Stability Results for Finite-Dimensional Discrete-Time Dynamical Systems Involving Non-Monotonic Lyapunov Functions

Anthony N. Michel and Ling Hou

Abstract—In [9] and in a more recent paper [7] we established results for the uniform stability and the uniform asymptotic stability in the large involving non-monotonic Lyapunov functions for continuous-time dynamical systems. In the present paper we continue this work by addressing finite-dimensional discrete-time dynamical systems. Similarly as in [9] and [7], we prove that in general, the results presented herein are less conservative than the corresponding standard Lyapunov stability results (henceforth called classical Lyapunov stability results) for finite-dimensional discrete-time dynamical systems. We present two specific examples to demonstrate the applicability of our results.

I. INTRODUCTION

In [9], and more recently, in [7], we established stability results for continuous-time dynamical systems involving non-monotonic Lyapunov functions. Systems of this type are frequently not tractable by the Principal Lyapunov Stability Results (see, e.g., [3], [8]) which henceforth will be called the classical Lyapunov stability results. We proved that the results in [9] and [7] are in general less conservative than the corresponding classical Lyapunov stability results [6]–[8].

In [4], [6] we show in a roundabout way how discrete-time finite-dimensional dynamical systems with non-monotonic Lyapunov functions can be analyzed by results for discontinuous continuous-time systems [9] (called discontinuous dynamical systems, or, DDS). In this approach, one associates an auxiliary piecewise continuous-time dynamical system with the discrete-time dynamical system under study, having identical stability properties. The stability analysis of the discrete-time system is then accomplished by applying the DDS stability results (given in [9]) to the auxiliary piecewise continuous system.

The results reported in [9] and [7], and in the present paper, pertain to general dynamical systems. Other existing stability results involving non-monotonic Lyapunov functions, albeit mostly for specialized classes of systems, are encountered in the extensive literature on switched and hybrid systems. A fairly complete survey of these results is given in [10], and in the references cited in [10]. Refer also to [11], [12] and to [1], where discrete-time dynamical systems determined by time-invariant difference equations are addressed.

2. PRELIMINARIES

A finite-dimensional discrete-time dynamical system is a four-tuple \( \{\mathbb{N}, \mathbb{R}^n, A, S\} \) where \( \mathbb{N} = \{0, 1, 2, \cdots\} \) denotes the time set, \( \mathbb{R}^n \) is the state space, \( A \subset \mathbb{R}^n \) is the set of initial states, and \( S \) denotes a family of motions. For a fixed \( x_0 \in A \), \( \tau_0 \in \mathbb{N} \), a mapping \( x(\cdot, x_0, \tau_0) : T_{x_0,\tau_0} \to \mathbb{R}^n \) is called a motion if \( x(\tau_0, x_0, \tau_0) = x_0 \), where \( T_{x_0,\tau_0} = \{\tau_0, \tau_1, \cdots\} \subset \mathbb{N} \) is a discrete set with \( \tau_1 > \tau_m \) when \( l > m \). Accordingly, \( S \) is a subset of the set

\[
\bigcup_{(x_0, \tau_0) \in A \times \mathbb{N}} \{T_{x_0,\tau_0} \to \mathbb{R}^n\}
\]

and for any \( x(\cdot, x_0, \tau_0) \in S \), we have \( x(\tau_0, x_0, \tau_0) = x_0 \).

In general, for each \( (x_0, \tau_0) \in A \times \mathbb{N} \), we allow more than one \( T_{x_0,\tau_0} \) to exist (i.e., we allow more than one motion to initiate from a given pair \( (x_0, \tau_0) \)).

We recall that for a dynamical system \( \{\mathbb{N}, \mathbb{R}^n, A, S\} \), a set \( M \subset A \) is said to be invariant with respect to system \( S \), if \( x_0 \in M \) implies that \( x(\tau_k, x_0, \tau_0) \in M \) for all \( \tau_k \in T_{x_0,\tau_0} \), \( \tau_0 \in \mathbb{N} \) and all \( x(\cdot, x_0, \tau_0) \in S \). If \( x_0 \in A \), and \( M = \{x_0\} \), we call \( x_0 \) an equilibrium (point) of the dynamical system. Throughout the present paper, we will assume that...
the dynamical system \( \{\mathbb{N}, \mathbb{R}^n, A, S\} \) has an equilibrium at the origin, i.e., at \( x = 0 \in \mathbb{R}^n \).

The motions of finite-dimensional discrete-time dynamical systems are usually characterized by the solutions of systems of first order ordinary difference equations given by (using the notation \( x(\tau_k, x_0, 0) = x(\tau_k) \))

\[
x(\tau_{k+1}) = h(x(\tau_k), \tau_k),
\]

where \( h : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^n \). However, we note that there are many finite-dimensional discrete-time dynamical systems which are not necessarily determined by equations of this form (2). For example, the scalar difference equation

\[
x(k + 1) = \begin{cases} 2x(k), & \text{if } k = k_0 + 2m, \\ 1/4x(k), & \text{if } k = k_0 + 2m + 1, \end{cases}
\]

with \( x(k_0) = x_0, m \in \mathbb{N} \), \( x_0 \in \mathbb{R} \) and \( k_0 \in \mathbb{N} \), is strictly speaking not a special case of (2) since its right hand side depends explicitly on the parameter \( k_0 \). However, the solutions of this system determine a finite-dimensional discrete-time dynamical system as defined above. Indeed, the unique solutions of (3) which determine the set \( S \) are given by

\[
x(k, x_0, k_0) = \begin{cases} x_0/2^m, & \text{if } k = k_0 + 2m, \\ x_0/2^{m-1}, & \text{if } k = k_0 + 2m + 1, \end{cases}
\]

\( m \in \mathbb{N} \), for each pair \( (x_0, k_0) \in \mathbb{R} \times \mathbb{N} \) and for all \( k \geq k_0 \). We note that \( x = 0 \) is an equilibrium of the dynamical system determined by (3).

We will have occasion to revisit system (3) in the subsequent presentation.

Before proceeding, we note that dynamical systems determined by (2) have an equilibrium at the origin \( x = 0 \) if \( h(0, \tau_k) = 0, k \in \mathbb{N} \).

In the interests of brevity, we will address only uniform asymptotic stability (see, e.g., [3], [8]).

**Definition 1** The equilibrium \( x = 0 \) of the dynamical system \( \{\mathbb{N}, \mathbb{R}^n, A, S\} \) is said to be stable if for every \( \varepsilon > 0 \) and every \( k_0 \in \mathbb{N} \), there exists a \( \delta = \delta(\varepsilon, k_0) > 0 \) such that \( |x(k, x_0, k_0)| < \varepsilon \) for all \( k \in T_{x_0, k_0} \) and for all \( x(\cdot, x_0, k_0) \in S \) whenever \( |x_0| < \delta \). The equilibrium \( x = 0 \) is said to be uniformly stable if \( \delta \) is independent of \( k_0 \), i.e., \( \delta = \delta(\varepsilon) \).

**Definition 2** The equilibrium \( x = 0 \) of the dynamical system \( \{\mathbb{N}, \mathbb{R}^n, A, S\} \) is said to be uniformly asymptotically stable in the large if

1) the equilibrium \( x = 0 \) is uniformly stable, and
2) for every \( \alpha > 0 \), for every \( \varepsilon > 0 \), and for every \( k_0 \in \mathbb{N} \), there exists a \( \tau = \tau(\varepsilon, \alpha, k_0) > 0 \) such that if \( |x_0| < \alpha \), then for all \( x(\cdot, x_0, n_0) \in S \), \( |x(k, x_0, k_0)| < \varepsilon \) for all \( k \in T_{x_0, k_0, +\tau} \).

In stating the classical Lyapunov results for uniform stability and uniform asymptotic stability, we will make use of functions of class \( K \) and class \( K_\infty \). A function \( \varphi \in \mathcal{C}([0, r], \mathbb{R}^+)(\text{resp., } \varphi \in \mathcal{C}([r, +\infty], \mathbb{R}^+)) \) is said to belong to class \( K \) (i.e., \( \varphi \in K \)) if \( \varphi(0) = 0 \) and if \( \varphi \) is strictly increasing on \([0, r]\) (resp., on \([r, +\infty]\)). If \( \varphi \in K \) and \( \lim_{r \to +\infty} \varphi(r) = +\infty \), then \( \varphi \) is said to belong to class \( K_\infty \).

**Theorem 1** [3], [8] Let \( \{\mathbb{N}, \mathbb{R}^n, A, S\} \) be a discrete-time dynamical system with an equilibrium at the origin \( x = 0 \). Assume that there exists a function \( V : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^+ \) and functions \( \varphi_1, \varphi_2 \in K \) defined on \( \mathbb{R}^+ \) such that

\[
\varphi_1(|x|) \leq V(x, k) \leq \varphi_2(|x|)
\]

for all \( x \in \mathbb{R}^n \) and \( k \in \mathbb{N} \). Assume that there exists a neighborhood \( U \) of the origin \( x = 0 \) such that for all \( x_0 \in U \) and for all \( x(x_0, k_0) \in S \), \( V(x(k, x_0, k_0), k) \) is nonincreasing for all \( k \geq k_0 \), \( k \in \mathbb{N} \). Then the equilibrium \( x = 0 \) of the system \( \{\mathbb{N}, \mathbb{R}^n, A, S\} \) is uniformly stable.

**Theorem 2** [3], [8] Let \( \{\mathbb{N}, \mathbb{R}^n, A, S\} \) be a discrete-time dynamical system with an equilibrium at the origin \( x = 0 \). Assume that there exists a function \( V : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^+ \) and functions \( \varphi_1, \varphi_2 \in K_\infty \) defined on \( \mathbb{R}^+ \) such that

\[
\varphi_1(|x|) \leq V(x, k) \leq \varphi_2(|x|)
\]

for all \( x \in \mathbb{R}^n \) and \( k \in \mathbb{N} \). Assume that there exists a function \( \varphi_3 \in K \) defined on \( \mathbb{R}^+ \) such that for all \( x_0 \in A \) and for all \( x(x_0, k_0) \in S \),

\[
V(x(k + 1, x_0, k_0), k + 1) - V(x(k, x_0, k_0), k) \leq -\varphi_3(|x(k, x_0, k_0)|)
\]

for all \( k \geq k_0 \), \( k \in \mathbb{N} \). Then the equilibrium \( x = 0 \) of the system \( \{\mathbb{N}, \mathbb{R}^n, A, S\} \) is uniformly asymptotically stable in the large.

3. **Stability Results Involving Non-Monotonic Lyapunov Functions**

As stated above, Theorems 1 and 2 constitute sufficient conditions. Although Converse Theorems (necessary conditions) have been established for Theorems 1 and 2 under reasonable conditions (see [8] and the references cited therein) there are nevertheless systems with known stability properties, where the classical Lyapunov results (such as Theorems 1 and 2) fail. This arises, e.g., in the case of non-monotonic Lyapunov functions. To be more specific, an examination of Theorems 1 and 2 indicates that when the hypotheses of these theorems are satisfied, then along the motions of the dynamical systems \( \{\mathbb{N}, \mathbb{R}^n, A, S\} \), the Lyapunov function will be nonincreasing or strictly decreasing, respectively. This is not the situation, e.g., in the case of system (3). In the next result, we prove that, in fact, there does not exist for system (3) a Lyapunov function which satisfies the hypotheses of Theorems 1 or 2. On the other hand, we will be able to prove that the equilibrium \( x = 0 \) of system (3) is uniformly asymptotically stable in
the large, using our stability results involving non-monotonic Lyapunov functions, given in Theorems 3 and 5. Of course the stability properties of system (3) may also be ascertained by applying the expression for the solutions of (3), given in (4), directly to the stability definitions given in Definitions 1 and 2.

**Proposition 1** For the discrete-time dynamical system \(\{\mathbb{N}, \mathbb{R}^n, A, S\}\) determined by (3), there does not exist a function \(V : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}^+\) which satisfies the hypotheses of Theorems 1 or 2. Therefore, Theorem 2 cannot be used to prove that the equilibrium \(x = 0\) of the dynamical system (3) is uniformly asymptotically stable in the large.

**Proof.** The following proof is adapted from the proof of a similar result in [4] and [6] and is presented here in the interests of completeness.

It suffices to prove that there does not exist a function \(V\) which satisfies the hypotheses of Theorem 1. For purposes of contradiction, assume that there exist a function \(V : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}^+\) and functions \(\varphi_1, \varphi_2 \in \mathcal{K}\) defined on \(\mathbb{R}^+\) such that

\[
\varphi_1(|x|) \leq V(x, k) \leq \varphi_2(|x|)
\]

(8)

for all \((x, k) \in \mathbb{R} \times \mathbb{N}\), and there exists a neighborhood \(U\) of the origin \(x = 0\) such that for all \(x_0 \in U\) and for all \(x(\cdot, x_0, k_0) \in S\), \(V(x(k, x_0, k_0), k)\) is nonincreasing for all \(k \geq k_0, k \in \mathbb{N}\). Without loss of generality, we assume that \(k = 1 \in U\). From (4), \(x(k_0 + 1, x_0, k_0) = 2x_0\) for any \((x_0, k_0) \in \mathbb{R} \times \mathbb{N}\). In particular, for any \(m \in \mathbb{N}\), since

\[
\begin{align*}
V\left(\frac{1}{2m}, 1\right) &\geq V\left(\frac{1}{2m-1}, 2\right) = V\left(\frac{1}{2m-1}, 1\right), \\
V\left(\frac{1}{2m-1}, 2\right) &\geq V\left(\frac{1}{2m-2}, 3\right) = V\left(\frac{1}{2m-2}, 2\right), \\
V\left(\frac{1}{2m-2}, 3\right) &\geq V\left(\frac{1}{2m-3}, 4\right) = V\left(\frac{1}{2m-3}, 3\right), \\
\end{align*}
\]

and since \(V(x(k, x_0, k_0), k)\) is nonincreasing for all \(k \geq k_0\) and all \(x(\cdot, x_0, t_0) \in S\), we have that

\[
\begin{align*}
\varphi_1(1) &\leq \varphi_2(1), \\
\varphi_1(2) &\leq \varphi_2(2), \\
\varphi_1(3) &\leq \varphi_2(3), \\
\varphi_1(m) &\leq \varphi_2(m),
\end{align*}
\]

for all \(m \in \mathbb{N}\), which implies

\[
\varphi_2(0) = \lim_{m \to \infty} \varphi_2\left(\frac{1}{2m}\right) \geq \varphi_1(1) > 0.
\]

However, by the assumption that \(\varphi_2 \in \mathcal{K}\), we know that \(\varphi(0) = 0\). We have arrived at a contradiction. Therefore, there does not exist a Lyapunov function that satisfies the hypotheses of the classical Lyapunov Theorem for uniform stability, Theorem 1, or for uniform asymptotic stability in the large, Theorem 2.

In a recent paper [7], we established Lyapunov stability results involving non-monotonic Lyapunov functions for continuous finite-time dimensional dynamical systems \(\{\mathbb{R}^+, \mathbb{R}^n, A, S\}\), including dynamical systems determined by ordinary differential equations given by

\[
\dot{x} = h(x, t),
\]

(9)

where \(h : \mathbb{R}^n \rightarrow \mathbb{R}^n\).

In what follows, we address Lyapunov stability result involving non-monotonic Lyapunov functions for discrete-time finite-dimensional dynamical systems. Our results are in the spirit of the results established in [7] for continuous systems.

In other recent results [1], the asymptotic stability of discrete-time dynamical systems of the form

\[
x_{k+1} = h(x_k),
\]

(10)

where \(h : \mathbb{R}^n \rightarrow \mathbb{R}^n\), were established. We will compare these results with our results subsequently.

**Theorem 3** Let \(\{\mathbb{N}, \mathbb{R}^n, A, S\}\) be a discrete-time dynamical system with an equilibrium at the origin \(x = 0\). Assume that there exist a function \(V : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^+\) and functions \(\varphi_1, \varphi_2 \in \mathcal{K}\) defined on \(\mathbb{R}^+\) such that

\[
\varphi_1(|x|) \leq V(x, k) \leq \varphi_2(|x|)
\]

(11)

for all \(x \in \mathbb{R}^n\) and \(k \in \mathbb{N}\). Assume that there exists a neighborhood \(U\) of the origin \(x = 0\) such that for all \(x_0 \in U\) and for all \(x(\cdot, x_0, t_0) \in S\), \(V(x(t, x_0, t_0), k)\) is nonincreasing for all \(k \geq k_0, k \in \mathbb{N}\). Without loss of generality, we assume that \(k = 1 \in U\). From (4), \(x(k_0 + 1, x_0, k_0) = 2x_0\) for any \((x_0, k_0) \in \mathbb{R} \times \mathbb{N}\). In particular, for any \(m \in \mathbb{N}\), since

\[
\begin{align*}
\varphi_1(1) &\leq \varphi_2(1), \\
\varphi_1(2) &\leq \varphi_2(2), \\
\varphi_1(3) &\leq \varphi_2(3), \\
\varphi_1(m) &\leq \varphi_2(m),
\end{align*}
\]

for all \(m \in \mathbb{N}\), which implies

\[
\varphi_2(0) = \lim_{m \to \infty} \varphi_2\left(\frac{1}{2m}\right) \geq \varphi_1(1) > 0.
\]

However, by the assumption that \(\varphi_2 \in \mathcal{K}\), we know that \(\varphi(0) = 0\). We have arrived at a contradiction. Therefore, there does not exist a Lyapunov function that satisfies the hypotheses of the classical Lyapunov Theorem for uniform stability, Theorem 1, or for uniform asymptotic stability in the large, Theorem 2.

Theorem 3 Let \(\{\mathbb{N}, \mathbb{R}^n, A, S\}\) be a discrete-time dynamical system with an equilibrium at the origin \(x = 0\). Assume that there exist a function \(V : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^+\) and functions \(\varphi_1, \varphi_2 \in \mathcal{K}\) defined on \(\mathbb{R}^+\) such that

\[
\varphi_1(|x|) \leq V(x, k) \leq \varphi_2(|x|)
\]

(11)

for all \(x \in \mathbb{R}^n\) and \(k \in \mathbb{N}\). Assume that there exists a neighborhood \(U\) of the origin \(x = 0\) such that for all \(x_0 \in U\) and for all \(x(\cdot, x_0, t_0) \in S\), \(V(x(t, x_0, t_0), k)\) is nonincreasing for all \(k \geq 0\), where \(E = \{\tau_1, \tau_2, \ldots : \tau_k > \tau_0\} \subset \mathbb{N}\) is an increasing, infinite sequence which may depend on \(x(\cdot, x_0, t_0)\). Furthermore, assume that there exists a function \(f \in C[\mathbb{R}^+, \mathbb{R}^+]\), independent of \(x(\cdot, x_0, t_0) \in S\), such that \(f(0) = 0\) and such that

\[
V(x(m, x_0, \tau_0), m) \leq f(V(x(\tau_k, x_0, \tau_0), \tau_k)),
\]

(12)

for all \(\tau_k < m < \tau_{k+1}, k \in \mathbb{N}\).

Then the equilibrium \(x = 0\) of the system \(\{\mathbb{N}, \mathbb{R}^n, A, S\}\) is uniformly stable.

**Proof.** Since \(f\) is continuous and \(f(0) = 0\) then for any \(\varepsilon > 0\) there exists a \(\delta = \delta(\varepsilon) > 0\) such that \(f(r) \leq \varphi_1(\varepsilon)\) as long as \(0 \leq r < \delta\). We assume that \(\delta < \varphi_1(\varepsilon)\). Thus, for any motion \(x(\cdot, x_0, \tau_0) \in S\), as long as the initial condition
\(|x_0| < \varphi^{-1}_2(\delta)\) is satisfied, then from the assumption we have,

\[
V(x(\tau_k, x_0, \tau_0), \tau_k) \leq V(x(\tau_0, x_0, \tau_0), \tau_0) \\
\leq \varphi_2(|x_0|) < \delta < \varphi_1(\varepsilon)
\]

for \(k \in \mathbb{N}\). Furthermore, for any \(m \in (\tau_n, \tau_{n+1})\) we can conclude in view of (12) that

\[
V(x(m, x_0, \tau_0), m) \leq f(V(x(\tau_k, x_0, \tau_0), \tau_k)) < \varphi_1(\varepsilon).
\]

Thus, we have shown that \(V(x(m, x_0, \tau_0), m) < \varphi_1(\varepsilon)\) is true for all \(m \geq \tau_0, m \in \mathbb{N}\). In view of (11) we have

\[
|x(m, x_0, \tau_0)| \leq \varphi_1^{-1}(V(x(m, x_0, \tau_0), m)) < \varepsilon.
\]

Therefore, the equilibrium \(x = 0\) of the dynamical system \(\{\mathbb{N}, \mathbb{R}^n, A, S\}\) is uniformly stable.

**Theorem 4** In addition to the assumptions given in Theorem 3, assume that \(\varphi_1, \varphi_2 \in \mathcal{K}_\infty, U = \mathbb{R}^n\), and there exists a function \(\varphi_3 \in \mathcal{K}\) defined on \(\mathbb{R}^+\) such that

\[
V(x(\tau_{k+1}, x_0, \tau_0), \tau_{k+1}) - V(x(\tau_k, x_0, \tau_0), \tau_k) \\
\leq -\varphi_3(|x(\tau_k, x_0, \tau_0)|)
\]

(13)

for all \(k \in \mathbb{N}\) and all \(x(\cdot, x_0, \tau_0) \in S\). Then the equilibrium \(x = 0\) of the system \(\{\mathbb{N}, \mathbb{R}^n, A, S\}\) is asymptotically stable in the large.

**Theorem 5** In addition to the assumptions given in Theorem 3, assume that \(\varphi_1, \varphi_2 \in \mathcal{K}_\infty, U = \mathbb{R}^n\), and there exists a function \(\varphi_3 \in \mathcal{K}\) defined on \(\mathbb{R}^+\) such that

\[
V(x(\tau_{k+1}, x_0, \tau_0), \tau_{k+1}) - V(x(\tau_k, x_0, \tau_0), \tau_k) \\
\leq -\left(\tau_{k+1} - \tau_k\right)\varphi_3(|x(\tau_k, x_0, \tau_0)|)
\]

(14)

for all \(k \in \mathbb{N}\) and all \(x(\cdot, x_0, \tau_0) \in S\). Then the equilibrium \(x = 0\) of the system \(\{\mathbb{N}, \mathbb{R}^n, A, S\}\) is uniformly asymptotically stable in the large.

The proofs of Theorems 4 and 5 follow along the same lines. We present only the proof of Theorem 5 below.

**Proof of Theorem 5.** For any \(x_0 \in A\) and for any \(x(\cdot, x_0, \tau_0) \in S\), letting \(z_k = V(x(\tau_k, x_0, \tau_0), \tau_k), k \in \mathbb{N}\), we obtain from the assumptions of the theorem that

\[
z_{k+1} - z_k \leq -\left(\tau_{k+1} - \tau_k\right)\varphi_3(|x(\tau_k, x_0, \tau_0)|) \\
\leq -\left(\tau_{k+1} - \tau_k\right)(\varphi_3 \circ \varphi_2^{-1})(z_k)
\]

for all \(k \in \mathbb{N}\). Noting that \(\varphi \triangleq \varphi_3 \circ \varphi_2^{-1} \in \mathcal{K}\), the above inequality becomes

\[
z_{k+1} - z_k \leq \varphi(z_k).
\]

Since \(\{z_k\}\) is nonincreasing and \(\varphi \in \mathcal{K}\), it follows that

\[
z_{m+1} - z_m \leq -\varphi(z_m)(\tau_{m+1} - \tau_m) \leq -\varphi(z_k)(\tau_{m+1} - \tau_m)
\]

for all \(m \leq k\). We thus obtain that

\[
z_{k+1} - z_0 \leq -(\tau_{k+1} - \tau_0)\varphi(z_k),
\]

which in turn yields

\[
\varphi(z_k) \leq \frac{z_0 - z_{k+1}}{\tau_{k+1} - \tau_0} \leq \frac{z_0}{\tau_{k+1} - \tau_0},
\]

(15)

for all \(k \in \mathbb{N}\).

Now consider a fixed \(\delta > 0\). For any given \(\varepsilon > 0\), we can choose a \(\gamma > 0\) such that

\[
\max\left\{\varphi_1^{-1}\left(\frac{\varphi^{-1}_2(\varepsilon)}{\gamma}\right), \varphi_1^{-1}\left(\frac{\varphi^{-1}_2(\delta)}{\gamma}\right)\right\} < \varepsilon
\]

(16)

since \(\varphi_1, \varphi_2 \in \mathcal{K}\) and \(f(0) = 0\). For any \(x_0 \in U\) with \(|x_0| < \delta\) and any \(\tau_0 \in \mathbb{R}^+\), we are now able to show that \(|x(m, x_0, \tau_0)| < \varepsilon\) whenever \(m \geq \tau_0 + \gamma\). This is true because for any \(m \geq \tau_0 + \gamma, m\) must belong to the interval \([\tau_{k+1}, \tau_{k+2})\) for some \(k \in \mathbb{N}\). Therefore we know that \(\tau_{k+1} - \tau_0 \geq m - \tau_0 \geq \gamma\). It follows from (15) that

\[
\varphi(z_{k+1}) \leq \frac{z_0}{\gamma} = \frac{V(x_0, \tau_0)}{\gamma} \leq \frac{\varphi_2(\delta)}{\gamma}.
\]

(17)

In the case when \(m = \tau_{k+1}\), it follows from (17) that

\[
|x(m, x_0, \tau_0)| \leq \varphi_1^{-1}(V(x(m, x_0, \tau_0), m)) < \varepsilon,
\]

noticing that (16) holds. In the case when \(m \in (\tau_{k+1}, \tau_{k+2})\), we can conclude from (12), (16) and (17) that

\[
|x(m, x_0, \tau_0)| \leq \varphi_1^{-1}(V(x(m, x_0, \tau_0), m)) \\
\leq \varphi_1^{-1}(f(z_{k+1})) < \varepsilon.
\]

This proves that the equilibrium \(x = 0\) of the dynamical system \(\{\mathbb{R}^+, \mathbb{R}^n, A, S\}\) is uniformly asymptotically stable in the large.

**Theorem 5** states that along the motions of the dynamical system \(\{\mathbb{N}, \mathbb{R}^n, A, S\}\), the Lyapunov function \(V\) is required to decrease only on a subsequence of time instants \(E\), while on the set \(\mathbb{N} - E, V\) may actually increase, resulting in a non-monotonic Lyapunov function. Inequality (14) in Theorem 5 guarantees that the motions approach the origin uniformly, while inequality (13) in Theorem 4 cannot determine how fast or slow each motion approaches the origin. If (13) is satisfied and \(\sup_{k \in \mathbb{N}}(|\tau_{k+1} - \tau_k| < \infty\), then (14) is true.

In the next section we apply Theorem 5 to prove that the equilibrium \(x = 0\) of system (3) is uniformly asymptotically stable in the large. Before concluding this section, we demonstrate the applicability of Theorem 5 by means of a specific example.

**Example 1** In [5], it is established that a switched linear system given by

\[
x(k + 1) = A(k)x(k),
\]

(18)

where \(A(k) \in \{A_1, A_2, \ldots, A_N\}\), is globally asymptotically stable under arbitrary switching if and only if there exists a finite \(p\) such that

\[
1 > \alpha = \max\left\{\|A_{i_1}A_{i_2} \cdots A_{i_p}\|: A_{i_k} \in \{A_1, A_2, \ldots, A_N\}\right\}
\]

(19)

\(k = 1, 2, \ldots, p\).
where $\| \cdot \|$ denotes either the 1 norm or the $\infty$ norm. The sufficiency is shown by viewing the above system as a subsystem of a polytopic uncertain linear system
\[ x(k + 1) = \Phi(k)x(k), \]
where $\Phi(k)$ is any matrix in the convex hull of $A_1, A_2, \ldots, A_N, k \in \mathbb{N}$, and then invoking a result in [2].

We will present a direct proof as follows. For any $x(\cdot, x_0, \tau_0)$, let $E = \{ \tau_1 = k_0 + p, \tau_2 = k_0 + 2p, \cdots \}$. Choosing the Lyapunov function $V(x) = |x|$, where $| \cdot |$ is the vector norm on $\mathbb{R}^n$ which induces the matrix norm $\| \cdot \|$ given in (19), we have
\[
V(x(\tau_{k+1})) - V(x(\tau_k)) = |x(k_0 + (k+1)p)| - |x(k_0 + kp)|
\]
\[
= |A(k_0 + (k+1)p - 1)\cdots A(k_0 + kp + 1)x(k_0 + kp)|
\]
\[
- |x(k_0 + kp)|
\]
\[
\leq \left( \| A(k_0 + (k+1)p - 1)\cdots A(k_0 + kp + 1)\| \right)^{-1}|x(\tau_k)|
\]
\[
\leq (\alpha - 1)V(x(\tau_k)). \tag{20}
\]
Next, let $f(s) = \beta s, s \geq 0$, where $\beta = \max\{ \| A_{1_1}A_{1_2}\cdots A_{1_q} \| : A_{1_i} \in \{ A_1, A_2, \cdots, A_N \}, 1 \leq q \leq p \}$. Then it is easily seen that for all $\tau_k < m < \tau_{k+1}$
\[
V(x(m)) = |A(m)|\cdots A(\tau_k + 1)x(\tau_k) \leq |\beta x(\tau_k)|.
\]
Therefore, all the hypotheses of Theorem 4 are satisfied.

### 4. The Stability Results Involving Non-monotonic Lyapunov Functions Are Less Conservative Than the Classical Lyapunov Stability Results

In Proposition 1 we proved that for the dynamical system determined by (3) there does not exist a Lyapunov function which satisfies Theorems 1 or 2. In the following, we will use Theorem 5 to prove that the equilibrium of system (3) is uniformly asymptotically stable in the large.

**Proposition 2** For system (3) there exists a Lyapunov function $V: \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}^+$ which satisfies Theorem 5, and therefore, the equilibrium $x = 0$ of system (3) is uniformly asymptotically stable in the large.

**Proof.** For system (3) we choose as a Lyapunov function $V(x) = |x|$ for all $x \in \mathbb{R}$. For any solution $x(\cdot, x_0, \tau_0)$, we choose the set $E = \{ \tau_k : \tau_k = k_0 + 2k \}$. In view of (4), we have
\[
V(x(\tau_k, x_0, \tau_0)) = |x(\tau_k)| \tag{21}
\]
and
\[
V(x(\tau_k + 1, x_0, \tau_0)) \leq 2V(x(\tau_k, x_0, k_0)) \tag{22}
\]
for all $k \in \mathbb{N}$. Therefore, all the conditions of Theorems 3 and 5 are satisfied and we conclude that the equilibrium $x = 0$ of system (3) is uniformly asymptotically stable in the large.

Note that in Proposition 2, $V(x(k, x_0, \tau_0))$ is non-monotonic along $x(\cdot, x_0, \tau_0)$.

To prove that the stability results involving non-monotonic Lyapunov functions (Theorems 3 and 5) are less conservative than the classical Lyapunov stability results (Theorems 1 and 2), we will require the following results.

**Theorem 6** (a) If the hypotheses of Theorem 1 are true for a dynamical system $\{ \mathbb{N}, \mathbb{R}^n, A, S \}$, then the hypotheses of Theorem 3 hold for the same dynamical system. Therefore, Theorem 1 reduces to Theorem 3.

(b) If the hypotheses of Theorem 2 are true for a dynamical system $\{ \mathbb{N}, \mathbb{R}^n, A, S \}$, then the hypotheses of Theorem 5 hold for the same dynamical system. Therefore, Theorem 2 reduces to Theorem 5.

**Proof.** Omitted due to space limitations.

From Proposition 2 and Theorem 6, there now follows the result below.

**Theorem 7** The classical Lyapunov stability results for uniform stability (Theorem 1) and uniform asymptotic stability in the large (Theorem 2) are more conservative than the stability results involving non-monotonic Lyapunov functions for uniform stability (Theorem 3) and uniform asymptotic stability in the large (Theorem 5), respectively.

### 5. Concluding Remarks

We conclude with a few observations concerning extensions and generalizations of the results addressed herein and with a few observations concerning a result reported in [1].

First of all, we note that it is not difficult to establish results involving non-monotonic Lyapunov functions for exponential stability and for Lagrange stability (uniform boundedness and uniform ultimate boundedness of motions) for the class of systems considered herein. Next, we note that stability results of the type given in Theorems 3 – 5 can readily be generalized to finite-dimensional as well as infinite-dimensional discrete-time dynamical systems defined on metric spaces. Finally, similarly as in [6], it should be possible to establish Converse Theorems for Theorems 3 – 5.

We now compare Theorems 3 and 5 with one of the results established in [1]. (There are additional results in [1].)

**Theorem A** [1] If there exist $m − 1$ nonnegative scalars $\sigma_i, i = 1, 2, \cdots, m − 1$, and a continuous radially unbounded function $V: \mathbb{R}^n \rightarrow \mathbb{R}$, such that $V(x) > 0$ for any $x \neq 0, V(0) = 0$, and
\[
\sigma_m (V(x(k + m)) - V(x(k))) + \sigma_{m-2} (V(x(k + m - 1)) - V(x(k))) + \cdots + (V(x(k + 1)) - V(x(k))) < 0, \tag{23}
\]
$k \in \mathbb{N}$, then the equilibrium $x = 0$ of (10) is globally asymptotically stable.

The present results (Theorems 3 – 5) are clearly applicable to a much larger class of systems than Theorem A. However, a generalization of Theorem A to this larger class of systems would be an easy matter. A more significant difference between Theorem 5 and Theorem A is that (23) is required to be true for all $k \in \mathbb{N}$, while (14) in Theorem 5 only needs to hold for $k \in E$. When applying Theorem A, for example, to Example 1, the fact that $V(x(k_0 + kp))$ decreases as $k$ increases is not sufficient, and one still needs to find a constant $\sigma_{i-1} > 0$, large enough such that the decrement term $\sigma_{i-1}(V(x(k_0 + (k+1)p)) - V(x(k_0 + kp)))$ dominates any possible increment of $V(x(k_0 + kp+1)) - V(x(k_0 + kp))$. It follows from (23) that

$$V(x(k + 1)) \leq \left( \sum_{j=1}^{m-1} \sigma_j + 1 \right)V(x(k))$$

and if for some $i$, $\sigma_{i-1} > 0$, then

$$V(x(k + i)) \leq \frac{1}{\sigma_{i-1}} \left( \sum_{j=1}^{m-1} \sigma_j + 1 \right)V(x(k)).$$

Then we can show that

$$V(x(k + j)) \leq \gamma V(x(k))$$

for all $j = 1, 2, \cdots, m$, for some constant $\gamma > 0$, which implies that (12) is satisfied with a linear function $f(s) = \gamma s$. Finally, we note that Theorem A reduces to the classical Lyapunov result for system (10) when in (23) $m = 1$ and $\sigma_0 = 1$ while Theorem 5 reduces to the classical Lyapunov stability results for the dynamical system $\{\mathbb{N}, \mathbb{R}^n, A, S\}$ when $E = \mathbb{N}$.

---

**REFERENCES**


