Abstract—This paper presents a locally stable nonlinear sliding mode control law for a rotary inverted pendulum with starting configurations above the horizontal line. An approximate nonlinear solution of the system's closed-loop loop dynamics during the sliding phase of the response is presented. The controller is tested experimentally and the approximate analytical solution is verified through comparison with the numerical solution. The nonlinear solution provides guidelines to select parameters that result in better controller performance.

I. INTRODUCTION

Sliding mode control (SMC) [1] has been shown to be a robust and effective control approach for nonlinear systems. The approach is based on defining exponentially stable (sliding) surfaces as a function of tracking errors and using Lyapunov theory to ensure all system trajectories reach these surfaces in finite time and since the surfaces are asymptotically stable the system trajectory slides along them till it reaches the origin. Sliding mode controllers have been successfully developed for some underactuated problems. Bergerman and Xu [2] introduce variable structure controllers for robots by physically locking joints which are not being controlled using an iterative process. Lee, et al. [3] developed set point sliding control laws for planar 2-link and 3-link robots under the influence of gravity. Su and Stepanenko [4] discuss SMC for serial underactuated robots under the influence of gravity. Ramos et al. [5] applied a combination of nonlinear regulation and sliding modes approach to a rotary inverted pendulum. Recently, Riachy et al. [6] introduced a second order sliding mode control approach with application to the inverted pendulum on a cart problem.

The more general sliding mode control approaches to underactuated systems include works by Olfati-Saber [7], Xu and Ozguner [8], Sankaranarayanan and Mahindrakar [9], and Ashrafiuon and Irwin [10]. The authors in [10] introduced a SMC approach that can be adapted to several classes of nonlinear systems. In this approach, first-order sliding surfaces are defined as a linear combination of actuated and unactuated states and the parameters are selected to satisfy the established stability criteria which are derived based on linearization. While, the control law is shown to guarantee the convergence of the reaching phase of the sliding mode in finite time, it does not provide a clear proof of asymptotic stability of the sliding phase. The inverted pendulum on a cart is used as a simulation example to verify the approach and its significance. The authors do not, however, explain how to choose the control parameters to not only guarantee stability of the nonlinear system but also provide good performance.

In this work, we determine a control law for a rotary inverted pendulum based on the method presented in [10]. Asymptotic stability of the sliding phase is established through introduction of a Lyapunov candidate function where the parameters of the function must be selected based on the system physical parameters as well as the controller’s surface parameters. A numerical simulation example is presented and experimentally implemented to verify the controller. An approximate nonlinear solution is then presented based on the non-dimensional system equations and parameters which provides an insight into the selection of control parameters for stable and effective closed-loop system response.

II. THE ROTARY INVERTED PENDULUM

The rotary inverted pendulum system shown in Fig. 1 has two degrees of freedom, the pendulum arm angle, \( \phi \) and the rotating arm angle \( \theta \) which is actuated by a DC motor with a gearbox.

Fig. 1. The rotary inverted pendulum.

Denoting the dimensional time as \( \bar{t} \), the equations of motion of the system are written as:

\[
-b \cos \phi D^2_t \theta + c D^2_t \theta = d \sin \phi \\
- b D^2_t \theta - b \cos \phi D^2_t \phi = b_v - b \sin (D_t \phi)^2 - c_e D_t \theta
\]

(1)

where \( a = J_e + m r^2, \ b = m l r, \ c = \frac{1}{2} m l^2, \ d = m g l, \ r \) is the rotating arm length, \( l \) half of pendulum arm length, \( m \) the pendulum arm mass, \( g \) acceleration due to gravity, \( J_e \) the equivalent inertia of the rotating arm, motor and
gearbox, \( c_p \) the equivalent system damping coefficient, and \( b_c \) is a function of motor torque constant and other parameters converting input voltage \( v \) to actuation torque.

III. SLIDING MODE CONTROL

The goal of the control system is to move the pendulum arm from its initial angle of \( \phi(0) \) to the upright position while stabilizing the rotating arm at any desired angle. The sliding mode control law may be derived by defining a sliding surface as a function of the tracking errors as [10]:

\[
s = \alpha_{\phi} D_{\phi} \phi + \lambda_{\phi} \phi + \alpha_{\theta} D_{\theta} \theta + \lambda_{\theta} \theta
\]

(2)

where \( \alpha_{\phi}, \lambda_{\phi}, \alpha_{\theta}, \) and \( \lambda_{\theta} \) are the surface parameters. Defining the Lyapunov candidate function as \( \frac{1}{2}s^2 \), it can be shown that all system trajectories reach the surfaces in Eq. (2) in finite time if the following reaching condition is satisfied [1]:

\[
s D_{\phi} s \leq -\eta |s|, \quad \eta > 0
\]

(3)

In the above equation, \( \eta \) determines how fast the trajectory reaches the surface. Substituting from Eqs. (1) and (2) into Eq. (3), the SMC can be derived as:

\[
v = \dot{\phi} - k \text{sgn}(s), \quad k > 0
\]

(4)

where \( k \) is a function of \( \eta \) and uncertainty in the system model and \( \dot{\phi} \) is the nominal control determined as:

\[
\dot{\phi} = b \sin \phi (D_{\phi} \phi)^2 + c_1 D_{\phi} \theta - \frac{(\alpha_{\phi} + \alpha_{\beta} b \cos \phi) \sin \phi + \alpha_{\beta} \theta \cos \phi + \lambda_{\phi} D_{\phi} \phi + \lambda_{\theta} \theta}{b_c (\alpha_{\phi} + \alpha_{\beta} b \cos \phi)}
\]

(5)

Since we are only interested in the upward stabilization of the pendulum and to avoid singularity, it is assumed that the above control law is only valid for starting position above the horizontal such that:

\[
\alpha_{\phi} c + \alpha_{\beta} b \cos \phi > 0
\]

(6)

Note that, the closer the arm is to the horizontal plane, the larger the magnitude of \( \alpha_{\phi} \) must be relative to \( \alpha_{\theta} \).

A. Stability of SMC

While the SMC in Eqs. (4) and (5) guarantees all trajectories reach the surface presented by Eq. (2), the stability of the sliding phase depends on the selected surface parameters. Substituting from Eq. (5) into the equations of motion and letting \( s = 0 \), the system closed-loop dynamics during the sliding phase can be represented by following equations:

\[
\alpha_{\phi} D_{\phi} \phi + \lambda_{\phi} \phi + \alpha_{\theta} D_{\theta} \theta + \lambda_{\theta} \theta = 0
\]

\[
D_{\phi}^2 \phi = \frac{\alpha_{\beta} c \sin \phi - b \cos \phi (\lambda_{\phi} D_{\phi} \phi + \lambda_{\theta} \theta)}{\alpha_{\phi} c (\alpha_{\phi} + \alpha_{\beta} b \cos \phi)}
\]

(7)

Linearization can be used as a guideline for the selection of surface parameters since it provides the necessary conditions (though not sufficient) for stability. Linearizing Eq. (7) about the origin and these equations in order to select appropriate surface parameters for a stable response.

\[
\alpha_{\theta}, \lambda_{\theta} < 0, \quad \alpha_{\phi}, \lambda_{\phi} > 0, \quad \frac{\lambda_{\theta}}{\alpha_{\theta}} < \frac{\lambda_{\phi}}{\alpha_{\phi}}
\]

(8)

Equations (6) and (8) can be used for preliminary selection of the surface parameters. To prove closed-loop stability of the sliding phase, we propose the following Lyapunov candidate function:

\[
V = \frac{1}{2} (a_1 \phi^2 + a_3 (D_{\phi} \phi)^2 + a_3 \theta^2) + a_4 \phi D_{\phi} \phi + a_5 \phi \theta - \frac{a_2 a b}{\alpha_{\phi} \phi} \min\left(\frac{\alpha_{\phi} c + \alpha_{\beta} b \cos \phi}{\alpha_{\phi} c} \right) > 0
\]

(9)

where the parameters \( a_1 \geq 0, a_2 \geq 0, a_3 \geq 0, a_4 > 0, \) and \( a_5 \) must be derived based on the physical parameters of the system as well as the surface parameters such that \( D_{\phi} V < 0 \).

B. Example and Verification

We have implemented the sliding mode control law presented in the previous section for the Quanser® rotary inverted pendulum. The following data are available from the company: \( l = .1675 \) m, \( r = .158 \) m, \( m = .125 \) kg, \( J_c = .0036 \) kg.m², \( c_c = .073 \) N.m.s/rad, and \( b_c = .1285 \) N.m/V. We selected \( k = \eta = 20 \) and approximated the sign function with a saturation function of boundary layer thickness of 1 to avoid large discontinuous control inputs and chattering normally associated with SMC [11].

The surface parameters may be selected based on the guidelines provided in Eqs. (6) and (8) as \( \alpha_{\phi} = 2, \lambda_{\phi} = 12, \alpha_{\theta} = -1, \) and \( \lambda_{\theta} = -1.6 \). Hence, it can be easily shown that \( V > 0 \) and \( D_{\phi} V < 0 \) for \( a_1 = 10, a_2 = 2, a_3 = .5, a_4 = 6, \) and \( a_5 = -2 \). Note that based on the condition in Eq. (6), stability is only guaranteed if \( |\phi| < 45.03^\circ \). Sliding mode controller performance is presented in Fig. 2 where the experimental and simulation results show good agreement in stabilizing the pendulum arm starting from an initial angle of \( \phi_0 = 45^\circ \).

![Fig. 2. Comparison of simulation and experimental results; (top) pendulum arm angle, (middle) rotating arm angle, (bottom) control input voltage.](image-url)
to iteratively change them until the stability is guaranteed through the use of the Lyapunov candidate function presented in Eq. (9) and a good performance is achieved. In this section, we introduce an approximate nonlinear solution of the system closed-loop response during the sliding phase. The analytical solution provides the insight required for selecting surface parameters that guarantee stability and good performance.

A. Dimensionless Equations

The closed-loop dynamics during the sliding phase represented by Eq. (7) can be formulated as a single third order system in terms of $\phi$:

\[
(\alpha_\theta c + \alpha_\phi b \cos \phi) D^3 \phi - \alpha_\phi d \sec \phi D_\zeta \phi - \lambda_\phi d \sin \phi + (\lambda_\theta c + \lambda_\phi b \cos \phi + \alpha_\theta \tan \phi D_\zeta \phi) D^2 \phi = 0
\]

with initial conditions

\[
\phi(0) = \phi_0, \quad D_\zeta \phi(0) = 0, \quad D^2 \phi(0) = 0
\]

Note that, $\alpha_\phi$ and $\alpha_\theta$ have dimensions of frequency while $\lambda_\phi$ and $\lambda_\theta$ are dimensionless. The value of $\phi$ is restricted to the range $|\phi| < \pi/2$ and its initial value, $\phi_0$, has an arbitrary value in this range.

We render these equations dimensionless by using $\omega_n = \sqrt{d/c} = \sqrt{3g/4l}$ to define a dimensionless time $t = \omega_n \bar{t}$ and $u = \phi/\phi_0$ to define a dimensionless angle. Then dividing Eq. (10) by $-\alpha_\phi \omega_n^3$, we get

\[
[\Delta \cos(\phi_0 u) - 1] \ddot{u} + \sec(\phi_0 u) \dot{u} + \frac{\epsilon \sin(\phi_0 u)}{\phi_0} + [\epsilon \delta \cos(\phi_0 u) - \epsilon - \dot{u} \phi_0 \tan(\phi_0 u)] \dot{u} = 0
\]

where the overdot denotes differentiation with respect to $t$ and the parameters $\Delta$, $\delta$, and $\epsilon$ are dimensionless control variables defined by

\[
\Delta = -\frac{\alpha_\phi b}{\alpha_\phi c}, \quad \epsilon = \frac{\lambda_\phi}{\alpha_\phi \omega_n}, \quad \delta = -\frac{\lambda_\theta}{\lambda_\phi c}
\]

The dimensionless control variables defined above also depend on the system parameters, $b, c$, and $\omega_n$.

Note that when we linearize Eq. (12) we obtain

\[
(\Delta - 1) \ddot{u} + \epsilon (\delta - 1) \dot{u} + \dot{u} + \epsilon u = 0
\]

and the necessary and sufficient conditions for stability of this equation are

\[
\epsilon > 0, \quad \delta > \Delta > 1
\]

which are the same as the ones in Eq. (8).

The dimensionless initial conditions are, using the definitions in Eqs. (11)

\[
u(0) = 1, \quad \dot{u}(0) = 0, \quad \ddot{u}(0) = 0
\]

We seek an approximate solution to Eq. (12) with Eqs. (15) as initial conditions when $\phi_0 = O(1)$. Two forms of the solution are considered here, one for $\epsilon << 1$ and the second for $\epsilon >> 1$. Note that, these two solutions should give us an idea about the system response for $\epsilon = O(1)$.

B. Solution for $\epsilon << 1$

In this case we write Eq. (12) as

\[
[\Delta \cos(\phi_0 u) - 1] \ddot{u} + \sec(\phi_0 u) \dot{u} = -\epsilon [\delta \cos(\phi_0 u) - 1] \dot{u} + \frac{\sin(\phi_0 u)}{\phi_0}
\]

where the right hand side is a small correction. Note then that the unique solution to the dominant order (left hand side set equal to zero) of Eq. (16) that satisfies the initial conditions is $u = 1$. Clearly, this is not physically acceptable; however, if we continue by iterating Eq. (16) we find the inhomogeneous differential equation

\[
[\Delta \cos(\phi_0 u) - 1] \ddot{u} + \sec(\phi_0 u) \dot{u} = -\epsilon \frac{\sin(\phi_0)}{\phi_0}
\]

which has the solution

\[
\bar{u} \sim 1 + \epsilon \frac{\sin(2\phi_0)}{2\phi_0} \left[ \frac{\sin(\omega t)}{\omega} - t \right]
\]

While this solution decays from $u(0) = 1$ it clearly becomes invalid at long times, $t = O(\epsilon^{-1})$.

This behavior suggests the multiple scale method [12] in which we seek a solution with the form of a power series

\[
u = \sum_{k=0}^{K} \epsilon^k u_k(T_n)
\]

but where the $u_k$ are functions of various time scales $T_n = \epsilon^n t$. Thus, for example, the derivative

\[
\dot{u}_k = \sum_{n=0}^{N} \epsilon^n \partial_n u_k
\]

where $\partial_n$ denotes the partial derivative with respect to the time scale $T_n$. In this multiple scale analysis, we will need as many scales as orders that we solve for, $N = K$. The flexibility provided by the scales allows us to cancel resonant forcing terms so that the asymptotic expansion in Eq. (17) has no secular terms (i.e. terms containing $\epsilon^N t$), and is thus valid to longer times as we increase the order.

1) Dominant Order Solution: Substituting Eqs. (17) and (18) as well as expressions for the higher derivatives into Eq. (16), we obtain for the dominant order, $\epsilon^0$, equation

\[
[\Delta \cos(\phi_0 u_0) - 1] \partial^3_0 u_0 - \phi_0 \tan(\phi_0 u_0) \partial_0 u_0 \partial^2_0 u_0 + \sec(\phi_0 u_0) \partial_0 u_0 = 0
\]

The dominant order initial conditions are, from Eqs. (15),

\[
u_0(0) = 1, \quad \partial_0 u_0(0) = \partial^2_0 u_0(0) = 0
\]

The solution of Eq. (19) that satisfies the initial conditions is simply $u_0 = C_{100}(T_1)$ where $T_1$ is the shortest scale of variation of the function, and $C_{100}(0) = 1$. Thus, to dominant order, the solution is a constant, $u = 1$ as we noted before.

This is clearly inaccurate for $t >> 1$ so we need to consider higher order terms. However, first note that because $\partial^m_0 u_0 = 0$ for all $m > 0$ the linear operator that acts on all higher order terms is simply

\[
\hat{L} = [\Delta \cos(\phi_0 C_{00}) - 1] \partial^3_0 + \sec(\phi_0 C_{00}) \partial_0
\]
and that since $C_{00}$ is independent of $T_0 = t$ this is a constant coefficient differential operator.

The first order, $\varepsilon^1$, equation is then, ignoring terms that are identically zero,

$$\hat{L}u_1 = -\frac{\sin(\phi_0 C_{00})}{\phi_0} - \frac{\partial_t C_0}{\cos(\phi_0 C_{00})}$$  \(20\)

If the right hand side of Eq. (20) is not zero $u_1$ will grow linearly in time (see the solution for $u$ after Eq. (16)) so the solution for $u$ will fail to be asymptotic at $t = O(\varepsilon^{-1})$. This is the reason we introduced the additional time scales. Setting the right hand side of Eq. (20) equal to zero, and integrating yields

$$u_0 = C_{00} = \frac{\tan^{-1}[\tan(\phi_0 C_{01})e^{-T_1}]}{\phi_0} = \frac{\xi}{\phi_0}$$ \(21\)

Here $C_{01}$ is a function of scales $T_2$ and longer, and the initial condition requires $C_{01}(0) = 1$. If we stop at this point $C_{01} = 1$ and the solution is simply

$$u \sim \frac{\tan^{-1}[\tan(\phi_0 e^{-\varepsilon t})]}{\phi_0} = \frac{\xi_0}{\phi_0}$$ \(22\)

Note that this is asymptotically stable for $\varepsilon > 0$, which is one of the linear stability conditions. Moreover, this a good approximation when $\varepsilon$ is small enough and $\phi_0$ is close enough to $\phi_{max} = \cos^{-1}(1/\Delta)$. Figure 3 shows a plot of Eq. (22) along with a numerical solution of Eq. (12) with the values $\Delta = 1.415$, $\delta = 5.306$, and $\phi_0 = 45^\circ$. The agreement is good but there is still a sizable error since $\varepsilon = 0.2415$, is not very small. The agreement is excellent for very small $\varepsilon = .1$. However, reducing $\varepsilon$ results in a slower response.

![Fig. 3. Comparison with exact solution for small $\varepsilon$ starting from $\phi_0 = 45^\circ$](image)

When $\Delta \cos(\xi) - 1 > 0$ for all $0 \leq \xi \leq \phi_0$, the solution of Eq. (20) is

$$u_1 = C_{10} + A_{10} \cos(\omega t + \Phi_{10})$$ \(23\)

where $C_{10}$, $A_{10}$ and $\Phi_{10}$ are functions of scales $T_1$ and longer. Like $C_{00}$ they are determined by preventing higher order, $O(\varepsilon^2)$, secular terms, and

$$\omega = \left\{[\Delta \cos(\xi) - 1] \cos(\xi)\right\}^{-1/2}$$ \(24\)

Note that when $\Delta \cos(\xi) - 1 > 0$ is not satisfied the solution of Eq. (20) contains a growing exponential and is not asymptotically stable. Hence, $\Delta \cos(\xi) - 1 > 0$ is a necessary condition for stability, and since the left side is a monotone decreasing function of $\xi$, its minimum value produces the condition

$$\Delta > \sec(\phi_0) > 1$$ \(25\)

This subsumes the linear stability condition, $\Delta > 1$. Note that for the response shown in Fig. 3, the limiting condition of Eq. (25) $\sec(\phi_0)_{max} = \Delta$ produces $(\phi_0)_{max} = 45.03^\circ$. This means that the initial angle is very close to its maximum possible value for a stable response.

2) $\varepsilon$ Order Solution: The next order, $\varepsilon^2$, equation is

$$\hat{L}u_2 = -\cos \xi_1 u_1 - [\delta \cos \xi - 1] \partial_0^2 u_1 - 3[\Delta \cos(\xi) - 1] \partial_0^2 u_1 + \phi_0 A \sin \xi_1 \partial_0^3 u_1 + \phi_0 \tan \xi \partial_0^3 u_1 \left(\partial_1 u_1 + \partial_1 u_0\right)$$ \(26\)

$$- \sec \xi \left[\partial_1 u_1 + \partial_2 u_0 + \phi_0 \tan \xi \left(\partial_1 u_1 + \partial_1 u_0\right)\right]$$

Substituting Eqs. (21) and (23) into the right hand side of Eq. (26), collecting terms and setting the coefficients of $\sin(\omega t + \Phi)$ and $(\omega t + \Phi)$ and the constant terms (independent of $T_0$) equal to zero to eliminate the secular terms:

$$\{3(\Delta \cos(\xi) - 1)\omega^2 - \sec \xi \partial_1 A_{10} + \{(\delta \cos \xi - 1)\omega^2 \cos \xi - \phi_0 \tan \xi (\omega^2 + \sec \xi) \partial_1 C_{00}\} A_{10} = 0$$ \(27\)

$$\{-3[\Delta \cos(\xi) - 1] \omega^2 + \sec \xi\} A_{10} \partial_1 \Phi_{10} + \phi_0 [\Delta \omega^2 \sec(\xi) \tan(\xi) + \sin(\xi)] A_{10} C_{10} \omega = 0$$ \(28\)

$$- \sec \xi \left(\partial_1 C_{10} + \partial_2 C_{00} + \phi_0 \tan \xi \partial_1 C_{10} - \cos(\xi) C_{10} = 0\right)$$ \(29\)

We can solve Eq. (29) by setting $C_{10} = 0$ and $\partial_2 C_{00} = 0$. The second of these means that $\xi = \xi_0$. Then Eq. (28) becomes $\partial_1 \Phi_{10} = 0$ or $\Phi_{10} = \Phi_{10}(0)$, and finally, on substituting Eqs. (22) and (24) and doing some algebra, Eq. (27) becomes

$$\partial_1 A_{10} + \sigma(T_1) A_{10} = 0$$ \(30\)

where

$$\sigma = \frac{(\delta - \Delta) \cos(\xi_0) + [2\Delta \cos(\xi_0) - 1] \sin^2(\xi_0)}{2(\Delta \cos(\xi_0) - 1) \cos(\xi_0)}$$ \(31\)

and $\xi_0 \equiv \xi_0(T_1)$ is defined in Eq. (22). The solution of Eq. (28) is

$$A_{10} = A_{10}(0) e^{-\int_0^{\xi_1} \sigma_0(\xi_1) d\xi_1}$$ \(32\)

This is asymptotically stable as long as the numerator of Eq. (31) is positive for all $\xi_0$. The minimum value occurs for $\xi_0 = 0$ so $\delta - \Delta > 0$, which is the linear stability condition. Now the initial conditions, to this order, are

$$u_1(0) = \partial_0^2 u_1(0) = 0 \quad \partial_0 u_1(0) + \partial_1 u_0(0) = 0$$

and on satisfying them we find $\Phi_{10}(0) = \pi/2$ and $A_{10}(0) = -\sin(2\phi_0)/(2\phi_0 \omega)$. The solution, valid to $O(\varepsilon)$ is then

$$u \sim u_0 + \varepsilon u_1 = \frac{\tan^{-1}(\tan(\phi_0 e^{-\varepsilon t}))}{\phi_0} + \varepsilon \frac{\sin(2\phi_0)}{2\phi_0 \omega} e^{-\int_0^{\xi_1} \sigma_0(\xi_1) d\xi_1} \sin(\omega t)$$ \(33\)
In this equation the function \( s(\epsilon t) \) is the integral that appears in Eq. (32). Note that the decay to equilibrium occurs on the scale \( \epsilon t \) and the sinusoidal variation is small, \( O(\epsilon) \), and occurs at frequency \( \omega \).

The accuracy of the approximate solution can be verified more effectively for larger values of \( \Delta \) since smaller values result in a narrow range for the initial angle \( \phi_0 \). Increasing the value \( \Delta \) will only increase the range of stable solution for \( \phi_0 \) but also affects how good of an approximation we get based on the initial angle. Figure 4 shows that when \( \Delta = 4 \), the approximate solution is in excellent agreement with the numerical solution for small and large initial angle angles \( \phi_0 = 15^\circ \& 75^\circ \) but not the medium on \( \phi_0 = 45^\circ \). Note that \( \delta = 5.306 \) and \( \epsilon = 0.1 \). Hence, better control performance can be achieved if large \( \Delta \) values close to the stability limit of Eq. 25 (sec \( \phi_0 \)) are used.

\[ \text{Fig. 4. Large } \Delta, \text{ small } \epsilon \text{ solution starting from different initial angles} \]

If we use \( \Delta = 4 \) and \( \epsilon = 0.1 \) and start with \( \phi_0 = 45^\circ \) which is the worst of the three cases presented in Fig. 4 and increase \( \delta \) from 5.306 to 10 and 20. The oscillatory response vanishes and the approximate solution shows good agreement with the numerical solution, as shown in Fig. 5. Hence, if smaller \( \Delta \) values as recommended above are used, we need to use significantly larger values for \( \delta \).

\[ \text{C. Solution when } \epsilon > 1 \]

In this case, we rewrite Eq. (12) by defining a frequency \( \omega = (\delta - 1)^{-1/2} \) and assume \( \phi_0 << 1 \) to obtain an analytic approximate solution. Hence, writing only the dominant order terms on the left hand side, and only the leading order in \( \phi_0^2 \) and \( \epsilon^{-1} \), on the right hand side gives

\[ u'' + \omega^2 u \sim \frac{\phi_0^2 \omega^2}{6} \left[3\delta u^2 \ddot{u} + u^3\right] - \frac{\omega^2}{\epsilon} \left[(\Delta - 1) \dddot{u} + \dot{u}\right] \]  \hspace{1cm} (34)

where we have neglected terms of \( O(\phi_0^3 \epsilon^{-2}) \) for \( i > 0 \) and \( j > 0 \) on the right. Equation (34) governs the motion in the main region, that is away from the origin where the boundary conditions must be satisfied. There is an initial layer in time over which the solution decays to match with the main region.

\[ \text{Fig. 5. Effect of } \delta \text{ on the response and the approximate solution, } \delta = 1 \text{ (top), } \delta = 10 \text{ (middle) } \delta = 20 \text{ (bottom)} \]

1) Initial Layer Solution: The scaling that takes us into this layer is \( t' = \epsilon t \) and on making this transformation and defining \( \Omega = (\Delta - 1)^{-1/2} \), Eq. (15) becomes

\[ u'' + \gamma u' \sim \frac{\phi_0^2 \Omega^2}{6} \left[3\Delta u^2 u'' + (3\delta u^2 + 6 uu')u''\right] - \frac{\omega^2}{\epsilon^2} (u' + u) \]  \hspace{1cm} (35)

Here we have written \( \gamma = \Omega^2 / \omega^2 \) and as we did in Eq. (34) have only written the dominant nonlinear terms on the right. Now the solution of the dominant order of Eq. (35) that satisfies the initial conditions is \( u = 1 \). We use this to obtain a first order correction by substituting it on the right side of Eq. (35). Doing this gives

\[ u'' + \gamma u' \sim -\Omega^2 / \epsilon^2 \]  \hspace{1cm} (36)

The solution of Eq. (36) that satisfies all the initial conditions is

\[ u \sim 1 + \frac{\omega^2}{(\gamma \epsilon)^2} \left[e^{-\gamma t'} - 1\right] + \frac{\omega^2 t'}{\gamma \epsilon^2} - \frac{\omega^2 t'^2}{2 \epsilon^2} \]  \hspace{1cm} (37)

Note that stability requires \( \gamma > 0 \), which is a subset of the linear stability conditions.

Transforming back to \( t \) gives a representation of Eq. (37) in the overlap region

\[ u \sim 1 - \frac{\omega^2 t^2}{2} + \frac{\omega^2 t}{\epsilon \gamma} \]  \hspace{1cm} (38)

We will use Eq. (38) to match the representation of the solution of the main region in this overlap region.

2) Main Region Solution: We return to Eq. (34) and note that its dominant order solution is

\[ u \sim A \cos(\omega t + \Phi) \]  \hspace{1cm} (39)

In order to generate a first order correction and in this process determine the stability of this system, we need to do a multiple scale analysis. Thus, in Eq. (34), we introduce a second time scale \( \tau = t / \epsilon \), assume that the solution depends
on both $t$ and $\tau$, and write $\dot{u} = \partial_t u + \epsilon^{-1} \partial_\tau u$. In this way we obtain

$$\partial_t^2 u + \omega^2 u \sim -\frac{\phi^2}{\epsilon}(\delta - \Delta) \partial_t u - \frac{2}{\epsilon} \partial_\tau \partial_t u \quad (40)$$

We need to do all this because otherwise resonant forcing terms would produce secular solutions. Assuming a solution of the form

$$u = A(\tau) \cos(\omega t + \Phi(\tau)) + \epsilon u_1 \quad (41)$$

we find, on substituting into Eq. (40) that $u_1$ satisfies

$$\partial_t^2 u_1 + \omega^2 u_1 \sim \left[2\omega A' + \omega^5(\delta - \Delta) A\right] \sin(\omega t + \Phi) - \frac{2\omega A \Phi'}{\epsilon_0^2} \frac{(2\delta + 1) A^3}{8} \cos(\omega t + \Phi) - \frac{\epsilon \phi^2}{2\epsilon} \omega^4(2\delta + 1) A^3 \cos 3(\omega t + \Phi) \quad (42)$$

where prime here denotes differentiation with respect to $\tau$. Now the forcing terms with frequency $\omega$ must be zero for $u_1$ to be bounded. This produces the two equations

$$2A' + \omega^4(\delta - \Delta) A = 0 \quad (43)$$

$$2\Phi' - \frac{\epsilon \phi^2}{8} (2\delta + 1) \omega^3 A^2 = 0 \quad (44)$$

On integrating we find

$$A = A_0 e^{-2\sigma t} \quad (45)$$

$$\Phi = \Phi_0 - \frac{\epsilon \phi^2}{32\sigma} \omega^3 (2\delta + 1) A_0^2 (e^{-2\sigma t} - 1) \quad (46)$$

with $\sigma = \omega^4(\delta - \Delta)/2$. Thus we can write the dominant order of Eq. (41) as

$$u \sim A_0 e^{-\sigma t} \cos(\omega t + \Phi(\tau)) \quad (47)$$

We see that the linear stability conditions, Eqs. (14), are sufficient for asymptotic stability, $\sigma > 0$, and that the nonlinearities, to this order, only modify the frequency. In order to determine values for the constants $A_0$ and $\Phi_0$ we consider the representation of Eq. (47) in the overlap region. Substituting $\tau = t'/\epsilon^2$ and $t = t'/\epsilon$, expanding for large $\epsilon$, and regarding $\Phi_0 = a/\epsilon$ as small gives for the dominant two orders

$$u \sim A_0 \left[1 - \frac{\omega^2 t'^2}{2\epsilon^2} - \frac{\omega t' a}{\epsilon} \right] \quad (48)$$

Writing Eq. (48) in terms of $t$ in order to compare with Eq. (38) gives the dominant two terms as

$$u \sim A_0 \left[1 - \frac{\omega^2 t^2}{2} - \frac{\omega t a}{\epsilon} \right] \quad (49)$$

Thus in order to match with Eq. (38), we must have $A_0 = 1$ and $a = -\omega^2/\gamma$. Using this analytic solution and the values $\delta = 5.306$, $\Delta = 1.415$, and $\phi_0 = 45^\circ$, we have plotted Eq. (47) together with a numerical solution of Eq. (12), as shown in Fig. 6. The plots are for a large value of $\epsilon = 10$ and another one very close to 1, $\epsilon = 1.1$. It can be seen that the agreement is excellent even for smaller $\epsilon$ values. It is interesting to note that increasing $\Delta$ or $\delta$ will result in a more oscillatory response.

V. Conclusions

A sliding mode control law is developed for a rotary inverted pendulum and experimentally verified. Since the selection of the control parameters for good stable performance is not readily available, an approximate nonlinear solution of the closed-loop response is presented. The analytical solution shows excellent agreement with the numerical results and provides a mechanism for control parameter selections resulting in stable and effective system response.

REFERENCES


