Analytic Conditions for Spontaneous Self-Excitation in Induction Generators

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Abstract—Spontaneous self-excitation in induction generators is a fascinating phenomenon triggered by the instability of a zero equilibrium state. Prediction of this condition for various values of free parameters requires many computations of the eigenvalues of a $6 \times 6$ matrix over a large space. The paper uses a novel approach to derive analytic conditions and predict precisely when instabilities occur. The formulas give the values of the minimum load resistance, the range of capacitor values, and the range of speeds for which spontaneous self-excitation appears. The paper concludes with an illustration of the theoretical results on an example.

Keywords: induction machines, electric generators, self-excitation, unstable systems, complex Hurwitz test.

I. INTRODUCTION

Self-excitation in induction generators refers to a mode of operation where voltages are generated without a connection of the machine to a voltage source or grid. An overview of various aspects of self-excited generators, with an extensive list of references, is provided in [1]. Induction generators have gained interest in recent years because of their ruggedness and low-cost, and applicability in renewable energy applications [2], [8], [13], specifically microhydro electric power [16] and wind energy [11]. A recent area of research is the design of electronic load controllers that regulate the voltage and frequency of the generated signals [4], [14], [15].

The understanding of self-excitation is also important for the design of electronic load controllers that regulate the reactive power [16] and wind energy [11]. A recent area of research is the design of electronic load controllers that regulate the voltage and frequency of the generated signals [4], [14], [15].

Self-excitation in induction generators is an unusual phenomenon in electric machines, and a fascinating example of nonlinear dynamic systems. The steady-state operation is characterized by a limit cycle deep into the magnetic saturation region. Analysis of the steady-state can be found, for example, in [3], [9], [10]. It assumes the existence of an operating point around which the linearized system has two imaginary poles. The stability of the limit cycle, and the determination of the equilibrium point itself, mandate that the nonlinear magnetic characteristics be accounted for.

Regardless of the existence of a stable limit cycle generating power, there always exists a stable equilibrium with zero states that generates no power. It is only for certain conditions that operation at some non-zero steady-state or transfer to that steady-state is possible. In particular, capacitors must be placed in parallel with the load to provide the reactive power. In some fortunate cases, instability of the zero equilibrium triggers a departure from the zero state. We will refer to such situation as spontaneous self-excitation. It is an unusual condition where instability is wanted to achieve the desired result. In the case of spontaneous self-excitation, the voltages grow exponentially until magnetic saturation is reached, due to small initial conditions originating, for example, to residual magnetization. Fortunately, the signals do not grow indefinitely, but rather converge to the limit cycle of steady-state operation in the magnetic saturation region.

The transient phenomenon of spontaneous self-excitation is nicely described in [7], where it is shown that the condition is related to the existence of unstable eigenvalues in a $6 \times 6$ matrix. Due to the size of the matrix, analytic conditions for stability are not found in [7] or elsewhere, and the determination of stability has been performed numerically. While the computations can be performed rapidly with modern computers, the search may span a large space, with various parameters such as load resistance, capacitor values, and speed to be varied, and no a priori knowledge of the location, shape and number of the possible unstable regions.

The contribution of this paper is to show that, by using a novel approach, analytic conditions for stability can be obtained. As a result, self-contained formulas give an understanding of exactly when spontaneous self-excitation occurs, and how it is affected by various parameters.

II. SPONTANEOUS SELF-EXCITATION IN INDUCTION GENERATORS

Consider the following model of a two-phase induction generator

$$L_S \frac{d}{dt} i_{SA} + R_S i_{SA} + M \frac{d}{dt} i_{RA} = v_{SA}$$
$$L_S \frac{d}{dt} i_{SB} + R_S i_{SB} + M \frac{d}{dt} i_{RB} = v_{SB}$$
$$M \frac{d}{dt} i_{SA} + n_P \omega M i_{SB} + L_R \frac{d}{dt} i_{RA} = 0$$
$$-n_P \omega M i_{SA} + M i_{SB} - n_P \omega L_R i_{RB} = 0$$

$$L_R \frac{d}{dt} i_{RB} + R_R i_{RB} = 0$$ (1)

where $v_{SA}$, $v_{SB}$ are the stator voltages, $i_{SA}$, $i_{SB}$ are the stator currents, $i_{RA}$, $i_{RB}$ are the rotor currents transformed...
into the stator frame of reference (or equivalent rotor currents in the case of a squirrel-cage generator), and $\omega$ is the speed of the generator. For the purpose of this analysis, the speed is assumed constant. The parameters of the generator are $L_s$, the stator inductance, $L_r$, the rotor inductance, $M$, the mutual inductance between the stator and rotor windings, $R_s$, the stator resistance, $R_r$, the rotor resistance, and $n_p$, the number of pole pairs. In the case of a three-phase motor, a three-phase to two-phase transformation should be used first to apply the results.

Attached to each stator winding is a load, as well as a capacitor $C$ that is added to provide the required reactive power. The load is assumed to be purely resistive, with resistance $R_L$. The capacitor $C$ is placed in parallel with the load. For convenience, we derive the results in terms the admittance $Y_L = 1/R_L$. This is simply so that the no-load case corresponds to $Y_L = 0$ instead of $R_L = \infty$. We have

$$
C \frac{dv_{SA}}{dt} + i_{SA} + Y_L v_{SA} = 0
$$

$$
C \frac{dv_{SB}}{dt} + i_{SB} + Y_L v_{SB} = 0
$$

Applying the Laplace transform to the equations, and re-ordering the equations, we obtain

$$
A_R(s)\begin{pmatrix}
 i_{SA} \\
 i_{RA} \\
 v_{SA} \\
 i_{SB} \\
 i_{RB} \\
 v_{SB}
\end{pmatrix} = \begin{pmatrix}
 L_s i_{SA}(0) + M i_{RA}(0) \\
 M i_{SA}(0) + L_r i_{RA}(0) \\
 C v_{SA}(0) \\
 L_s i_{SB}(0) + M i_{RB}(0) \\
 M i_{SB}(0) + L_r i_{RB}(0) \\
 C v_{SB}(0)
\end{pmatrix}
$$

(3)

where

$$
A_R(s) = \begin{pmatrix}
 sL_s + R_s & sM & -1 \\
 sM & sL_r + R_r & 0 \\
 1 & 0 & sC + Y_L \\
 -n_p \omega M & -n_p \omega L_r & 0 \\
 0 & 0 & 0 \\
 n_p \omega M & n_p \omega L_r & 0 \\
 0 & 0 & 0 \\
 sL_s + R_s & sM & -1 \\
 sM & sL_r + R_r & 0 \\
 1 & 0 & sC + Y_L
\end{pmatrix}
$$

(4)

As observed in [7], it is possible for one or more roots of $\det A_R(s) = 0$ to lie in the right-half plane. In this case, although a zero solution exists, it is unstable. Therefore, a non-zero initial state will result in a growth of the voltages and currents, until magnetic saturation is encountered. A limit cycle of the nonlinear system results, and production of AC power is possible.

The prediction of spontaneous self-excitation can be achieved by computing the roots of the determinant of the $6 \times 6$ matrix. This computation can be brought into an equivalent eigenvalue problem. The limitation of the approach, however, is that the result can only be obtained numerically, and that the computation may have to be performed for various capacitor values and speeds, as well as load resistance if it is a free parameter. Whether one or more regions are possible is unknown. In theory, stability conditions could be derived by applying the Routh-Hurwitz test to the characteristic polynomial $\det A_R(s)$. Such task, however, is daunting given the dimension of the problem ($6 \times 6$ matrix and $6^{th}$ order polynomial). The paper shows that an approach similar in concept, but different in application, converts the problem into a tractable one.

III. ANALYTIC CONDITIONS FOR SPONTANEOUS SELF-EXCITATION

A. Pole locations as roots of a polynomial with complex coefficients

Define the matrix $A_C(s)$ as

$$
A_C(s) = \begin{pmatrix}
 sL_s + R_s & sM & \ & \ & \ & -1 \\
 sM & sL_r + R_r & \ & \ & \ & 0 \\
 1 & 0 & sC + Y_L & \ & \ & \ \\
 -n_p \omega M & -n_p \omega L_r & \ & \ & \ & 0 \\
 0 & 0 & 0 & \ & \ & \ \\
 n_p \omega M & n_p \omega L_r & \ & \ & \ & 0 \\
 0 & 0 & 0 & \ & \ & \ \\
 sL_s + R_s & sM & \ & \ & \ & -1 \\
 sM & sL_r + R_r & \ & \ & \ & 0 \\
 1 & 0 & sC + Y_L
\end{pmatrix}
$$

(5)

The matrix satisfies

$$
T_C A_R(s) = \begin{pmatrix}
 A_C(s) & j A_C(s)
\end{pmatrix}
$$

(6)

where

$$
T_C = \begin{pmatrix}
 1 & 0 & 0 & j & 0 & 0 \\
 0 & 1 & 0 & 0 & j & 0 \\
 0 & 0 & 1 & 0 & 0 & j
\end{pmatrix}
$$

(7)

Note that the matrix $A_C(s)$ is comparable to $A_R(s)$, in the sense that

$$
A_C(s) \begin{pmatrix}
 i_s \\
 i_r \\
 v_s
\end{pmatrix} = \begin{pmatrix}
 L_s i_s(0) + M i_r(0) \\
 M i_s(0) + L_r i_r(0) \\
 C v_s(0)
\end{pmatrix}
$$

(8)

where $i_s = i_{SA} + i_{jSB}$, $i_r = i_{RA} + i_{jRB}$, and $v_s = v_{SA} + j v_{SB}$. Accordingly, we consider the polynomial with complex coefficients $\det A_C(s)$, which is of degree 3. In contrast, $\det A_R(s)$ has degree 6, but its coefficients are real. The polynomial $\det A_R(s)$ has exactly 6 roots, which must be either real or appear as complex pairs. The polynomial $\det A_C(s)$ has 3 roots, but the roots can lie anywhere in the complex plane.

The following fact shows that there is strong connection between the roots of the two polynomials.

**Fact 1:** any root of $\det A_C(s) = 0$ is a root of $\det A_R(s) = 0$. On the other hand, if $s_0$ is a root of $\det A_R(s) = 0$, then either $s_0$ or its complex conjugate $s_0^*$ is a root of $\det A_C(s) = 0$.

**Proof of Fact 1:**

**Part 1:** If $\det A_C(s_0) = 0$, there exists $z_A \in C^3$, such that $z_A \neq 0$ and $z_A^T A_C(s_0) = 0$. Letting $z_A^T = z_A^T C = (-z_A^T j z_A^T)$. (6) shows that $z_0 \neq 0$ and $z_A^* z_A R_A(s_0) = 0$. Therefore, $\det A_R(s_0) = 0$.

**Part 2:** If $\det A_R(s_0) = 0$, there exists $z_0 \in C^6$, such that $z_0 \neq 0$ and $A_R(s_0) z_0 = 0$. Let $z_A \in C^3$, $z_B \in C^3$ such that
\[ z_0^T = (z_A^T, z_B^T). \] Then, (6) implies that \( A_C(s_0)(z_A + jz_B) = 0. \) If \( z_A + jz_B \neq 0, \) it follows that \( \text{det} A_C(s_0) = 0. \) On the other hand, if \( z_A + jz_B = 0, \) one must have \( z_A - jz_B \neq 0, \) or else \( z_0 = 0. \) If \( z_A - jz_B \neq 0, \) \( A_C(s_0^0)(z_A - jz_B) = 0, \) which implies that \( \text{det} A_C(s_0^0) = 0. \)

The fact implies some properties of the roots of \( \text{det} A_R(s) = 0. \)

**Corollary 1:** the roots of \( \text{det} A_R(s) = 0 \) must be either complex pairs or double real pairs.

In other words, there cannot be single real roots. Each root of \( \text{det} A_C(s) = 0 \) is one of the roots in a pair of roots of \( \text{det} A_R(s) = 0. \) Thus, \( A_C(s) \) contains the full information about the dynamics of the original system, since all poles of the original system can be obtained for the roots of \( \text{det} A_C(s). \) These roots are the roots of the third-order polynomial with complex coefficients

\[
p(s) = \text{det} A_C(s) = a_0 s^3 + (a_1 - jn_p \omega b_1)s^2 + (a_2 - jn_p \omega b_2)s + (a_3 - jn_p \omega b_3)
\]

where

\[
\begin{align*}
a_0 &= C(L_SL_R - M^2) \\
a_1 &= Y_L(L_SL_R - M^2) + C(L_SR_R + L_RS_L) \\
a_2 &= Y_L(L_SR_R + L_RS_L) + (CR_SR_R + L_R) \\
a_3 &= R_R(Y_L R_S + 1) \\
b_1 &= a_0 \\
b_2 &= Y_L(L_SL_R - M^2) + CL_SR_S \\
b_3 &= L_R(Y_L R_S + 1)
\end{align*}
\]

Sometimes, one defines the leakage factor

\[
\sigma = \frac{L_SL_R - M^2}{L_SL_R}
\]

where \( 1 > \sigma > 0 \) is a small constant. Given that machine parameters are positive, it follows that the \( a_i's \) and \( b_i's \) are all positive. We are thus left with the problem of determining whether the polynomial with complex coefficients (9) has any root in the right-half plane. If the coefficients were real, this could be done using the classical Routh-Hurwitz test found in textbooks. Fortunately, a solution also exists when the coefficients are complex.

**B. Hurwitz criterion for polynomials with complex coefficients**

The extension of the standard Routh-Hurwitz criterion to polynomials with complex coefficients is an old result of the literature [6], possibly not well-known due to the lack of relevant applications (until now). We give here the test for a third-order polynomial, which is the case under consideration.

**Lemma 1:** the roots of a third-order polynomial with complex coefficients

\[
p(s) = s^3 + (p_1 + jq_1)s^2 + (p_2 + jq_2)s + (p_3 + jq_3)
\]

are in the open left-half plane if and only if \( \Delta_1 > 0, \Delta_2 > 0, \) and \( \Delta_3 > 0, \) where

\[
\begin{align*}
\Delta_1 &= p_1 \\
\Delta_2 &= \text{det} \begin{pmatrix} p_1 & p_3 & -q_2 \\ p_2 & 0 & q_1 \end{pmatrix} \\
\Delta_3 &= \text{det} \begin{pmatrix} p_1 & p_3 & 0 & -q_2 \\ p_2 & 0 & -q_1 -q_3 \end{pmatrix}
\end{align*}
\]

The case that concerns us is a polynomial (9) whose first coefficient is not equal to 1. However, the coefficient \( a_0 \) is real and positive, so that Lemma 1 can be applied with the following three test variables

\[
\begin{align*}
\Delta_1 &= a_1 \\
\Delta_2 &= \text{det} \begin{pmatrix} a_1 & a_3 & n_p \omega b_2 \\ a_2 & 0 & n_p \omega b_1 \end{pmatrix} \\
\Delta_3 &= \text{det} \begin{pmatrix} a_1 & a_3 & 0 & n_p \omega b_2 \\ a_2 & 0 & 0 & n_p \omega b_1 \\
0 & -n_p \omega b_2 & 0 & a_3 \\
0 & 0 & -n_p \omega b_1 & -n_p \omega b_3 & a_3 \end{pmatrix}
\end{align*}
\]

**C. Analytic conditions derived from the Hurwitz criterion**

We will show that application of the complex Hurwitz test yields the following analytic condition for spontaneous self-excitation.

**Fact 2:** spontaneous self-excitation occurs if and only if the parameters of the induction generator satisfy

\[
\beta < -2\sqrt{\alpha \gamma}
\]

where

\[
\begin{align*}
\alpha &= (a_1 - b_2)b_1 b_2^2 b_3 \\
\beta &= a_1^2 a_2 b_2 b_3 - a_1^3 b_2^2 - 3a_1 a_3 b_1 b_2 b_3 + 2a_1^2 a_3 b_1 b_3 + a_1 a_3 b_1 b_2 - a_1 a_3 b_1^2 - a_2 a_3 b_1 b_2 + a_3^2 b_1^2 b_2 \\
\gamma &= a_3(a_1 a_2 - a_3 b_1)^2
\end{align*}
\]

and the constants \( a_i's \) and \( b_i's \) are defined in (10).

**Proof of Fact 2:** Since \( a_1 > 0, \) it is trivial that \( \Delta_1 > 0. \) Next, using the fact that \( b_1 = a_0, \) one has

\[
\Delta_2 = a_1(a_1 a_2 - a_3 b_1) + b_1 b_2(a_1 - b_2)(n_p \omega)^2
\]

Since

\[
a_1 - b_2 = CL_SR_R
\]
the second term of $\Delta_2$ is positive. As for the first term of $\Delta_2$, one obtains, after simplifications

$$a_1a_2 - a_3b_1 = Y_L^2(LSLR - M^2)(LSLR + LRR_S)$$

$$+ Y_L(LSR^2 - M^2)L_R$$

$$+ CY_L(LSR^2 + LRR_S)^2$$

$$+ C^2RS_R(LSR^2 + LRR_S)$$

which is positive. One may conclude that $\Delta_2 > 0$ and that the stability of the generator is determined solely by the condition $\Delta_3 > 0$.

Expanding the determinant $\Delta_3$, again using $b_1 = a_0$, one finds, after simplifications, that

$$\Delta_3 = \alpha(n_p\omega)^4 + \beta(n_p\omega)^2 + \gamma$$

(20)

where $\alpha$, $\beta$, and $\gamma$ are given by (16). Given that $a_1 - b_2 > 0$, $\alpha > 0$ and $\gamma > 0$, a necessary condition for instability is therefore that the parameter $\beta$ must be negative. If it is the case, one also needs to find a speed such that $\Delta_3 < 0$. Note that $\Delta_3$ is a quadratic function of $(n_p\omega)^2$, which is positive for $\omega^2 = 0$ and for large $\omega^2$. Therefore, there must be two real positive roots of $\Delta_3 ((n_p\omega)^2 = 0$, in order to have a range of speeds for which $\Delta_3 < 0$. The condition on $\beta$ (15) is necessary and sufficient for it to be the case.

The proof of fact 2 allows us to establish other new results. They are summarized in the following corollary.

**Corollary 2:** Spontaneous self-excitation can only occur for a single range of speeds $\omega \in (\omega_{\text{min}}, \omega_{\text{max}})$, such that

$$\omega_{\text{min}} = \sqrt[n_p]{\frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}}$$

$$\omega_{\text{max}} = \sqrt[n_p]{\frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}}$$

(21)

The polynomial $A_C(s)$ can only have one root in the right-half plane. Therefore, the original system can only have one pair of unstable poles.

The speed range is the direct result of the proof of Fact 2. The number of unstable roots can be deduced because the Hurwitz array, whose leading column is composed of $\Delta_1$, $\Delta_2$, $\Delta_3$, predicts the number of right-half plane roots as for the classical Routh-Hurwitz test. Since there can only be one sign change in the first column of the Hurwitz array, the system with complex poles can only have one unstable pole. Therefore, the original system can only have one pair of unstable complex poles. The result is enlightening because it shows that the origin of spontaneous self-excitation is a growing oscillation whose rate of growth is determined by the real part of a complex pole, and whose frequency is determined by the imaginary part of the pole. There cannot be two competing oscillations of different frequencies. The overall power of the result is that it gives a direct computation of the speed range for which spontaneous self-excitation will occur. The critical speeds are obtained by solving a single quartic equation, rather than computing the eigenvalues of a $6 \times 6$ matrix for a large number of speeds.

**D. Special case: no load and zero stator resistance**

For $R_S = 0$ and $Y_L = 0$, the parameters become

$$a_0 = C(LSLR - M^2) = b_1$$

$$a_1 = CLSR_R, \quad a_2 = L_R, \quad b_2 = 0$$

$$a_3 = R_R, \quad b_3 = L_R$$

(22)

and, after simplifications

$$\alpha = 0$$

$$\beta = -C^3LM^4R^3_R$$

$$\gamma = C^2M^4R^3_R$$

$$\Delta_3 = C^2M^4R^3_R(1 - L_SC(n_p\omega)^2)$$

(23)

The condition for self-excitation is simply

$$\omega > \sqrt[n_p]{\frac{1}{L_SC}}$$

(24)

In other words, the condition for self-excitation is that the electrical frequency corresponding to the mechanical speed must be greater than the natural frequency of the $LC$ circuit composed of the stator inductance and of the capacitor. There is no upper limit to the range in this case, because the $(n_p\omega)^2$ term drops out of $\Delta_3$.

**E. Example**

Consider the generator of [12], with $R_S = 1.7\Omega$, $R_R = 2.7\Omega$, $L_S = L_R = 191.4\mu H$, and $M = 180\mu H$. Let $R_L = \infty$ and $C = 300\mu F$. The computations give $\omega_{\text{min}} = 66.7$ rad/s and $\omega_{\text{max}} = 465.3$ rad/s. The minimum speed of 640 rpm is consistent with Fig. 2 of the paper [12]. At $\omega = 100$ rad/s, the roots of $P$ are given by $-73.4 \pm 377.9j$, $-142.1 + 389.1j$, and $16.6 + 188.7j$. The roots of $A(s)$ are given by $-73.4 \pm 377.9j$, $-142.1 + 389.1j$, and $16.6 + 188.7j$, with the last pair being the unstable pair of poles leading to spontaneous self-excitation. For $R_L = 25\Omega$, the range is reduced to 86.9-265.2 rad/s and the range vanishes for $R_L < 14.5\Omega$. Lower resistance values require a larger capacitor although, for low enough resistance, no capacitor produces spontaneous self-excitation.

Using the results, one may plot the combinations of speed and capacitor values where spontaneous self-excitation occurs. Fig. 1 shows the upper and lower limits of speed as functions of the capacitor value, for several values of load resistance. The figure is also consistent with the figures of [12].

**IV. Analytic Conditions for Spontaneous Self-Excitation Based on Singularity**

**A. Conditions for singularity**

A further set of conditions can be obtained based on the results. The idea is that, since instability is caused by the crossing of a single root of the complex polynomial across the imaginary axis, a condition for spontaneous self-excitation is that, for some $\omega_e$,

$$\det A_C(j\omega_e) = 0$$

(25)
Therefore, stability boundaries can be obtained from the solutions of this equation. Expansion of (25) will allow us to prove the following fact.

**Fact 3:** spontaneous self-excitation is possible if and only if

\[ R_L > \frac{4\sqrt{\sigma}}{(\sqrt{\sigma} - 1)^2} R_S \]  

(26)

If \( R_L \) satisfies (26), the range of capacitor values for which self-excitation occurs is given by the limits

\[ C_{\text{min}} = \frac{-g_2 - \sqrt{g_2^2 - 4g_1g_3}}{2g_1}, \quad C_{\text{max}} = \frac{-g_2 + \sqrt{g_2^2 - 4g_1g_3}}{2g_1} \]  

(27)

where

\[ g_1 = R_S^2 \]
\[ g_2 = 2L_S(R_3Y_L + 1)\sqrt{\sigma} - (\sigma + 1)L_S \]
\[ g_3 = \sigma L_S^2 Y_L^2 \]  

(28)

For any value of \( C \) in the range defined by (27), the range of electrical frequencies for which self-excitation occurs is given by

\[ \omega_{e,\text{min}} = \sqrt{\frac{-f_2 - \sqrt{T_2^2 - 4f_1f_3}}{2f_1}} \]
\[ \omega_{e,\text{max}} = \sqrt{\frac{-f_2 + \sqrt{T_2^2 - 4f_1f_3}}{2f_1}} \]  

(29)

where

\[ f_1 = C^2 L_S (L_S L_R - M^2) \]
\[ f_2 = Y_L^2 L_S (L_S L_R - M^2) + C^2 R_S^2 L_R - C(2L_S L_R - M^2) \]
\[ f_3 = L_R(Y_L R_S + 1)^2 \]  

(30)

The range of mechanical speeds for which self-excitation occurs is obtained by replacing \( \omega_e \) by \( \omega_{e,\text{min}} \) and \( \omega_{e,\text{max}} \) in the following equation

\[ \omega = \omega_e - \frac{Y_L R_S R_R - \omega_e^2 C R_S L_S + R_R}{\omega_e (Y_L (L_S L_R - M^2) + R_S L_R C)} \]  

(31)

**Proof of Fact 3:** The real and imaginary parts of (25) yield the following two conditions

\[ Y_L R_S R_R - \omega_e^2 C R_S L_S + R_R = (\omega_e - n_P \omega) \omega_e (Y_L (L_S L_R - M^2) + R_S L_R C) \]
\[ \omega_e (R_S L_S Y_L + C R_S R_R) = (\omega_e - n_P \omega) (\omega_e^2 C (L_S L_R - M^2) - R_S L_R Y_L - L_R) \]  

(32)

The first equation leads to (31). Eliminating \( \omega_e - n_P \omega \) from the two equations gives, after simplifications, the quartic equation in \( \omega_e \)

\[ f_1 \omega_e^4 + f_2 \omega_e^2 + f_3 = 0 \]  

(33)

where \( f_1, f_2, \) and \( f_3 \) are given by (30), and \( f_1 \) and \( f_3 \) are both positive. The quartic equation is a quadratic equation in \( \omega_e^2 \), which has a positive real root if and only if has two positive real roots (given that \( f_1 > 0 \) and \( f_3 > 0 \)). Thus, a solution exists for \( \omega_e \) if and only if

\[ f_2 < -2\sqrt{f_1f_3} \]  

(34)

If the condition is satisfied, the solutions for \( \omega_e \) are given by (29).

After simplifications, (34) gives the following inequality

\[ g_1 C^2 + g_2 C + g_3 < 0 \]  

(35)

where the parameters \( g_1, g_2, \) and \( g_3 \) are given by (28). Given that \( g_1 > 0 \) and \( g_3 > 0 \), we again find ourselves in a situation where the inequality can have a solution if and only if the quadratic equality has two positive real roots, which requires that

\[ g_2 < -2\sqrt{g_1g_3} \]  

(36)

If the condition is satisfied, the roots of the quadratic equality associated with (35) are given by (27) and specify the range of capacitor values. After simplifications, (36) gives

\[ \frac{\sigma + 1 - 2\sqrt{\sigma}}{4\sqrt{\sigma}} > R_S Y_L \]  

(37)

which yields (26). \( \square \)

The approach using the singularity condition (25) is similar to the derivation of steady-state conditions for self-excited induction generators except that, in that case, the differential equations are those obtained from linearization around some equilibrium in the magnetic saturation region. By itself, this approach only gives the boundaries for stability. The fact that instability occurs between the capacitor values is known from the previous analysis given in this paper.

The singularity approach gives the same boundaries as the stability approach, but additional limits were derived that were not immediately obvious with the stability approach. It is surprising that the minimum load resistance is only dependent on two parameters: the stator resistance and the leakage factor. The other limits are convenient in terms of bounding the search space for possible operating regions. Analytic formulas eliminate the need for a large number of iterations to obtain these regions.


**B. Example**

We return to the example discussed earlier. For the example, (26) gives a minimum value of $R_L = 5.3064\Omega$. For this extreme case, (27) gives both minimum and maximum values of the capacitor as $C = 7.2131mF$. For this value of capacitor, the range of electrical frequencies is also a single frequency $\omega_c = 53.04$ rad/s, and the mechanical frequency is $\omega = 47.27$ rad/s. This is a limit case. Note that the slip $\omega_c - n_p \omega$ (with $n_p = 2$) is $-41.5$ rad/s, which is negative, as must be for the generator mode. For $R_L = \infty$, (27) gives $C_{\text{min}} = 0$ and $C_{\text{max}} = 28.9$ mF. For $C_{\text{max}}$, (29) gives a single frequency $\omega_c = 23.1$ rad/s, and a mechanical frequency $\omega = 23.64$ rad/s. Again, this is a limit case. The results are shown on Fig. 2, which is an expanded view of Fig. 1. The region of spontaneous self-excitation for infinite load resistance is delimited by the outside curve, while the region for $R_L = 5.31\Omega$ (which is slightly greater than the minimum resistance) is inside the tiny spot under the $\Omega$ symbol on Fig. 2.

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**VI. CONCLUSIONS**

The paper presented conditions for spontaneous self-excitation in induction generators. The results originated from conditions for the instability of the electrical equations of the generator. Using a novel approach based on the Hurwitz test for polynomials with complex coefficients, analytic conditions were derived that conveniently replace the exhaustive numerical search that was required before. The results provided the justification for a second approach based on a singularity test for a complex matrix, which gave further results. Overall, the paper gives new, compact analytic formulas that specify the minimum load resistance, the range of capacitor values, and the range of speeds for which self-excitation is possible. These results will make it easier for engineers to design off-grid induction generator applications, in particular for renewable energy.

**REFERENCES**