Approximate Off-Line Receding Horizon Control of Constrained Nonlinear Discrete-Time Systems: Smooth Approximation of the Control Law

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Abstract—In this work, the off-line approximation of state-feedback nonlinear model predictive control laws by means of smooth functions of the state is addressed. The idea is to investigate how the approximation errors affect the stability of the closed-loop system, in order to derive suitable bounds which have to be fulfilled by the approximating function. This analysis allows to conveniently set up the characteristic parameters of some techniques such as Neural Networks which can be used to implement the control law, in order to render the system Input-to-State Practically Stable while satisfying, in addition, hard constraints on the trajectories; both the amount of data storage and the computational time result strongly reduced with respect to Nearest Neighbor or Set Membership approaches, which have been recently proposed to obtain effective off-line approximation of nonlinear MPC. The provided simulations confirm the validity of the method.

I. INTRODUCTION

The possibility to apply Model Predictive Control (MPC) to constrained nonlinear plants with fast dynamics is an interesting challenge in control engineering. The computational burden required to solve an optimization problem at each sampling instant represents indeed a major limitation for its real-time implementation. Recent studies have focused on efficient hardware solutions (such as parallel FPGAs architectures: for a tutorial see [1]) or algorithms to obtain explicit RH control laws recurring to parametric quadratic programming techniques (see e.g., [2], [3] and [4], [5], [6]). As regards nonlinear systems, the explicit solution of MPC problems is a really hard task: some recent works give efficient formulations for special classes of constrained uncertain nonlinear systems ([7], [8], [9]).

An alternative approach relies on the off-line approximation of the exact RH control law by means of different techniques like Neural Networks, Set Membership approximators ([10], [11], [12], [13]). Nevertheless, the obtained approximated control law is required to guarantee stability for the system enforcing the robust constraint satisfaction.

In the recent work [14], the effects of the approximation errors introduced by a generic function approximator on the robustness of the closed-loop (c-l) system have been investigated in presence of strong nonlinearities and hard constraints on state and input variables. Remarkably, one of the most important contributions introduced in the aforementioned work consists in the removal of the assumption on the continuity of the MPC law, still guaranteeing robust constraints enforcement. Moreover, in [14] it has been shown that a Nearest Neighbor Search method [15] can be used for the approximation of a discontinuous control law.

In this paper we are going to study the case in which the MPC state-feedback is approximated by a smooth function; in particular, following the early papers [11], [12], we will focus on Neural Networks with smooth activation functions for their favorable approximation properties. The use of smooth functions is reasonable when the RH control law is continuous and in particular when it is locally Lipschitz (the reader can refer to the recent work [16] and the references therein to get deep insight on the sufficient conditions to obtain a smooth MPC feedback).

Compared to the Nearest Neighbour approach, the use of smooth approximators leads to a dramatic reduction of the requested storage capacity for on-line implementation; furthermore, the on-line computational time decreases in view of the fact that the extensive norm-distance evaluations required by the Nearest Neighbor Search are not needed anymore. The time depends now polynomially only on the number of neurons of the neural network.

Bounds on admissible errors, corresponding to the $\epsilon$-tube in network training, will be given in order to maintain the practical stability of the c-l system and to enforce the fulfillment of hard state constraints.

The paper is organized as follows. After giving main notations and preliminary definitions and results in Sections II and III, respectively, in Section IV, sufficient conditions for practical stability are provided. Furthermore, in Section V, the smooth approximation of the MPC control law is addressed and in Section VI simulation results are given showing the effectiveness of the methodology.

II. NOTATION AND BASIC DEFINITIONS

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, and $\mathbb{Z}_+$ denote the real, the non-negative real, the integer, and the non-negative integer sets of numbers, respectively. The Euclidean norm is denoted as $|\cdot|$. For any discrete-time sequence $\xi: \mathbb{Z}_0 \rightarrow (\mathbb{R}^n)^{\mathbb{Z}_0}$, $r \in \mathbb{Z}_0$, let $\|\xi\|_{\sup_{\tau \in \Omega}} \{\|\xi_k\|\}$ and $\|\xi\|_{\sup_{\tau \in \Omega}} \{\|\xi_k\|\}$, where $\xi_k$ denotes the value that the sequence $\xi$ takes on in correspondence with the index $k$. The set of discrete-time sequences $\xi$ taking values in some subset $\Omega \subseteq \mathbb{R}^r$ is denoted by $\mathcal{M}_\Omega$, while $\sup_{\tau \in \Omega} \{\|\xi_k\|\}$. The symbol $id$ represents the identity function from $\mathbb{R}$ to $\mathbb{R}$, while $\gamma_1 \gamma_2$ is the composition of two functions $\gamma_1$ and $\gamma_2$ from $\mathbb{R}$ to $\mathbb{R}$. Given a set $A \subseteq \mathbb{R}^n$, int$(A)$ denotes the interior of $A$. A vector $x \in \mathbb{R}^n$, $d(x, A)$ $\triangleq$ inf $\{\|x - \xi\| : \xi \in A\}$ is the point-to-set distance from $x \in \mathbb{R}^n$ to $A$. Given two sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, dist$(A, B) \triangleq$ inf $\{d(\xi, A) : \xi \in B\}$ is the minimal set-to-set distance. The difference between two given sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, with $B \subseteq A$, is denoted as $A \setminus B \triangleq \{x : x \in A, x \notin B\}$. Given two sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, the Pontryagin difference set $C$ is defined as $C = A \setminus B \triangleq \{x \in \mathbb{R}^n : x + \xi \in A, \forall \xi \in B\}$, while the Minkowski sum set is defined as $S = A + B \triangleq \{x \in \mathbb{R}^n : \exists \xi \in A, \eta \in B, x = \xi + \eta\}$. Given a vector $\eta \in \mathbb{R}^n$ and
a positive scalar $\rho \in \mathbb{R}_{>0}$, the closed ball in $\mathbb{R}^r$ centered in $\eta$ and of radius $\rho$, is denoted as $B^r(\eta, \rho) \triangleq \{\xi \in \mathbb{R}^r : ||\xi - \eta|| \leq \rho\}$. The shorthand $B^r(\rho)$ is used when the ball is centered in the origin. The domain of a generic function $\kappa : \mathbb{R}^n \rightarrow \mathbb{R}$ will be denoted as dom($\kappa$). The notions of functions of class $\mathcal{K}$, class $\mathcal{K}_\infty$, and class $\mathcal{KL}$ are utilized to characterize stability properties. A function $\alpha : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. It belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ belongs to class $\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{t \rightarrow -\infty} \beta(s, t) = \lim_{t \rightarrow -\infty} \beta(t, s) = 0$.

Let us consider the discrete-time dynamic system

$$x_{t+1} = g(x_t, \xi_t), \quad t \in \mathbb{Z}_{\geq 0}, \quad x_0 = \bar{x},$$

(1)

with $g(0, 0) = 0$ and where $x_t \in \mathbb{R}^n$ and $\xi_t \in \mathcal{Y} \subseteq \mathbb{R}^r$ denote the state and the bounded input of the system, respectively. The discrete-time state trajectory of the system (1), with initial state $\bar{x}$ and input sequence $\xi \in \mathcal{M}_T$, $\xi = \{\xi_t, t \in \mathbb{Z}_{\geq 0}\}$, is denoted by $x_t \triangleq (\bar{x}, \xi_t, \xi_{t+1}, \cdots, \xi_{T-1})\subset \mathcal{M}_T$.

Definition 2.1 (RPI set): A set $\Xi \subseteq \mathbb{R}^n$ is a Robust Positively Invariant (RPI) set for system (1) if $g(x_t, \xi_t) \in \Xi, \forall x_t \in \Xi$ and $\forall \xi_t \in \mathcal{Y}$. In the following, the Regional Input-to-State Stability property [17], is recalled. To this end, let us first consider the following definition.

Definition 2.2 (ISS-Lyapunov Function [18], [17]): Given system (1) and a pair of compact sets $\Xi \subseteq \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$, with $0 \in \Omega$, a function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is called a (Regional) ISS-Lyapunov function in $\Xi$, if there exist some $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha_3$, and a $\mathcal{K}$-function $\sigma$ such that

i) the following inequalities hold $\forall \xi \in \mathcal{Y}$

$$V(\xi) \geq \alpha_1(|\xi|), \quad \forall \xi \in \Xi,$$

(2)

$$V(\xi) \leq \alpha_2(|\xi|), \quad \forall \xi \in \Omega,$$

(3)

$$V(g(\xi, \xi')) - V(x) \leq -\alpha_3(|\xi|) + \sigma(|\xi|), \quad \forall \xi \in \Xi,$$

(4)

ii) there exists a suitable $\mathcal{K}_\infty$-function $\rho$ (with $\rho$ such that $(id - \rho)$ is a $\mathcal{K}_\infty$-function, too) such that the following compact set $\Theta \subset \{\xi : \xi \in \Omega, d(\xi, \Theta) > c\}$ can be defined for some arbitrary constant $c \in \mathbb{R}_{>0}$

$$\Theta \triangleq \{\xi : V(\xi) \leq b(\xi^\text{up}),\}$$

where $b(s) \triangleq \alpha_4^{-1} \circ \rho^{-1} \circ \sigma(s), \alpha_4 \triangleq \alpha_3 \circ \alpha_2^{-1}$. □

Definition 2.3 (Regional ISS and ISP s): Given a compact set $\Xi \subseteq \mathbb{R}^n$, if $\Xi$ is RPI for (1) and if there exist a $\mathcal{KL}$-function $\beta$, a $\mathcal{K}$-function $\gamma$ and a positive number $\eta \in \mathbb{R}_{>0}$ such that

$$|x(\bar{x}, \nu, t)| \leq \max \left\{ \beta(|\bar{x}|, t), \gamma(||\nu||) \right\} + \eta, \quad \forall t \in \mathbb{Z}_{\geq 0}, \forall \nu \in \Xi,$$

(5)

then the system (1), with $\nu \in \mathcal{M}_T$, is said to be Input-to-State Practically Stable (ISP s) in $\Xi$. If $\{\theta \} \in \Xi$ and inequality (5) is satisfied for $\eta = 0$, then system (1) is said to be Input-to-State Stable (ISS) for initial conditions in $\Xi$. □

The following important result can be proved.

Theorem 2.1 (Regional ISS [17]): If system (1) admits an ISS-Lyapunov function in $\Xi$, and $\Xi$ is RPI for (1), then it is Regional ISS in $\Xi$ and $\lim_{t \rightarrow \infty} d(x(\bar{x}, \nu, t), \Theta) = 0$. □

III. Preliminaries

In this section, for the reader’s convenience, we recall the main results presented in [14]: consider the nonlinear discrete-time perturbed dynamic system

$$x_{t+1} = f(x_t, u_t, \xi_t), \quad t \in \mathbb{Z}_{\geq 0}, \quad x_0 = \bar{x},$$

(6)

where $x_t \in \mathbb{R}^n$ denotes the system state, $u_t \in \mathbb{R}^m$ the control vector and $\xi_t \in \mathbb{R}^r$ an exogenous disturbance input; the state and control variables are subject to the following constraints

$$x_t \in X, \quad t \in \mathbb{Z}_{\geq 0},$$

(7)

$$u_t \in U, \quad t \in \mathbb{Z}_{\geq 0},$$

(8)

where $X$ and $U$ are compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, containing the origin as an interior point. Given the system (6), let $\tilde{f}(x_t, u_t)$, with $\tilde{f}(0, 0) = 0$, denote the nominal model used for control design purposes, such that

$$x_{t+1} = \tilde{f}(x_t, u_t) + d_t, \quad t \in \mathbb{Z}_{\geq 0}, \quad x_0 = \bar{x},$$

(9)

where $d_t \triangleq f(x_t, u_t) - \tilde{f}(x_t, u_t) \in \mathbb{R}^n$ denotes the discrete-time state transition uncertainty. Let us now introduce the following definition.

Definition 3.1 (C1(\Xi)): Given a set $\Xi \subseteq \mathbb{R}^n$, the one-step (Controllability Set to $\Xi$, denoted as $C_1(\Xi)$, is defined as $C_1(\Xi) \triangleq \{\xi \in \mathbb{R}^n : \exists u \in U : f(\xi, u) \in \Xi\}$, i.e., $C_1(\Xi)$ is the set of state vectors that can be steered to $\Xi$ by a control action under $f(\xi, u)$. □

In the sequel the following assumptions will be needed.

Assumption 1: The function $f : X \times U \rightarrow X$ is Lipschitz (L) continuous w.r.t. $x \in X$, with $L$ constant $L_f \in \mathbb{R}_{>0}$, uniformly in $u \in U$ (i.e., for any fixed $u \in U$, it holds that $|f(x, u) - f(x', u)| \leq L_f|x - x'|$ for all $(x, x') \in X^2)$.

Furthermore, the function $\tilde{f}$ is uniformly continuous in $u$: there exists a $\mathcal{K}$-function $\eta_u$ such that $|f(x, u) - f(x', u')| \leq \eta_u(||u - u'||)$ for all $x \in X$ and for all $(u, u') \in U^2$. □

Assumption 2 (Uncertainties): The additive transition uncertainty verifies $d_t \leq \mu(|\xi_t|), \forall t \in \mathbb{Z}_{\geq 0}$ where $\mu$ is a $\mathcal{K}$-function. Moreover, $d_t$ is bounded in a compact ball $D$, that is $d_t \in \mathbb{D} = \mathbb{B}(\bar{d}), \forall t \in \mathbb{Z}_{\geq 0}$, with $\bar{d} \in \mathbb{R}_{\geq 0}$ finite. □

Assumption 3 (Input-to-State Stabilizing controller): There exist a compact set $\Xi \subseteq X$, with $\{0\} \in \Xi$, and a state-feedback control law (possibly non-smooth)

$$u_t = \kappa(x_t), \quad \kappa(x_t) : \Xi \rightarrow U,$$

(10)

such that the following system, given by (9) in c-l with (10)

$$x_{t+1} = \tilde{f}(x_t, \kappa(x_t)) + d_t, \quad t \in \mathbb{Z}_{\geq 0}, \quad x_0 = \bar{x},$$

(11)

satisfies the properties:

i) it is ISS in $\Xi$ w.r.t. additive disturbances $d_t \in D$. In particular, there exists a ISS-Lyapunov function for which Points i) and ii) of Definition 2.2 hold.

ii) the set $\Xi$ is RPI for system (9) with additive disturbances $d_t \in D$. □

One of the most innovative contribution of [14] resides in giving a procedure to design a RH control law capable of guaranteeing the input-to-state stability of a c-l system not asymptotically stabilizable by continuous static state feedback, subject to bounded uncertainties. Let’s now resume this technique, starting from the formulation of the RH control policy, based on the following suitable Finite-Horizon Optimal Control Problem (FHOCP).
Definition 3.2 (FHOCP): Given a positive integer \( N_c \in \mathbb{Z}_{>0} \), at any time \( t \in \mathbb{Z}_{\geq 0} \), let \( u_{t,t+N_c−1|t} \triangleq \text{col}[u_t|t, \ldots, u_{t+N_c−1|t}] \) denote a sequence of input variables over the control horizon \( N_c \). Moreover, given \( x_t \) and \( u_{t,t+N_c−1|t} \) let \( \hat{x}_{t+j|t} \) denote the state “predicted” by means of the nominal model, such that
\[
\hat{x}_{t+j|t}=f(\hat{x}_{t+j−1|t}, u_{t+j−1|t}), \quad \forall j \in \{1, \ldots, N_c\}.
\]
Then, given a transition cost function \( h \), an auxiliary control law \( \kappa_f \), a terminal cost function \( h_f \), a terminal set \( X_f \) and a sequence of constraint sets \( X_{t+j|t} \subseteq \mathbb{X}, \forall j \in \{1, \ldots, N_c−1\} \), to be described later on, the FHOCP consists in minimizing, with respect to \( u_{t,t+N_c−1|t} \), the cost function
\[
J_{FH}(x_t, u_{t,t+N_c−1|t}, N_c) \triangleq \sum_{t=0}^{t+N_c−1} h(\hat{x}_{t|t}, u_{t|t}) + h_f(\hat{x}_{t+N_c|t})
\]
subject to
\begin{enumerate}
\item the nominal dynamics (12), with \( \hat{x}_{t|t}=x_t \);
\item the control and the state constraints \( u_{t+j|t} \in U, \hat{x}_{t+j|t} \in X_{t+j|t}, \forall j \in \{0, \ldots, N_c−1\} \);
\item the terminal state constraint \( \hat{x}_{t+N_c|t} \in X_f \).
\end{enumerate}
The usual RH control technique can now be stated as follows: given a time instant \( t \in \mathbb{Z}_{\geq 0} \), let \( \hat{x}_{t|t}=x_t \), and find the optimal control sequence \( u_{t,t+N_c−1|t} \) by solving the FHOCP. Then, according to the RH strategy, apply
\[
u_t=\kappa_{RH}(x_t),
\]
where \( \kappa_{RH}(x_t) \triangleq u_{t|t}^* \) and \( u_{t|t}^* \) is the first element of the optimal control sequence \( u_{t,t+N_c−1|t}^* \) (implicitly dependent on \( x_t \)).

It can be shown that the satisfaction of the original state constraints is ensured, for any admissible disturbance sequence, by imposing restricted constraints to the predicted open-loop trajectories. The tightened constraints can be computed as prescribed by the following lemma.

Lemma 3.1 (Constraints tightening [19]): Assuming to know an upper bound \( \delta \) on the uncertainty as specified by Assumption 2, given the state vector \( x_t \) at time \( t \), if a control sequence, \( u_{t,t+N_c−1|t} \), is feasible with respect to the state constraints \( X_{t+j|t} \), where
\[
\hat{x}_{t+j|t} \triangleq X \prec B^n \left( \frac{L_j^f}{L_{fs} - 1} \right),
\]
then, the control sequence \( u_{t,t+N_c−1|t} \), applied to the perturbed system (6) in open-loop, guarantees that \( x_{t+j} \in X, \forall j \in \{1, \ldots, N_c\} \).

In order to prove the ISS property for the c-l system, let us introduce the following assumptions.

Assumption 4: The transition cost function \( h \) is such that \( h((x,u)) \leq h(x,u), \forall x \in X, \forall u \in U \), where \( h \) is a \( K_\infty \)-function. Moreover, \( h \) is Lipschitz w.r.t. \( x \) uniformly in \( u \), with L. constant \( L_h > 0 \).

Assumption 5: A terminal cost function \( h_f \), an auxiliary control law \( \kappa_f \), and a set \( X_f \) are given such that
\begin{enumerate}
\item \( X_f \subset X, X_f \) closed, \( X_f \);
\item \( \exists \delta > 0: \kappa_f(x) \in U, \forall x \in X_f \oplus B^n(\delta); \)
\item \( f(x, \kappa_f(x)) \in X_f, \forall x \in X_f \oplus B^n(\delta); \)
\item \( h_f(x) \) is Lipschitz in \( X \), with L. constant \( L_{h_f} > 0; \)
\item \( h_f(f(x, \kappa_f(x)))−h_f(x) \leq −h(x, \kappa_f(x)), \forall x \in X_f \oplus B^n(\delta); \)
\end{enumerate}

With respect to previous works [19], [17], [20] concerning the design of input-to-state stabilizing RH controllers, in order to cope with possibly discontinuous auxiliary control laws, we do not require neither \( \kappa_f(x) \) nor the c-l map \( f(x, \kappa_f(x)) \) to be Lipschitz continuous w.r.t. \( x \in X_f \). In addition, in order to establish the ISS property for the c-l system, we require the following assumption to be verified together with Assumption 5.

Assumption 6 (\( X_{c,f} \)): Suppose that there exist a compact set \( X_{c,f} \subseteq X_f \) for which \( u_{t,t+N_c−1|t} \triangleq \text{col}[\kappa_f(\hat{x}_{t|t})], \kappa_f(\hat{x}_{t+j|t}), \ldots, \kappa_f(\hat{x}_{t+N_c−1|t}) \), being \( \hat{x}_{t|t}=x_t \in X_{c,1} \), is a feasible control sequence for the FHOCP and for which Points 1), 2) and 5) of Assumption 5 are satisfied.

Under the stated assumptions, the following theorem characterizes the ISS property of the c-l system with respect to bounded additive uncertainties. Moreover, in order to guarantee the feasibility of the FHOCP, an upper bound on the admissible uncertainty is introduced, which is shown to depend on the invariance properties of \( x_f \). This theorem represents the extension of the ISS result presented in [20] to the case of systems which are not asymptotically stabilizable by smooth feedback (see the proof in [21]).

Theorem 3.1 (Regional ISS): Let us denote as \( X_{RH} \subset \mathbb{R}^n \) the set of state vectors for which the FHOCP is feasible. Under Assumptions 1, 2, 5-6, the system (6), driven by the RH control law (14), is regional ISS in \( X_{RH} \) with respect to additive perturbations \( d_t \in D \), with \( D \subseteq B^m(d) \) and
\[
d \leq L_j^{−N_c} \text{dist}(\mathbb{R}^n\setminus C_1(X_f), X_f).
\]

To conclude, in the current section we recalled how to design an Input-To-State stabilizing exact RH control law \( \kappa_{RH} \) for system (9), which renders RPI the set \( X_{RH} \subset X \) with respect to additive disturbances \( d_t \in D \). Therefore, Assumption 3 is verified by the RH controller with \( \kappa=\kappa_{RH} \) and \( \Xi=X_{RH} \).

In the next section we are going to infer, from the stabilizing properties of \( \kappa \), the stability properties of the c-l system driven by an approximate control law \( \kappa^* \), satisfying suitable requirements to be specified later on. The results we are going to present are a generalization of those given in [14] and will allow to cope with smooth function approximations of the control law obtained i.e. by Neural Networks.

IV. PRESERVATION OF STABILITY UNDER APPROXIMATION: SUFFICIENT CONDITIONS FOR PRACTICAL STABILIZATION

Consider the following dynamic system:
\[
x_{t+1}=f(x_t, \kappa^*(x_t)) + w_t, \quad t \in \mathbb{Z}_{\geq 0}, \quad x_0=\bar{x}.
\]

where \( w_t \in W \triangleq B^n(\delta_w) \) is a disturbance input and the function \( \kappa^* : \mathbb{R}^n \to \mathbb{R}^m \) is an approximation of the given ISS stabilizing \( \kappa \) satisfying Assumption 3. We will show that the stability properties of (17) can be inferred from those of (11) provided that \( \kappa^* \) satisfies the following additional requirements.

Assumption 7: Let us define the \( K_\infty \)-function \( \eta_k(s) = L_{fs} s + s \) for \( s \geq 0 \) and let \( d_{q|t} \in \mathbb{R}_{\geq 0} \) and \( \hat{d}_{q|t} \in \mathbb{R}_{\geq 0} \) be two positive scalars satisfying the following inequality
\[
\tilde{d}_q+d_{v}+\tilde{d}_w \leq \tilde{d}.
\]

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Defining \( \mathcal{T} \triangleq \eta_x^{-1}(\overline{d}_q) \), assume that \( \forall \xi \in \text{dom}(\kappa), \exists \xi \in B^n(\xi, \mathcal{T}) \cap \text{dom}(\kappa) \) such that
\[ \eta_u(|\kappa^*(\xi) - \kappa(\xi)|) + \eta_x(|\xi - \xi|) + d_w \leq \overline{d}(\xi), \] (19)
where \( \overline{d}(\xi) \) is the local uncertainty bound under which the state can be driven from \( \xi \in \Xi \) by the control law \( \kappa \), i.e.
\[ \overline{d}(\xi) \triangleq \inf\{c \in \mathbb{R}_{>0} : \exists d \in B^n(c) : \hat{f}(\xi, \kappa(\xi)) + d \notin \Xi \}. \]

It is worth to notice that for the global uncertainty bound \( d \), it holds that \( \overline{d} \leq \overline{d}(\xi) \) and thus the conditions posed on the approximating function are less restrictive than those formulated in [14], in which a global bound has been used.

Then, the stability properties of the closed-loop system driven by the approximate control law \( \kappa^* \) can be established.

**Theorem 4.1:** Suppose that Assumptions 1-7 hold and let \( \Xi \triangleq \Xi \cup Q \), with \( Q \triangleq B^0(\mathcal{T}) \). Then, the following statements hold:

i) The set \( \Xi \subset X \) is RPI for the c-1 system (17) with \( w_i \in W \), where \( W \triangleq B^n(\overline{d}_w) \);

ii) The c-1 system (17) is ISS in \( \Xi \).

**Proof:** Points i) and ii) of Theorem 4.1 will be addressed separately in the following.

i) Let \( x_t \in \Xi, q_t \in Q \) and \( w_t \in W \). Now we will prove that \( \Xi \) is RPI for (17).

First, under assumption 7, \( \forall x \in \Xi \) there exists \( \zeta_x = q_x + x \), with \( q_x \in B^n(\mathcal{T}) \) such that (19) holds. Then let us consider that
\[ \hat{f}(x, \kappa^*(x)) + w + q_x = \hat{f}(x, \kappa(x)) - \hat{f}(x, \kappa(\zeta_x)) + \hat{f}(x, \kappa(\zeta_x)) + \hat{f}(x, \kappa(\xi)) + w + q_x, \]
which can be written in compact form as
\[ \hat{f}(x, \kappa^*(x)) + w + q_x = \hat{f}(x, \kappa(\xi)) + d_{x,w} \]
(20)
with
\[ d_{x,w} \triangleq \hat{f}(x, \kappa^*(x)) - \hat{f}(x, \kappa(\xi)) + \hat{f}(x, \kappa(\xi)) + \hat{f}(x, \kappa(\zeta_x)) + w + q_x. \]

Using Assumption 1 we can notice that, \( \forall w \in W \) and \( \forall x \in \Xi \),
\[ \left| d_{x,w} \right| = \left| \hat{f}(x, \kappa^*(x)) - \hat{f}(x, \kappa(\xi)) + \hat{f}(x, \kappa(\xi)) + \hat{f}(x, \kappa(\zeta_x)) + \hat{f}(x, \kappa(\zeta_x)) + w + q_x \right| \leq \left| \hat{f}(x, \kappa^*(x)) - \hat{f}(x, \kappa(\xi)) + \left| \hat{f}(x, \kappa(\zeta_x)) + w + q_x \right| \leq \eta_u(|\kappa^*(\xi) - \kappa(\xi)|) + L_f q_x + |w| + |q_x| \leq \eta_u(|\kappa^*(\xi) - \kappa(\xi)|) + \eta_x(|\xi - \xi|) + |w|. \]
(22)

In view of Assumption 7 it follows that
\[ \left| d_{x,w} \right| \leq \overline{d}(\xi). \]
(23)

Since \( \hat{f}(x, \kappa(\zeta_x)) + d \in \Xi, \forall d \in B^n(\overline{d}(\xi)) \), then (20) and (23) together imply that
\[ \hat{f}(x, \kappa^*(x)) + w + q_x \in \Xi, \forall q_x \in Q, \forall w \in W. \]
(24)

Then, under Assumptions 1 and 7, for any \( x \in \Xi \), \( f(x, \kappa^*(x)) + w + q_x \in \Xi, \forall q_x \in Q, \forall w \in W. \)

ii) The ISS property for the c-1 system can be straightforwardly proven considering that, in view of Theorem 3.1 and taking in account inequality (22), the optimal finite horizon cost function satisfies the condition
\[ V(\hat{f}(x, \kappa^*(x)) + w + q_x) \leq -\alpha_3(|x|) + \sigma(|d_{x,w} + q_x|) \leq -\alpha_3(|x|) + \sigma(\eta_u(|\kappa^*(\xi) - \kappa(\xi)|) + \eta_x(|\xi - \xi|) + |w|). \]

where \( V(x_t) \) is the value of the FHOCP cost function starting from state \( x_t \) when the optimal control sequence is applied, while \( \alpha_3 \) and \( \sigma \) are two opportunistically chosen K-functional. Now, posing \( v_x = \kappa^*(x) - \kappa(\xi) \), in view of (23) and being \( \Xi \) compact, it holds that \( |v_x| \leq \mathcal{T} \). \( \forall x \in \Xi \) for some \( \mathcal{T} \in \mathbb{R}_{>0} \). Then we can conclude that
\[ V(\hat{f}(x, \kappa^*(x)) + w + q_x) \leq -\alpha_3(|x_t|) + \sigma(3\eta_u(|v_x|)) + \sigma(3\eta_x(|q_x|)) + \sigma(3|w|) \leq -\alpha_3(|x_t|) + \sigma_v(|q_x|) + \sigma_u(|w|) \]
where \( \sigma_v(s) \triangleq \sigma(3\eta_u(s)), \sigma_u(s) \triangleq \sigma(3\eta_x(s)), \sigma(|w|) \triangleq \sigma(3s) \in \mathbb{R}_{>0} \).

V. SMOOTH APPROXIMATION OF THE CONTROL LAW

It has been shown in [14] that considering Assumption 7 and Theorem 4.1 together, it is possible to easily deduce sufficient conditions to be imposed to a Nearest Neighbor-based approximation in order to stabilize the system. Now we are going to address other types of approximators, such as Neural Networks with smooth activation functions and smooth output function.

As in the Nearest Neighbor case, the domain has to be conveniently grided in such a way that the error committed in the reconstruction of the control law admits a known bound in a given training set. In the sequel we will describe how such a (possibly non uniform) grid \( X_G \), covering the region \( X \), has to be constructed.

**Assumption 8:** Given the set \( X \) and \( \overline{d}_q \in \mathbb{R}_{>0} \) satisfying (18), the set \( X_G \) verifies

1) \( \forall x \in X, \exists x_G \in X_G : |x - x_G| \leq \eta_x^{-1}(\overline{d}_q) \); 
2) \( \exists \psi_{NN} \in \mathbb{R}_{>0} : |x' - x''| \geq \psi_{NN}, \forall (x', x'') \in X_G^2 \),

where \( \eta_{NN} \) and \( \psi_{NN} \) are referred respectively as knot density and knot separation parameters (see [15] and references therein).

**Remark 5.1:** Notice that being \( X \) compact, point 2 implies that \( X_G \) is made by a finite number of knots. The cardinality of the training set grows with the decrease of \( d_q \), but being a lower limit on this scalar imposed by (18), there exists a finite upper bound on the knot density \( \eta_{NN} \).

Once the quantization (to be intended as spatial sampling)

1See the proof of Theorem 3.1 in [14].
nonlinear transition map $\hat{f}$ we can easily compute a local linear bound on this function: indeed, the problem of finding a global $K$–function $\eta_a(\cdot)$ is simplified in that of computing a local Lipschitz bound $L_{f_a}(\xi)$ such that

$$|\hat{f}(\xi, u) - \hat{f}(\xi, u')| \leq L_{f_a}(\xi)|u - u'|,$$

$\forall \xi \in \mathcal{B}^n(\xi, \varepsilon)$, $\forall (u, u') \in U^2$. A conservative bound on $L_{f_a}$ can be evaluated as

$$L_{f_a}(\xi) = \max_{(\xi, u) \in \mathcal{B}^n(\xi, \varepsilon)} \sum_{j=1}^{m} \frac{|\partial \hat{f}(\xi)|}{\partial u(j, \xi)}$$

where $\hat{f}(i)$ and $u(j)$ are the $i$–th and the $j$–th components of $f$ and $u$ respectively.

To conveniently design the network, it is possible to resort to the method proposed in [12], which allows to optimize the number and the position of the neurons when the tolerance–tube (in our case $\epsilon(\xi)$) varies in the domain.

As far as implementation is concerned, it is important to notice that, for most applications, the use of a Neural Network allows to save memory resources compared to the Nearest Neighbor approach: the net needs to store in memory just $(\text{num of neurons}) \times \text{(parameters per neuron)}$ elements, whilst the latter requires $(n \times \text{(num of points of the grid)}) \times m$ elements, where $n$ and $m$ denote the dimensions of the state and input respectively. Since the number of neurons needed to obtain a suitable approximator is usually far smaller than the number of reference points in the training set (which coincides with the grid set of the Nearest Neighbor), then a properly designed Neural Network can reduce consistently the memory consumption and the time required for on-line computations.

VI. SIMULATION RESULTS

Consider the following system (undamped nonlinear oscillator):

$$\begin{align*}
    x(t+1) &= x(t) + 0.05[-x(2) + 0.5(1 + x(1))u_1] \\
    x(2) &= x(2) + 0.05[4x(1) + 0.5(1 - 4x(2))u_2]
\end{align*}$$

subjected to constraints (7) and (8), with $U \triangleq \{u \in \mathbb{R} : |u| \leq 2\}$. The Lipschitz constant for the system is $L_{f_1} = 1.1390$. An auxiliary control law can be designed following i.e. [12], obtaining the feedback control law $u = \kappa f(x) = k^T x$, with $k^T = [0.5955 \ 0.9764]$; other choices accomplishing to our Assumptions are $N_{\varepsilon} = 8$,

$$X_f = \left\{ x_t \in \mathbb{R}^2 : x_T^T \begin{bmatrix} 167.21 & -43.12 \\ -43.12 & 305.50 \end{bmatrix} x_t \leq 1 \right\},$$

$$X_{N_{\varepsilon}} = \left\{ x_t \in \mathbb{R}^2 : x_T^T \begin{bmatrix} 114.21 & -29.45 \\ -29.45 & 208.67 \end{bmatrix} x_t \leq 1 \right\}.$$

The stage cost is $h(x, u) = x^T Q x + u^T R u$, while the final cost is $h_f(x) = x^T P x$, with $Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $R = 1$ and $P = \begin{bmatrix} 91.56 & -23.61 \\ -23.61 & 167.28 \end{bmatrix}$.

The exact MPC feedback function is depicted in Figure 1. The following bounds on additive uncertainty can be evaluated considering the invariance properties of the chosen terminal set: $\delta = 5.94 \cdot 10^{-4}$ and $\tilde{\eta}_{NN} = \bar{\eta} = 2.75 \cdot 10^{-4}$.
To carry out the off-line approximation, a Neural Network with two layers, 453 centers (mostly concentrated where the control function experiences a significant increase of the slope) and sigmoidal activation function is sufficient to fulfill the error bounds in the whole training set (the feasible training points belonging to the intersection $X_G \cap X_{RH}$ are depicted by red dots in Figure 2). Sample c-1 trajectories are shown in Figure 2: notably, the application of the approximated control renders the system ISpS, that is, the system driven by the approximate feedback is not guaranteed to asymptotically converge to the origin: such a behavior is due to the fact that the approximation errors do not vanish along the trajectories.

VII. CONCLUSION

This paper focuses on the off-line approximation of MPC feedback laws for constrained nonlinear discrete-time systems. The devised method allows to avoid on-line optimizations by using a smooth function approximation of the exact MPC law, while guaranteeing the practical stability of the resulting closed-loop system and the fulfillment of hard constraints on input and state variables; to this task, Neural Networks can be properly designed and trained on a specified set of points sampled in the state domain. Due to the low computational load required for the output evaluation, the approximated Neural Network feedback can be applied online to systems with fast dynamics; indeed, with respect to the existing Nearest Neighbor and Set Membership approximation schemes, the proposed technique leads to a dramatic reduction of memory and computational consumption. An example is provided to show the effectiveness of the devised method.

REFERENCES