LQR and Receding Horizon Approaches to Multi-Dimensional Option Hedging under Transaction Costs

James A. Primbs

Abstract—In this paper we formulate the problem of dynamically hedging a basket option on multiple underlying stocks and in the presence of proportional transaction costs as a linear quadratic control problem subject to constraints. The linear structure is obtained by sampling over paths of the underlying stocks and linearly parameterizing control actions over basis functions. Two solutions are then proposed. The first involves quadratically penalizing transaction costs in the objective and allows the hedging problem to be solved as a standard unconstrained linear quadratic regulator problem. The second approach uses receding horizon control to solve a quadratic program over a specified prediction horizon, where the cost function utilizes the LQR solution from the first approach. A numerical example illustrates the methodology.

I. INTRODUCTION

This paper develops a control systems based method for the hedging of multi-dimensional options under proportional transaction costs. For concreteness, we develop the method for a European basket call option which is a derivative security that gives the holder the right (but not the obligation) to purchase a basket (or portfolio) of stocks at a specified price $K$ (known as the strike price) and time $N$ (known as the expiration date).

The dynamic hedging problem corresponding to an option is the problem faced by a trader who wishes to dynamically trade a portfolio of the stocks underlying the option (and a risk free asset) such that the value of his portfolio at expiration matches (or possibly super-replicates) the payoff value of the option. The idea that a trader may successfully replicate the payoff value of an option is the key idea behind modern derivative pricing theory [5], [15]. One may view the dynamic hedging problem as a stochastic optimal control problem. Given an initial wealth, one should trade that wealth in a self-financing manner in order to replicate the payoff of an option. In a control systems framework, the system dynamics corresponds to the stochastic evolution of the stock prices and hedger’s wealth, while the objective function is a measure of the distance between the hedger’s wealth and the payoff of the option at expiration.

When there is a transaction cost for trading the underlying stock, or when the option is on a basket of stocks, the dynamic hedging problem is known to be difficult. The case of proportional transaction costs has been considered in the work of Leland [11], Hodges and Neuberger [9], Davis, Panas, and Zariphopoulou [6] and others. Additionally, no exact closed form solution is known for the pricing or hedging of a basket option, even when transaction costs are not present and the stocks follow geometric Brownian motion [10].

The purpose of this paper is to develop a control systems approach to the hedging problem under proportional transaction costs, even on multiple underlyings. The approach is to capture the linear structure present in the system dynamics by sampling over paths of the stocks and linearly parameterizing hedging strategies (the control actions). Using a quadratic objective function subject to these linear dynamics leads to a control problem with linear quadratic structure. We are then able to use standard LQR and receding horizon control (RHC) methods to solve the hedging problem.

Note that the control systems approach to finance is a developing area that is bringing new tools, theory, and modeling paradigms to the subject. For example, Barmish [2] recently presented a new robust control paradigm for equity trading. Additionally, receding horizon and predictive control methods are being developed for classical financial engineering problems. Primbs and Sung [19] developed receding horizon methods for the constrained index tracking problem, while Piccoli and Marigo [16], Herzog, et. al. [7], [8], and Primbs [17] have used receding horizon methods for dynamic portfolio optimization. Meindl and Primbs [12], [13], [14], [18], and recently Bemporad et. al. [3] have developed stochastic receding horizon approaches to option hedging. The current paper is a contribution to this line of work.

The paper is organized as follows. In Section II we develop the system dynamics for a self financing portfolio of multiple stocks and a risk free asset. Section III converts the dynamics to the form of a linear system under constraints by sampling over stock paths and linearly parameterizing control actions. Section IV sets up the dynamic hedging problem for a basket option. Section V formulates the dynamic hedging problem as a linear quadratic regulator control problem, and in Section VI a receding horizon control approach is developed. Section VII provides a numerical example and Section VIII concludes.

II. SYSTEM DYNAMICS

A. Stock and Bond Model

We consider a market in which stocks and a bond may be traded. We begin by specifying the details of these basic tradable assets.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}, P)$ denote a filtered probability space where the filtration $\{\mathcal{F}_k\}$ represents the information available
to the hedger at time $k$.

Let the $\{\mathcal{F}_k\}$-adapted stochastic process $B(k,\omega)$ denote the price process of the bond in discrete time with random total return $R(k,\omega)$ per period satisfying

$$B(k+1,\omega) = R(k,\omega)B(k,\omega), \quad k = 0 \ldots N.$$  (1)

We assume that the price process of the bond is always positive. Additionally, let the $\{\mathcal{F}_k\}$-adapted stochastic processes $S_i(k,\omega)$, $i = 1 \ldots n$ for $k = 0, \ldots, N-1$ model the price processes of $n$ stocks.

B. Wealth Dynamics of a Hedger

We consider a hedger who is allowed to trade stocks and the bond in a self-financing manner (i.e. his portfolio can only be rebalanced, no money can be added or extracted) at times $k = 0 \ldots N-1$. The hedger chooses a dynamic trading strategy in which the first $m < n$ stocks are traded. Let $S(k,\omega) = [S_1(k,\omega), S_2(k,\omega), \ldots, S_m(k,\omega)]^T \in \mathbb{R}^m$ denote the vector containing the first $m$ stock prices, and let $h(k,\omega) = [h_1(k,\omega), h_2(k,\omega), \ldots, h_m(k,\omega)]^T \in \mathbb{R}^m$ denote the vector of “holdings” (i.e. number of shares held) of the $m$ stocks at time $k$. A trading strategy is an $\{\mathcal{F}_k\}$-adapted specification of $h(k,\omega)$ for $k = 0, \ldots, N-1$.

We let $W(k,\omega)$, $k = 0 \ldots N$ denote the hedger’s wealth corresponding to the trading strategy $h(k,\omega)$ immediately preceding his trade at time $k$.

Time $N$ is assumed to be the expiration of the option under consideration and we define $W(N+1,\omega)$ to be his wealth immediately following the final transaction cost at expiration.

It can be derived that the dynamics of wealth under the self-financing strategy $h(k,\omega)$ is given by

$$W(k+1,\omega) = R(k,\omega)W(k,\omega) + [S(k+1,\omega) - R(k,\omega)S(k,\omega)]^T h(k,\omega)$$

$$- R(k,\omega)\kappa^T (k,\omega)\Delta h(k,\omega)$$  (2)

for $k = 1 \ldots N$, where we define $S(N+1,\omega) = S(N,\omega)$ and $R(N,\omega) = 1$ so that $W(N+1,\omega)$ corresponds to the hedger’s wealth immediately following the transaction cost at expiration.

In these equations, the quantity $\kappa^T (k,\omega)\Delta h(k,\omega)$ represents the total transaction cost at time $k$ where $\kappa(k,\omega) \in \mathbb{R}^m$ is a vector containing the per-unit transaction costs of the stocks $S(k,\omega)$, and $\Delta$ is the backward difference operator so that $|\Delta h(k,\omega)| = |h(k,\omega) - h(k-1,\omega)|$ is the number of shares of each stock transacted at time $k$.

Remark 1 A common specification of transaction costs is as a percentage of the dollar amount transacted. In this case one would use the assignment $\kappa(k,\omega) = [\xi_1 S_1(k,\omega), \xi_2 S_2(k,\omega), \ldots, \xi_m S_m(k,\omega)]^T$, with $\xi_i$, $i = 1 \ldots m$ the percentage transaction cost levels of each stock.

In the following section, we reformulate the wealth dynamics as a linear system. This step will allow us to apply standard control systems methods to the hedging problem.

III. Modeling Wealth Dynamics as a Linear System

In this section we represent the system dynamics in (2) as a linear system. The rest of this section details these two steps.

A. Parameterization of Trading Strategy

Our first step is to replace $h(k,\omega)$ and $|\Delta h(k,\omega)|$ for $k = 1 \ldots N$ by linearly parameterized versions. That is we make the replacements

$$h(k,\omega) \rightarrow \Psi_h(k,\omega)u(k), \quad |\Delta h(k,\omega)| \rightarrow \Psi_{\Delta}(k,\omega)\tau(k)$$  (3)

where $\Psi_h(k,\omega) \in \mathbb{R}^{m \times p}$ and $\Psi_{\Delta}(k,\omega) \in \mathbb{R}^{m \times q}$ are matrices of $\{\mathcal{F}_k\}$-adapted basis functions, and $u(k) \in \mathbb{R}^p$, $\tau(k) \in \mathbb{R}^q$ are vectors of coefficients that do not depend on $\omega$.

When required, the relationship between $h(k,\omega)$ and $|\Delta h(k,\omega)|$ is enforced via the constraint

$$\Psi_{\Delta}(k,\omega)\tau(k) \geq |\Psi_h(k,\omega)u(k) - \Psi_h(k-1,\omega)u(k-1)|$$  (4)

which constrains $\Psi_{\Delta}(k,\omega)\tau(k)$ to be an upper bound on the true value of $|\Delta h(k,\omega)|$.

Note that at the current time $k = 0$ no basis functions are needed to represent the holding $h(0)$ or change in holding $|\Delta h(0)|$ since they are both deterministic vectors in $\mathbb{R}^n$. For consistency and ease of notation at time $k = 0$, we define $\Psi_h(0,\omega) = I$ and $\Psi_{\Delta}(0,\omega) = I$ so that at time $k = 0$ we have the direct replacements $h(0) \rightarrow u(0)$ and $|\Delta h(0)| \rightarrow \tau(0)$.

Upon using the substitutions in (3) the wealth dynamics in (2) can be written as

$$W(k+1,\omega) = R(k,\omega)W(k,\omega) + [S(k+1,\omega) - R(k,\omega)S(k,\omega)]^T \Psi_h(k,\omega)u(k)$$

$$- R(k,\omega)\kappa^T (k,\omega)\Psi_{\Delta}(k,\omega)\tau(k).$$  (5)

Remark 2 It is often the case that the holding of a particular asset at time $k$, say asset $S_i(k,\omega)$, will use its own set of basis functions and coefficients. That is, we might have the holding of asset $S_i(k,\omega)$ given by

$$h_i(k,\omega) = \sum_{i=1}^{p_i} \psi_{h_i}^{(i)}(k,\omega)u_i^{(i)}(k)$$  (6)

where this set of basis functions and coefficients is specific to asset $i$. In this case, the matrix of basis functions $\Psi_h^{(i)}(k,\omega)$ will be structured and take the form

$$\Psi_{h_i}^{(i)}(k,\omega) \triangleq \begin{bmatrix} \psi_{h_i}^{(1)}(k,\omega) & 0 & \ldots & 0 \\
0 & \psi_{h_i}^{(2)}(k,\omega) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \psi_{h_i}^{(p_i)}(k,\omega) \end{bmatrix} \in \mathbb{R}^{m \times p_i}$$  (7)

where each $\psi_{h_i} = [\psi_{h_i}^{(1)}, \psi_{h_i}^{(2)}, \ldots, \psi_{h_i}^{(p_i)}]^T$, $i = 1 \ldots p$ is a vector of the basis functions for the holdings $h_i(k,\omega)$. The overall vector of coefficients $u(k) \in \mathbb{R}^p$ will then contain all the coefficients $u_i^{(i)}(k)$, $i = 1 \ldots m$, $i = 1 \ldots p$, with $p = \sum_{i=1}^{p_i} p_i$. 

6892
Example 1 A simple example of a parameterization is to let \( h_1(k, \omega) \) be linear feedback on the stock prices. That is, choose \( \Psi_h = [1, S_1(k, \omega), S_2(k, \omega), \ldots, S_n(k, \omega)]^T \) for \( k = 1, \ldots, N - 1 \) in which case,

\[
\begin{align*}
h_1(k, \omega) &= \Psi_h^T(k, \omega) u(k) \\
&= [1, S_1(k, \omega), S_2(k, \omega), \ldots, S_n(k, \omega)] u(k).
\end{align*}
\]

While more sophisticated basis functions are easily used, the numerical example in Section VII demonstrates that this simple linear parameterization performs remarkably well.

B. Sampled Wealth Dynamics

Next, we sample the dynamics in (5) over paths of the underlying stocks and bond. Thus, we refer to each realization as a "path" or "scenario" and enumerate them with slight abuse of notation) by \( \omega = 1, \ldots, M \).

We define \( \mathbf{W}(k) \) to be a vector with each element corresponding to the wealth for the sampled paths \( \omega = 1, \ldots, M, \)

\[
\mathbf{W}(k) = \begin{bmatrix} W(k, 1) \\ \vdots \\ W(k, M) \end{bmatrix} \in \mathbb{R}^M, \quad k = 1, \ldots, N + 1. \tag{8}
\]

That is, \( \mathbf{W}(k) \) is a sampled version of the wealth random variable \( W(k, \omega) \). At time \( k = 0 \) we define \( \mathbf{W}(0) = \mathbf{W}(0) \mathbf{I} \) where \( \mathbf{I} \in \mathbb{R}^M \) is a column vector with all elements 1 and \( \mathbf{W}(0) \) is the current wealth.

From (5) we can write dynamics for \( \mathbf{W}(k) \) as the linear system

\[
\mathbf{W}(k+1) = F(k) \mathbf{W}(k) + G_u(k) u(k) + G_\tau(k) \tau(k), \tag{9}
\]

with \( F(k) \triangleq \text{diag}([R(0, 1), \ldots, R(0, M)]) \),

\[
G_u(k) \triangleq \begin{bmatrix} (S(k+1, 1) - R(k, 1) S(k, 1))^T \Psi_h(k, 1) \\ \vdots \\ (S(k+1, M) - R(k, M) S(k, M))^T \Psi_h(k, M) \end{bmatrix}
\]

\[
G_\tau(k) \triangleq \begin{bmatrix} R(k, 1) \kappa_1^T(k, 1) \Psi_\Delta(k, 1) \\ \vdots \\ R(k, M) \kappa_1^T(k, M) \Psi_\Delta(k, M) \end{bmatrix}
\]

The wealth dynamics in (9) is a deterministic linear system when we view \( \mathbf{W}(k) \) as the state and both \( u(k) \) and \( \tau(k) \) as the control inputs.

Finally, we define sampled versions of the coefficient matrices

\[
\Psi_h(k) = \begin{bmatrix} \Psi_h(k, 1) \\ \vdots \\ \Psi_h(k, M) \end{bmatrix}, \quad \Psi_\Delta(k) = \begin{bmatrix} \Psi_\Delta(k, 1) \\ \vdots \\ \Psi_\Delta(k, M) \end{bmatrix}
\]

where the absence of the \( \omega \) argument indicates it is sampled over \( M \) paths.

IV. Dynamic Hedging of Options

In this section, we formulate the problem of dynamically hedging a European call option on a basket of stocks. As stated previously, a European call option is a security that gives the holder the option (but not the obligation) to purchase a specified basket (or group) of stocks at a specified date \( N \) (known as the expiration date) and at a specified price \( K \) (known as the strike price).

For example, consider the basket consisting of a single share of three stocks whose prices at time \( k \) we denote by \( S_1(k, \omega), S_2(k, \omega), \) and \( S_3(k, \omega) \). Thus, the total value of this basket of stocks at time \( k \) is simply the sum: \( S_1(k, \omega) + S_2(k, \omega) + S_3(k, \omega) \). More generally, we will let the basket be made up of \( \alpha_i \) shares of stock \( S_i \), for \( i = 1 \ldots n \). Thus, the total value of the basket at time \( k \) will be \( \sum_{i=1}^n \alpha_i S_i(k, \omega) \).

Consider a European call option on this basket with strike price \( K \) and time to expiration \( N \). At expiration \( N \), the value of this basket option is \( \sum_{i=1}^n \alpha_i S_i(N, \omega) - K \) if \( \sum_{i=1}^n \alpha_i S_i(N, \omega) \geq K \), or \( 0 \) if \( \sum_{i=1}^n \alpha_i S_i(N, \omega) < K \). More generally, we denote this payoff value at time \( N \) as \( C(N, \omega) \). Thus, in the specific case of a European basket option we have

\[
C(N, \omega) = \max \left\{ \sum_{i=1}^n \alpha_i S_i(N, \omega) - K, \ 0 \right\}. \tag{10}
\]

The dynamic hedging problem is to trade a self financing portfolio such that its wealth at expiration \( W(N+1, \omega) \) replicates (or perhaps super-replicates) the payoff of the option \( C(N, \omega) \).

A. Hedging Objective Functions

In this paper we focus on objective functions for hedging that are quadratic in \( W(k, \omega) \). This will allow us to use LQR based techniques. Examples of such objective functions are given below.

When the goal is to replicate the payoff of the option, one may consider minimizing the mean squared error between final wealth and the payoff of the option

\[
\mathbb{E} \left[ (W(N+1, \omega) - C(N, \omega))^2 \right]. \tag{11}
\]

An alternative objective could be to minimize the variance of the difference between final wealth and the payoff of the option

\[
\mathbb{E} \left[ (W(N+1, \omega) - C(N, \omega))^2 \right]. \tag{12}
\]

or if super-replication is desired the objective may trade off the mean and the variance by maximizing

\[
\mathbb{E} \left[ (W(N+1, \omega) - C(N, \omega))^2 + \gamma \mathbb{VAR} \left[ (W(N+1, \omega) - C(N, \omega))^2 \right] \right] \tag{13}
\]

where \( \gamma \) is a risk aversion coefficient.

The objectives in (11), (12), and (13) can be approximated via sample averages by using

\[
\mathbb{E} \left[ (W(N+1, \omega) - C(N, \omega))^2 \right] \approx \frac{1}{M} \mathbf{1}^T (W(N+1) - C(N)), \tag{14}
\]

Note that the approach in this paper does not depend on the specific payoff structure of a basket option. It is only used for concreteness.
\[ E \left[ (W(N + 1, \omega) - C(N, \omega))^2 \right] \approx \frac{1}{M} W(N + 1) - C(N) \right)^T I_M (W(N + 1) - C(N)) \]

and

\[ \text{VAR}\left[(W(N + 1, \omega) - C(N, \omega))\right] \approx \frac{1}{M} (W(N + 1) - C(N))^T \left[ I_M - \frac{1}{M} 1 \cdot 1^T \right] (W(N + 1) - C(N)) \]

where again \( 1 \) denotes a column vector of appropriate length with all elements equal to 1, and \( I_M \) is the \( m \times m \) identity matrix. \( C(N) \in \mathbb{R}^m \) represents the sampled version of the payoff function (i.e. the payoff of the option for each path \( \omega \)). Thus, the objectives can be written in the form of a quadratic function of the sampled wealth \( W(N + 1) \).

V. AN LQR FORMULATION

Our first approach to the dynamic hedging problem is to use an LQR formulation that quadratically penalizes transactions in the objective rather than including transaction costs in the dynamics. In this case, \( W(N + 1) = W(N) \) so we will write the objective in terms of \( W(N) \). We consider objective functions of the form:

\[
J(W(k), u(k)) = \varphi^T [W(N) - C(N)] + \| (W(N) - C(N)) \|_\gamma^2 + \sum_{k=0}^{N} \| \Delta h(k) \|_{\Gamma(k)}^2
\]

where \( h(k) \) is the sampled version of the holdings, and \( \gamma \geq 0 \) and \( \Gamma(k) \geq 0 \). The final term in the objective is used to penalize transactions. Note that the objectives of the previous section all fit this structure. The corresponding control problem is given by

\[
\begin{align*}
\text{subject to} & \\
W(k + 1) &= F(k)W(k) + G_a(k)u(k) \\
W(0) &= W_0I, \quad h(-1) = h_{-1}, \quad h(N) = h_N \\
h(k) &= \Psi(k)u(k), \quad k = 0 \ldots N - 1
\end{align*}
\]

Remark 3 This control problem is also a stochastic program and thus statements concerning convergence in the number of samples for both the objective value and the minimizing control actions can be pursued. The interested reader is referred to standard stochastic programming references such as Birge and Louveaux [4].

This optimal control problem can be easily written in the form of a standard linear quadratic regulator (LQR) problem. Thus, it can be readily solved via dynamic programming and a Riccati difference equation. Details may be found in any standard optimal control text (e.g. [1]).

In the receding horizon scheme that follows, we will utilize the solution of the Riccati equation corresponding to problem LQR. That is, the optimal "cost-to-go" or "value function" at time \( k \) can be written compactly as

\[
V(W(k), u(k-1)) = \left[ \begin{array}{c} W(k) \\ u(k-1) \end{array} \right]^T P(k) \left[ \begin{array}{c} W(k) \\ u(k-1) \end{array} \right]
\]

with \( P(k) \) coming from the solution to the Riccati equation for problem LQR.

Remark 4 Note that the state in problem LQR can be quite large depending on the number of sample paths used, \( M \). However, the computational complexity of solving problem LQR grows only linearly in the horizon length \( N \), and each step of the Riccati difference equation involves only matrix multiplication (which can be quite large), and the inversion of a \( p \times p \) matrix where \( p \) is the number of basis function coefficients. Thus, despite the large size, it can be solved somewhat efficiently as demonstrated in the numerical example.

Remark 5 This formulation can also be used to price an option by minimizing the value function \( V(W(0), u(-1)) \) over the initial wealth \( W(0) \) when the objective is the mean squared error. One can argue that the minimizing initial wealth is a reasonable price for the option. While this approach can be pursued quite successfully, due to space considerations we do not elaborate on the pricing problem in this paper.

VI. A RECEDING HORIZON CONTROL APPROACH

In this section, we formulate on-line quadratic programs to be used in a receding horizon approach to the hedging problem. These quadratic programs correspond to a control problem that starts at the current time \( k \) and extends \( L \) time steps into the future, and we refer to \( L \) as the prediction horizon.

In what follows, the current time is always denoted by \( k \) and \( N \) represents the time remaining to expiration from \( k \). In the on-line quadratic programs, we use predicted variables that extend from the current time \( k \) to the end of the prediction horizon \( k + L \). To indicate variables over the prediction horizon, we use the argument \( j \). Thus, \( x(j) \) should be interpreted as \( x(k + j) \) in standard receding horizon notation.

A. Receding Horizon on-line Quadratic Programs

To create a receding horizon control method for the hedging problem, we will solve the following quadratic
programming problem with prediction horizon \(L\).

\[
\begin{align*}
\text{\textbullet Problem } QP(L, u_{-1}, W_0) : \\
\min_{u(j), \tau(j)} & \left[ \begin{array}{c}
w(l) \\
u(l) \end{array} \right]_1^T P(\hat{L}) \left[ \begin{array}{c}
\hat{w}(l) \\
u(l) \end{array} \right]_1 \\
\text{subject to} & \\
\hat{L} = \min(L, N+1) \\
u(-1) = u_{-1}, \quad u(N) = u_N, \quad W(0) = W_0 I \\
W(j+1) = F(j)u(j) + G\sigma(j)\tau(j) + F_T(j)\tau(j), \\
\psi_\Delta(j)\tau(j) \geq |\psi_h(j)u(j) - \psi_h(j-1)u(j-1)|, \\
j = 0, \ldots, \hat{L} - 1
\end{align*}
\]

Note that when \(L\) extends past the expiration time \(N\), we truncate the prediction horizon to \(\hat{L} = N + 1\). In this case, we define \(P(N + 1)\) so that it corresponds to the cost

\[
\begin{align*}
\left[ \begin{array}{c}
W(N+1) \\
u(N) \end{array} \right]_1^T P(N+1) \left[ \begin{array}{c}
W(N+1) \\
u(N) \end{array} \right]_1 = \\
\phi^T (W(N+1) - C(N)) + \|W(N+1) - C(N)\|^2_T.
\end{align*}
\]

Also note that the constraint \(u(N) = u_N\) which fixes the holdings at expiration has no effect on the optimization unless \(L > N\).

**Remark 6** We also allow the possibility of choosing the prediction horizon to be \(L = 0\). This is meant to correspond to a receding horizon strategy based solely upon the repeated solution of the LQR problem.

**Remark 7** Note that given the flexibility of receding horizon control, one can easily use an alternative objective function in problem \(QP\) (such as maximizing the minimum of \(\|W(N+1) - C(N)\|\), etc.).

Our receding horizon hedging algorithm is built upon these on-line optimizations as outlined below.

**B. The Receding Horizon Algorithm with \(QP\)**

To implement the receding horizon \(QP\) approach, we apply the following algorithm. Let \(N\) denote the steps to expiration from the current time and let \(L\) denote the prediction horizon.

1) Set \(k = 0\). Let \(W_0\) denote the initial wealth, let \(u_{-1}(-1) = u_{-1}\) denote the current holdings in the \(m\) tradable stocks, and let \(u_N\) correspond to the constraint on the holdings at expiration.

2) Generate \(M\) predicted paths to expiration of the bond \(B(k+j, \omega)\) and underlying assets \(S_j(k+j, \omega)\) for \(j = 0, \ldots, N+1\).

3) Set \(\hat{L} = \min(L, N+1)\) and compute \(P(\hat{L})\) via the Riccati equation corresponding to Problem LQR.

4) Solve the optimization problem \(QP(L, u_{rh}(k-1), W_k)\) and denote the initial optimizing control action by \(u_{rh}(k) = u^*(k)\).

5) Implement control action \(u_{rh}(k)\) and observe the actual new stock and bond prices \(S_j(k, \omega), B(k, \omega)\) and the resulting new wealth \(W(k+1)\).

6) Set \(k = k + 1, N = N + 1\).

7) If \(N = 0\), Then set \(u(N) = u_N\), compute \(W(N+1)\) after the final transaction cost, and exit, Else go to 2.

The following section tests this receding horizon approach via a numerical example.

**VII. Numerical Example**

This section provides a numerical example of the receding horizon algorithm on a 5-dimensional basket option under transaction costs. The 5 stocks are modeled as correlated geometric Brownian motions with the following drift and covariance parameters

\[
\mu = \begin{bmatrix}
0.0916 \\
0.1182 \\
0.1462 \\
0.0924 \\
0.1486
\end{bmatrix},
\]

\[
\Sigma = \begin{bmatrix}
0.0940 & 0.0137 & 0.0145 & 0.0124 & 0.0184 \\
0.0137 & 0.0506 & 0.0148 & 0.0273 & 0.0220 \\
0.0145 & 0.0148 & 0.0902 & 0.0103 & 0.0129 \\
0.0124 & 0.0273 & 0.0103 & 0.0992 & 0.0267 \\
0.0184 & 0.0220 & 0.0129 & 0.0267 & 0.0732
\end{bmatrix}.
\]

100 simulations were run for an at-the-money basket option with \(S(0) = 10\) and \(\omega = 1/5\) for \(i = 1, \ldots, 5\), and strike price \(K = 10\). Time to expiration was \(T = 0.2\) and \(N = 20\) equally spaced trading opportunities were allowed. The transaction cost for all stocks was taken to be 1% (i.e. \(\xi = 0.01\)) while the bond was assumed to grow as a risk free rate of \(r = 0.05\).

The initial wealth for the dynamic hedge was \(W(0) = 0.4787\) and was calculated using a heuristic Leland [11] adjusted method of moments approach [10]. The initial and final holding of the stock was assumed to be \(u_{-1} = 10\) and \(u_N = 0\). That is, there is no initial holding in the stocks, and all holdings in the stocks are liquidated at expiration.

Due to space considerations, we only show a few representative simulation results. In the receding horizon approach we used linear basis functions in the stock prices (as in Example 1) to represent both \(h(k, \omega)\) and \(|\Delta h(k, \omega)|\), a prediction horizon of \(L = 1\), and \(M = 500\) sample paths. Figure 1 shows final wealth plots (left) and histograms of the hedging error \((W(N+1, \omega) - C(N, \omega))\) under the mean-variance objective of the form (13) for \(\gamma_R = 5, 10, 100\). The histograms show that the receding horizon method effectively reflects the objective function by trading off mean and variance in the hedging strategy.

Table I gives sample means and variances along with the average computation time with 20 steps to expiration for the three cases considered the Figure 1. The simulations were run on a 1.66GHz Intel Core Duo laptop with 2GB of RAM. The quadratic programs in the receding horizon scheme were solved using the SPC quadratic programming in C [20]. The results show that the computation times are on the order of a couple of seconds in this case.

In general, using a prediction horizon of \(L = 1\) resulted in a noticeable improvement over the pure LQR based scheme with \(L = 0\). This is likely due to the fact that the LQR approach only approximates the transaction cost quadratically,
whereas the receding horizon optimization captures it more accurately. Extending the prediction horizon beyond \( L = 1 \) resulted in much smaller improvements. Additionally, use of other basis functions did not significantly alter performance. Finally, for this specific example using more than \( M = 500 \) Monte-Carlo samples in the RHC scheme did not change the results appreciably.

**VIII. Conclusions**

In this paper we formulated the dynamic hedging problem under proportional transaction costs as a linear quadratic control problem with constraints, and presented a solution using LQR and receding horizon methods. A numerical example demonstrated the calculations for a European basket call option on 5 stocks under proportional transaction costs. The results indicate that the linear control systems approach provides a flexible and relatively scalable methodology. Future research will explore the use of variance reduction methods in the Monte-Carlo step, and more sophisticated basis functions.

**IX. Acknowledgments**

The author would like to thank Li Xu, Jia Liu, Joo Hyung Lee, Minyong Shin, Wilfred Wong, and Alberto Bemporad for helpful discussions.

**References**


