_1}(t) > \epsilon / 2 \]
\[ \dot{x}_1(t) \geq \mu_1 \epsilon r^2 > 0, \quad \text{for } x_1^{1/r_1}(t) \leq -\epsilon / 2 \]

with
\[ \mu_1 = \frac{d_1^1 \beta_1 (1/2)^{r_2} - 1 - 2d_1}{d_1 (1/2)^{r_2} 2^{r_2}(1+2d_1)/d_1 - 1 - 2d_1} = 0 \]

guaranteed by (3.2). It follows that there exists a $t_n$ such that for $t \geq t_n$ we have $|x_1^{1/r_1}(t)| < \epsilon / 2 < \epsilon$. Then we have
\[ x_1(t) \in Q_1 = \left\{ x_1 : |x_1^{1/r_1}(t)| < \epsilon \right\}, t \geq t_n. \]

Therefore, when $t \geq t_n$, $x_n(t)$ will enter and stay in the set
\[ Q = \left\{ x_n : |x_n^{1/r_n}(t)| < \epsilon, |x_2^{1/r_2}(t) - u_1^{1/r_2}(x_1(t))| < \epsilon, \cdots, |x_n^{1/r_n}(t) - u_n^{1/r_n}(x_{n-1}(t))| < \epsilon. \right\} \]

By choosing a small enough parameter $\epsilon$ we can assure that
\[ Q \subset \Omega = \{ x_n : V_n(x_n) \leq \lambda_0 \} \]

where $\Omega$ is defined as in Theorem 2.1.

Therefore, when $t \geq t_n$, $x_n(t)$ will enter and stay in $\Omega$ where $u_i = x_{i+1}^* = \beta_i(x_i^{1/r_i} - x_i^{1/r_i})^{r_i+1}, i = 1, \cdots, n$, with $x_i^*$ defined the same as in Theorem 2.1. It means that after $t_n$ the saturated control law (3.1) becomes the unsaturated control law (2.5) which will stabilize the system in $\Omega$. As a result the global finite-time stability for the closed-loop system (1.1)-(3.1) is proved.

In what follows, we use an example to illustrate this new feature.

**Example 3.1:** Consider the following nonlinear system:
\[ \dot{x}_1 = x_2 + x_3^2 + x_3^3, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u. \] (3.49)

For system (3.49), when we choose $\tau = -2/15$, Assumption 2.1 holds due to the fact that $q_{12} = 2 > 1, q_{13} = 2 > (1 + \tau)/(1 + 2\tau) = 13/11$. Hence, according to Theorem 3.1, a global finite-time stabilizer can be designed as follows
\[ u = -\beta_3 \sigma \left( x_{12}^{15} + \beta_2 \sigma \left( x_{12}^{15} + \beta_1 \sigma \left( x_{12}^{15} + \beta_1 \sigma \left( x_{12}^{15} \right) \right) \right) \right) \]

with appropriate gains $\beta_{1,2,3}$ and $\epsilon$.

**REFERENCES**