Static Output Feedback Design using Asymptotic Properties of LQ Regulators

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Abstract – Asymptotic properties of linear quadratic regulators are exploited to aid in the design of static output feedback controllers, (an open problem in control engineering). Specifically, we offer a constructive proof to show that any controllable observable minimum-phase linear time-invariant dynamics with a non-zero high frequency gain can be stabilized using static output feedback.

I. INTRODUCTION

In this paper, we propose a constructive design method to address the static output feedback problem for linear time-invariant (LTI) multi-input-multi-output (MIMO) dynamical systems. According to [1], “... the problem of static output feedback is still open.” Our interest in the design of output feedback controllers stems from aerospace applications where vehicle flexibility effects cannot be ignored. Dynamics of these vehicles exhibit no frequency separation between their primary and flexible modes. The latter often have low damping ratios and as such must be actively controlled / stabilized. Using dynamic output feedback (such as state observers) is often undesirable since it brings complexity into the overall system architecture and, as a result, significantly increases costs in the controller implementation, validation and verification. Because of these issues, static output feedback controllers gain interest and their formal design methods become relevant once again.

Main ideas presented here, are motivated by the now-classical paper [2], written by Huibert Kwakernaak and Raphael Sivan in 1972, where the authors investigated “... the maximal achievable accuracy of linear optimal regulators.” They have shown that the linear quadratic (LQ) optimal cost tends to zero, as the control weight tradeoff parameter \( \rho \) decreases. What is interesting to note is the fact that the LQ optimal feedback gain \( K_\rho \) approaches a scaled version of the system output matrix \( C \). So it becomes natural to investigate if a static output feedback approximation of the LQ state-feedback regulator can be constructed. This is precisely the question that is addressed in this paper. The answer is Yes. In fact, we propose a constructive design method to compose static output feedback controllers for controllable observable minimum-phase systems, with a nonzero high frequency gain.

We immediately note that a similar topic was discussed in [6], where necessary and sufficient conditions were given for the existence of an optimal static output feedback. Our work is complementary and yet different from the above. Instead of constructing an LQ optimal output feedback, we choose to approximate the corresponding LQ optimal state-feedback controller. This results in a much simplified design procedure.

The rest of the paper is organized as follows. Section II presents a brief overview of the well-known asymptotic properties of optimal LQ regulators. A modification to the quadratic optimality index is proposed in Section III, and related asymptotic properties of the LQ controllers are derived. Based on the latter, a constructive design solution to the static output feedback problem is proposed in Section IV, followed by practical design considerations in Section V, and filter-duality discussions in Section VI. Conclusions are given in Section VII.

II. ASYMPTOTIC PROPERTIES OF LQ REGULATORS – AN OVERVIEW

Consider the LTI MIMO dynamical system:

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

where \( x \in \mathbb{R}^n \) is the \( n \) - dimensional state, \( u \in \mathbb{R}^m \) is the \( m \) - dimensional control input, and \( y \in \mathbb{R}^p \) is the \( p \) - dimensional controlled / regulated output. It is assumed that in (2.1), the matrix pairs \((A, B)\) and \((A, C)\) are controllable and observable, respectively. Also, it is assumed that the number of control inputs and controlled outputs are the same, that is: \( m = p \).

In [2, 3, 4], the now-classical and well-known asymptotic properties of optimal LQ regulators are derived using the quadratic cost function,

\[
J_\rho = \int_0^\infty (y^T y + \rho u^T Ru) dt
\]

where \( R = R^T > 0 \) and \( \rho > 0 \) represents the tradeoff parameter between the regulator performance and the control effort expended. Specifically, under the following two assumptions: a) \( \text{rank}(CB) = m \), and b) the roots of the polynomial

\[
\psi(s) = \det \left[ C(sI_{n\times n} - A)^{-1} B \right] \det \left[ sI_{n\times n} - A \right]
\]

have non-positive real parts, (i.e., the transfer function \( C(sI_{n\times n} - A)^{-1} B \) is minimum-phase), it is shown that the LQ optimal gain,

\[
K_\rho = R^{-1} B^T P_\rho
\]

behaves asymptotically,

\[
\sqrt{\rho} K_\rho \to R^{1/2} WC
\]

as the tradeoff parameter \( \rho \) tends to zero, where \( W \in \mathbb{R}^{m \times m} \).
is an arbitrary unitary matrix, and \( P_\rho = P_\rho^T > 0 \) is the unique positive-definite symmetric solution of the Algebraic Riccati Equation (ARE):

\[
P_\rho A + A^T P_\rho - \frac{1}{\rho} P_\rho B R^{-1} B^T P_\rho + C^T C = 0 \tag{2.6}
\]

Substituting (2.4) into (2.5), gives:

\[
\frac{1}{\sqrt{\rho}} R \frac{1}{\rho} B^T P_\rho \rightarrow WC \tag{2.7}
\]

or, equivalently,

\[
P_\rho B \rightarrow \sqrt{\rho} \left( C^T W^T \sqrt{R} \right) \tag{2.8}
\]

In [2], it is also shown that,

\[
P_\rho = O(\sqrt{\rho}) \tag{2.9}
\]

and consequently, for any initial condition \( x(0) \), the minimum / optimal cost value is:

\[
\min_{\rho} J_\rho = x(0)^T P_\rho x(0) = O(\sqrt{\rho}) \tag{2.10}
\]

Hence, for any given \( x(0) \), \( \min_{\rho} J_\rho \) tends to 0, as \( \rho \to 0 \), with the \( \sqrt{\rho} \) rate of convergence. This asymptotic relation represents the “…maximally achievable accuracy…” of the LQ regulators, as it was defined and proved in [2].

Referring to this result in their textbook [7, Th. 3.14, pp. 306 – 307], the authors noted that the corresponding formal proof was “… long and technical …”. We offer an alternative proof here, which will be employed in the next section to derive our main results. Our approach is based on the theory of asymptotic expansions [8]. In particular, we establish the validity of (2.8) and (2.9) with the help of the following asymptotic expansion,

\[
P_\rho = P_0 + P_1 \sqrt{\rho} + O(\rho) \tag{2.11}
\]

as \( \rho \to 0 \), where \( P_0 \) and \( P_1 \) are symmetric semi-positive definite matrices. In order to calculate these three quantities, we substitute (2.11) into (2.6), and get:

\[
\rho \left[ \left( P_0 + P_1 \sqrt{\rho} + O(\rho) \right) A + A^T \left( P_0 + P_1 \sqrt{\rho} + O(\rho) \right) \right]
- \left( P_0 + P_1 \sqrt{\rho} + O(\rho) \right) B R^{-1} B^T \left( P_0 + P_1 \sqrt{\rho} + O(\rho) \right)
+ \rho C^T C = 0 \tag{2.12}
\]

Collecting the \( \rho^0 \) terms in (2.12), results in:

\[
P_0 B R^{-1} B^T P_0 = 0 \tag{2.13}
\]

Therefore, \( P_0 B = 0 \) and, consequently all the \( \sqrt{\rho} \) – terms in (2.12), vanish. We then proceed to collect the \( \rho^1 \) – terms to obtain:

\[
P_1 A + A^T P_0 - P_1 B R^{-1} B^T P_1 + C^T C = 0 \tag{2.14}
\]

It is not difficult to show that (2.14), in combination with \( P_0 B = 0 \) and the assumed controllability of \( (A,B) \), implies \( P_0 = 0 \), and

\[
P_1 B = C^T W^T \sqrt{R} \tag{2.15}
\]

where \( W \) is a unitary matrix, with \( W^T W = I \). In fact, since \( P_0 B = 0 \) then left-right-multiplying (2.14) by \( B^T \) and \( B \), gives

\[
\left( B^T P_1 B \right) R \left( B^T P_1 B \right) = \left( B^T C^T \right) \left( C B \right) \tag{2.16}
\]

Now, the validity of (2.15) can be easily checked by its direct substitution into (2.16). Then,

\[
P_0 A + A^T P_0 = 0, \quad P_0 B = 0 \tag{2.17}
\]

Furthermore, since \( P_0 B = 0 \) then right-multiplying the \( 1^\text{st} \) relation in (2.17) by \( B \), gives \( P_0 AB = 0 \). Right-multiplying (2.17) by \( AB \), yields \( P_0 A^2 B = 0 \). Continue using induction, implies \( P_0 \left[ B \ AB \ldots A^{n-1} B \right] = 0 \), and hence \( P_0 = 0 \).

Multiplying (2.11) by \( B \), setting \( P_0 = 0 \), and substituting (2.15) proves the following statement: As \( \rho \to 0 \), the asymptotic expansion

\[
P_\rho B = \sqrt{\rho} \left( C^T W^T \sqrt{R} \right) + O(\rho) \tag{2.18}
\]

takes place, which in addition to validating (2.8) and (2.9), also determines the rate of convergence \( O(\rho) \).

Because of (2.9), the ARE matrix solution \( P_\rho \) becomes singular, in the limit as \( \rho \to 0 \). This singularity phenomenon is undesirable, since it can potentially cause numerical problems in implementing the corresponding LQ controllers for small values of \( \rho \). In what follows, we propose a modification to the performance cost index that will guarantee strict positive definiteness of \( P_\rho \), uniformly in \( \rho \).

### III. QUADRATIC COST MODIFICATION

For the system dynamics (2.1), consider the following quadratic cost,

\[
J_\rho = \int_0^t \left( \frac{1}{2} v^T (y^T y) + v^T (x^T Q_x x) + v^2 \left( u^T R u \right) \right) dt \tag{3.1}
\]

where \( Q_x \in R^{n_x \times n_x} \), \( R \in R^{n_u \times n_u} \) are symmetric positive definite weighting matrices, and \( v > 0 \) is the tradeoff parameter between the system performance and the control utilization. In comparison to (2.2), the cost (3.1) contains two additional quadratic components, related to the system output and the state, whose weights are \( v^{-2} \) and \( v Q_x \), respectively. Also here, we choose the tradeoff parameter \( \rho = v^{-1} \). It is easy to see that as \( v \) becomes smaller, and hence a larger control effort is allowed, the system output performance is weighted...
higher. This tradeoff, between large control efforts and small output errors, guarantees the inform non-singularity of the corresponding ARE solution, in the limit as $\rho \to 0$. In fact, the cost (3.1) can be represented as a product of two functions of $v^2$:

$$I_v = \frac{1}{v^2} \int (v^2 y + v^2 (x^T Q x) + v^2 (u^T R u)) \, dt = \frac{1}{v^2} \tilde{I}_v$$

(3.2)

Consequently, for any $v > 0$

$$x_0^T P_v x_0 = \min\limits_{x \in \mathbb{R}^n} I_v = \frac{1}{v^2} \min\limits_{u^2} \tilde{I}_v$$

(3.3)

where $P_v$ is the unique positive definite symmetric solution of the ARE

$$P_v A + A^T P_v + \frac{1}{v^2} \left( C^T C - P_v B R^{-1} B^T P_v \right) + v^2 Q_v = 0$$

(3.4)

Following [2], one can show that as $v \to 0$, 

$$\min\limits_{u^2} \tilde{I}_v = O(v^2)$$

(3.5)

Then because of (3.3),

$$x_0^T P_v x_0 = \frac{1}{v^2} O(v^2) = O(1)$$

(3.6)

This relation proves the existence of a symmetric positive definite matrix $P_v$, where

$$\lim\limits_{v \to 0} P_v = P_0$$

(3.7)

The corresponding LQ optimal solution is,

$$u_v = -\left[ \left( v^2 R \right)^{-1} B^T P_v \right] x = -K_v x$$

(3.8)

where $K_v \in \mathbb{R}^{m \times n}$ is the optimal LQ gain matrix. Next, we shall investigate the limiting behavior and the asymptotic approximation rates of the LQ regulator $u_v$, as $v \to 0$. Towards that end, rewrite (3.4) as,

$$v^2 \left( P_v A + A^T P_v \right) - P_v B R^{-1} B^T P_v + C^T C + v^3 Q_v = 0$$

(3.9)

and consider the asymptotic expansion,

$$P_v = P_0 + P_1 v + O(v^2)$$

(3.10)

where $P_0$ and $P_1$ are symmetric positive definite matrices. Substituting (3.10) into (3.9), gives:

$$v^2 \left( \left( P_0 + P_1 v + O(v) \right) A + A^T \left( P_0 + P_1 v + O(v) \right) \right) - \left( P_0 + P_1 v + O(v^2) \right) B R^{-1} B^T \left( P_0 + P_1 v + O(v^2) \right) + C^T C + v^3 Q_v = 0$$

(3.11)

Matching the $0^\text{th}$ order terms in $v$, results in:

$$P_0 B R^{-1} B^T P_0 = C^T C$$

(3.12)

Consequently, $P_0 = P_0^T$ is a constant matrix. It is easy to see (by direct substitution) that the relation

$$P_0 B = C^T W^T \sqrt{R}$$

(3.13)

is valid for any unitary matrix $W \in \mathbb{R}^{m \times m}$, which satisfies:

$$B^T P_0 B = B^T C^T W^T \sqrt{R} > 0$$

(3.14)

Next, we show constructively how to select $W$. Let $(U, V) \in \mathbb{R}^{m \times m}$ denote the two unitary matrices, from the singular value decomposition of $(C B \sqrt{R})$, that is:

$$C B \sqrt{R} = U \Lambda V$$

(3.15)

It is easy to see that (3.16) defines a unitary matrix. Substituting (3.15) and (3.16) into (3.14), yields the needed symmetric positive definite matrix:

$$\sqrt{R} W C B = R^2 W \left( C B \sqrt{R} \right)^{1/2} \left( \sqrt{R} \right)^{-1/2}$$

(3.17)

$$= R^2 V^T U^T U \Lambda V R \sqrt{R}^{-1} = R^2 V^T \Lambda V R \sqrt{R}^{-1} > 0$$

Next, matching the $1^\text{st}$ – order terms in (3.11), gives:

$$P_1 B R^{-1} B^T P_1 = 0$$

(3.18)

which is equivalent to:

$$P_1 B = 0 \lor P_1 B = 0$$

(3.19)

This directly follows from left-right-multiplying (3.18) by $B^T P_0$ and $P_1 B$, respectively,

$$\left( B^T P_0 B \right) R^{-1} \left( B^T P_1 B \right) = 0$$

(3.20)

and comparing the matrix diagonal elements. Moreover, because of (3.13), $P_0 B \neq 0$ and consequently:

$$P_1 B = 0$$

(3.21)

Therefore,

$$P_1 B = P_0 B + P_1 B v + O(v^2)$$

(3.22)

and consequently,

$$P_v = P_0 + O(v^2)$$

(3.23)

where $P_0$ is a constant symmetric positive-definite matrix independent of $v$. Substituting (3.22) into (3.8), gives the asymptotic expansion for the LQ optimal feedback gain:

$$K_v = \frac{1}{v^2} R^{-1} \left( \sqrt{R} W C + O(v^2) \right)$$

(3.24)

and hence, the optimal control solution takes the form,
\[
\begin{align*}
  u_v &= -K_v x = -\left(\frac{1}{\nu^2} R^{1/2} W C + R^{-1} O(1)\right)x \\
  &= -\frac{1}{\nu^2} \left( R^{1/2} W \right)y + R^{-1} O(1)x
\end{align*}
\]
(3.25)

where \( O(1) \in \mathbb{R}^{m \times m} \) is a constant matrix, whose induced norm is upper-bounded by \( k \), uniformly in \( \nu \).

We now summarize and state our formally proven 1st main result.

**Theorem 3.1**
Consider the LQ optimal control problem for the controllable and observable LTI MIMO system (2.1) and the performance cost (3.1). Suppose that the system is minimum-phase and \( \det(CB) \neq 0 \). Then as \( \nu \to 0 \), the following asymptotic expansions take place,

\begin{align*}
  a) \quad P_v &= P_0 + O(\nu^2) \\
  b) \quad P_v B &= C^T W^T \sqrt{R} + O(\nu^2) \\
  c) \quad K_v &= \frac{1}{\nu^2} R^{1/2} W C + R^{-1} O(1)
\end{align*}

(3.26)

where \( P_0 = P_0^T > 0 \), and \( (U, V, W) \) are the unitary matrices, independent of \( \nu \), defined as in (3.15) and (3.16), respectively. Moreover, starting with any initial condition \( x_0 \in \mathbb{R}^n \), the total optimal cost is given by:

\[
I_v = x_0^T P_0 x_0 + O(\nu^2)
\]
(3.27)

where \( O(\nu^2) \in \mathbb{R}^{m \times m} \) is a matrix, whose norm upper bound satisfies:

\[
\|O(\nu^2)\| \leq k \nu^2
\]
(3.28)

with a positive constant \( k > 0 \).

**Remark 3.1**
Since \( B^T P_v B > 0 \) then (3.22) implies

\[
B^T P_v B = B^T C^T W^T \sqrt{R} + B^T R^{-1} O(\nu^2) > 0
\]
(3.29)

Then in the limit as \( \nu \to 0 \),

\[
B^T P_v B = B^T C^T W^T \sqrt{R} > 0
\]
(3.30)

Relation (3.30) shows that for \( P_0 \) to exist, the requirement \( \det(CB) \neq 0 \) is both sufficient and necessary. Note that for scalar input systems (\( m = 1 \)), the inequality (3.30) becomes \( W CB > 0 \), and therefore \( W = \text{sgn}(CB) \).

**Remark 3.2**
Necessity of the minimum phase requirement can be established using the technique from [2]. This method shows that those closed-loop system eigenvalues that stay finite as \( \nu \to 0 \) approach the open-loop system transmission zeros.

**Remark 3.3**
Rewrite (3.9) as,

\[
\begin{align*}
  P_v (A - BK_v) + (A - BK_v)^T P_v \\
  &= -\left(\frac{1}{\nu^2} P_v B R^{-1} B^T P_v + \frac{1}{\nu^2} C^T C + v Q_v\right) < 0
\end{align*}
\]
(3.31)

and let,

\[
A_v = A - BK_v
\]
(3.32)

denote the system closed-loop matrix. It is easy to see that \( A_v \) is Hurwitz. Merging (3.26) with (3.31), gives:

\[
\begin{align*}
  P_v A_v + A_v^T P_v &\leq -\frac{1}{\nu^2} C^T C - v Q_v \\
  P_v B &= C^T W^T \sqrt{R} + O(\nu^2)
\end{align*}
\]
(3.33)

Define,

\[
M^T = W^T \sqrt{R}
\]
(3.34)

Then,

\[
\begin{align*}
  P_v A_v + A_v^T P_v &\leq -\frac{1}{\nu^2} C^T C - v Q_v \\
  P_v B &= (M C) + O(\nu^2)
\end{align*}
\]
(3.35)

Relations (3.35) imply that the matrix triplet \( (A_v, B, C_v) \) satisfies Positive Real (PR) Lemma [Ref. 9, Lemma 6.2, p. 240] asymptotically, as \( \nu \to 0 \). For this very reason, the asymptotic relation (3.26) will be referred to as the “Asymptotic PR Recovery”, or simply APR. Furthermore, the 2\textsuperscript{nd} equation in (3.35) indicates that the APR error is of the order \( \nu^2 \):

\[
P_v B - C_v = O(\nu^2)
\]
(3.36)

Note that the redefined APR output signal

\[
y_{APR} = C_{APR} x = MCx = M y
\]
(3.37)

consists of \( m \) linear combinations of the original system output components. Such a signal can be measured online.

**IV. STATIC OUTPUT FEEDBACK**

The corresponding LQ regulator can be written as

\[
\begin{align*}
  u_v &= -\frac{1}{\nu^2} \left( R^{1/2} W \right)y + R^{-1} O(1)x
\end{align*}
\]
(4.1)

Static Output Feedback

The 1\textsuperscript{st} term in (3.25) represents a static output feedback. We argue that for sufficiently small \( \nu > 0 \), the resulting static output feedback controller

\[
u_{off} = -\frac{1}{\nu^2} \left( R^{1/2} W \right)y = -K_v y
\]
(4.2)

stabilizes the original system (2.1). Towards that end, note that since,
we can substitute (4.1) into (2.1), and compute the closed-loop system dynamics:
\[
\dot{x} = \left( A - B\left( v^2 R \right)^{-1} B^T P \right) x + B R^{-1} O(1) x
\]
\[
= A_j x + B R^{-1} O(1) x
\]
In order to assess closed-loop system stability, consider the following Lyapunov function candidate:
\[
V(x) = x^T P_v x
\]
Its time-derivative along the system (4.4) trajectories can be easily evaluated.
\[
\dot{V}(x) = x^T \left[ A_j^T P_v + P_v A_j \right] x + 2 x^T P_v B R^{-1} O(1) x
\]
Using (3.31), gives
\[
\dot{V}(x) = -x^T \left[ \frac{1}{v} P_v B R^{-1} B^T P_v + \frac{1}{v} C^T C + v Q_v \right] x
+ 2 x^T P_v B R^{-1} O(1) x
\]
Let,
\[
w = \| x^T P_v B \|
\]
Via (4.7) and (4.8), an upper bound for \( \dot{V}(x) \) can be computed as follows:
\[
\dot{V}(x) \leq -\frac{1}{v^2} \lambda_{\min} \left( R^{-1} \right) w^2 - \frac{1}{v^2} \| y \|^2
- v \lambda_{\min} \left( Q_v \right) \| x \|^2 + 2 k w \| R^{-1} \| \| x \|
= -\left( \frac{1}{\sqrt{v}} \lambda_{\min} \left( R^{-1} \right) w - \sqrt{v} \lambda_{\min} \left( Q_v \right) \| x \| \right)^2
- 2 \left( \frac{\lambda_{\min} \left( R^{-1} \right) \lambda_{\min} \left( Q_v \right)}{v} - k \lambda_{\min} \left( R^{-1} \right) \right) w \| x \| - \frac{1}{v^2} \| y \|^2
\]
Consequently, if
\[
\frac{\lambda_{\min} \left( R^{-1} \right) \lambda_{\min} \left( Q_v \right)}{v} > k^2 \lambda_{\min} \left( R^{-1} \right)
\]
or, equivalently, if
\[
0 < v < \frac{\lambda_{\min} \left( R^{-1} \right) \lambda_{\min} \left( Q_v \right)}{\lambda_{\min} \left( R^{-1} \right) k^2} = \frac{2 \lambda_{\min} \left( R \right) \lambda_{\min} \left( Q_v \right)}{\lambda_{\min} \left( R \right) k^2}
\]
then
\[
\dot{V}(x) \leq -\frac{1}{v^2} \| y \|^2 \leq 0
\]
By means of the LaSalle’s Invariance Theorem [9], the upper bound (4.12) implies global exponential stability of the closed-loop system (4.4). We now summarize and state our 2nd main result.

**Theorem 4.1**
Consider the controllable and observable LTI MIMO dynamical system (2.1). Suppose that the system is minimum-phase and det \((CB) \neq 0\). Then for a sufficiently small positive scalar \( v \), the static output feedback controller (4.2) renders the corresponding closed-loop system dynamics globally exponentially stable.

**V. PRACTICAL CONSIDERATIONS**
In the previous two sections, we have shown that any controllable observable minimum-phase dynamics (2.1), with a nonzero high frequency gain \( \text{det}(CB) \neq 0 \), can be stabilized by static output feedback. Our formal proof was based on the inverse optimal control design approach. In other words, we have started with an LQ optimal problem formulation and then utilized the theory of asymptotic expansions, along with the Lyapunov arguments, to show closed-loop global exponential stability.

At this point, we note that the established static output regulation comes at a “steep” price – the resulting solution (4.2) represents a high gain controller, whose feedback gain matrix
\[
K_v = \frac{1}{v^2} R^{-2} WC = O \left( \frac{1}{v^2} \right)
\]
tends to infinity, as the tradeoff parameter \( v \) tends to zero. In the limit, the system maximum crossover frequency \( \omega_{\text{max}} \), evaluated at the system input break point, becomes infinite. This can be shown as follows. Let,
\[
G(s) = K_v \left( s I_{n_n} - A \right)^{-1} B
\]
denote the system open-loop transfer matrix, evaluated at the system input break point. Similar to [Ref. 5, Eq. (48), p. 13], we evaluate \( G(s) \) at high frequencies \( s = j \omega \), with
\[
\omega_v = \frac{\omega}{\sqrt{v}}
\]
and as \( v \to 0 \). To simplify derivations, we set \( R = I_{n_n} \).
Using (3.26), (5.2) and (5.3), we get:
\[
\left( j \frac{\omega}{\sqrt{v}} \right)^{\frac{1}{2}} = K_v \left( j \frac{\omega}{\sqrt{v}} I_{n_n} - A \right)^{-1} B
\]
\[
= v^2 K_v \left( j \omega I_{n_n} - v^2 A \right)^{-1} B \to \frac{1}{j \omega} WC
\]
We now study asymptotic behavior of the singular values of \( G(j \omega_v) \). Using (3.26) and (5.4), yields:
\[
\sigma \left( G(j \omega_v) \right) \to \sigma \left( \frac{1}{j \omega} WC \right) = \frac{\sigma(WC)}{\omega} = \frac{\sigma(CB)}{\omega} = \frac{\sigma(CB)}{\omega}
\]
The loop transfer maximum crossover frequency \( \omega_{\text{max}} \) of the LQ regulator is defined as:
\[
\omega_{\text{max}} = \max \left\{ \omega_v : \sigma \left[ G(j \omega_v) \right] = 1 \right\}
\]
where \( \bar{\sigma}(X) \) denotes the maximum singular value of a matrix \( X \). Based on (5.5), we get:

\[
\begin{align*}
1 = \bar{\sigma} \left( G \left( j \frac{\partial}{\partial v} \right) \right) & \Rightarrow 1 = \frac{\bar{\sigma}(CB)}{\omega_0} \Rightarrow \{ \omega_0 = \bar{\sigma}(CB) \} (5.7)
\end{align*}
\]

Finally, using (5.3) we compute the maximum crossover frequency:

\[
\omega_{c, \text{max}} = \omega_0 \frac{v^2}{v^2} = \frac{\bar{\sigma}(CB)}{v^2} (5.8)
\]

Thus, as \( \nu \rightarrow 0 \), the maximum crossover frequency of the regulator tends to infinity and as such, the static output feedback solution becomes a high gain controller. This asymptotic behavior is very undesirable, since if \( \nu \) is chosen too small then the corresponding regulator will lose its robustness to noise and modeling uncertainties that may influence the plant dynamics at high frequencies. In order to avoid large crossover frequencies, one needs to treat (5.8) as a constraint imposed on the selection of the tradeoff parameter \( \nu \), for a given crossover frequency value.

VI. EXTENSIONS TO FILTERING PROBLEMS

The theoretical results presented in this paper address the static output regulator problem. These statements can be easily reformulated in terms of the optimal filtering problem, which is dual to the regulator problem [2]. In this case, the system dynamics are

\[
\begin{align*}
\dot{x} &= Ax + Bu + \xi \\
y &= Cx + \eta
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the state, \( y \in \mathbb{R}^r \) is the measured output, and \( (\xi, \eta) \in \mathbb{R}^d \) represent two uncorrelated white noise processes. We assume that the noise autocorrelation and intensity matrices are defined so that:

\[
\begin{align*}
R_\xi(\tau) &= E[(\xi(t) + \xi(t + \tau))^2] = \nu Q_\xi + \frac{1}{v^2} BB^T \delta(\tau) \\
R_\eta(\tau) &= E[(\eta(t) + \eta(t + \tau))^2] = v^2 R \delta(\tau)
\end{align*}
\]

where \( \delta(\tau) \) is the Dirac delta function. According to the well-known Kalman – Bucy Filter theory, the steady-state error covariance related filter performance index

\[
I_c = \lim_{\tau \rightarrow \infty} E[\epsilon^T(t)\epsilon(t)] = \text{trace}(P_c)
\]

is minimized, if \( P_c \) is chosen to be the unique symmetric positive definite solution of the filter ARE:

\[
AP_c + P_c A^T + \frac{1}{v^2}(BB^T - P_c C^T R^{-1} C P_c) + vQ_\xi = 0 \quad (5.12)
\]

One can repeat the analysis steps from the previous sections and arrive at the results which are dual to those stated in Theorems 1 and 2. In particular, it is easy to show that the following asymptotic relations take place:

\[
\begin{align*}
a) \quad P_c = P_0 + O(\nu^2) \\
b) \quad P_c C^T = B W \sqrt{R} + O(\nu^2) \\
c) \quad I_c = \text{trace}(P_0) + O(\nu^2)
\end{align*}
\]

as \( \nu \rightarrow 0 \). These equalities are valid under the same assumptions as before, that is: a) \( \det(CB) \neq 0 \), and b) the system \( C(sI_{n \times n} - A)^{-1} B \) is minimum-phase.

We note that the noise data in (5.10) closely resemble the LQG / LTR selections from [5]. In fact, the only difference between the two sets is the term \( \nu Q_\nu \), whereby in [5] a constant matrix \( Q \), without the multiplier \( \nu \), is utilized. For this reason, the asymptotic relations (5.13) can be exploited to aid in the design procedure of the LQG / LTR observers and controllers. Vice versa, the results presented in this paper, can be treated as the LQG / LTR duality consequences that are applicable to the design of static output regulators. Due to space limitations, detailed elaboration of this duality phenomenon will be reported elsewhere.

VII. CONCLUSIONS

Motivated by [2], in this paper we presented a constructive method to design static output feedback controllers for LTI MIMO systems. Our proposed design is based on the well-known asymptotic properties of LQ regulators. We have shown that any controllable observable minimum-phase LTI dynamics, with a non-zero high frequency gain, can be stabilized using static output feedback. These controllers were designed as asymptotic approximations of the LQ optimal state-feedback regulators. We have also discussed the duality between our reported asymptotic results and the LQG / LTR method.

REFERENCES