Considerations choosing the Optimal Equilibrium Point on the Rotational Sphere

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Abstract—In order to save energy during attitude maneuvers the choice of optimal rotation is imperative, especially when considering applications where resources are inaccessible such as orbiting spacecraft. Normally the shortest rotation is considered preferable but in this paper we derive an analytical expression where the optimal rotational direction is found by performing a simple check. A passivity based PD+ controller is derived and the equilibrium points of the closed-loop system are shown uniformly asymptotically stable. We apply the control law to achieve simulation results showing the performance of the presented scheme where different initial conditions are considered, comparing power consumption for rotational maneuvers in opposite directions.

I. INTRODUCTION

It is a fact that working with attitude on $SO(3)$ is a challenging task due to the parametrization of the attitude for both Euler angles and the unit quaternion, resulting in multiple equilibria and possibilities of the unwinding phenomenon. For control of a rigid body the demands are in contrast; faster and more accurate settling using minimum energy. Thus a lot of different controllers have been developed during the years focusing on enhancing the performance while guaranteeing robust stability and minimizing the control effort. Optimal control is a vital part of this field of study obtaining good results weighted between control input and performance, but at a cost: solving the optimal control problem for highly nonlinear systems is a resource consuming task and may not even be possible where resources are sparse such as on board a spacecraft and therefore other techniques must be considered.

The optimal control problem of a rigid body has received a great deal of attention mainly from the interest in the control of rigid spacecraft. Most of this research has been directed towards time, and control-input optimization, however such as in [1] geometric control theory was applied to reduce the overall angular velocity during a maneuver to avoid tumbling of the spacecraft using Pontryagin’s Maximum Principle. In [2] the problem of minimizing a quadratic cost function was solved for, obtaining conditions which guarantee the existence of linear stabilizing optimal and suboptimal controllers, but only for rotational motion and not attitude.

The problem was further pursued in [3] where optimal feedback regulation was applied for an axially-symmetric rigid body, and both angular velocity and orientation were regulated. The control laws only considered the location of the symmetry-axis along a specific direction such as spin-stabilized spacecraft. The axis-symmetric problem was also investigated in [4] where body fixed actuators and open end time is considered, performing reduction of the Bellman-Krotov equation using passive integrals and solving the optimal control in the form of the cyclic sliding mode. [5] implements the Successive Galerkin Approximation to approximate the optimal control problem with penalty on the angular velocity, attitude and control signal, while the problem of minimal fuel consumption with body-fixed directions of the control torques and unconstrained time was solved in [6] using the mentioned method of passive integrals. The time optimal control problem has been further studied such as in [7] to minimize time according to desired attitude and angular velocity while subject to constraints on the control input, presenting necessary conditions for optimality.

The nonlinear nature of the tracking control problem has been a challenging task in robotics and control research. The passivity-based approach to robot control have gained much attention, which, contrary to computed torque control, couple with the robot control problem by exploiting the physical structure of the robots [8]. A simple solution to the robot position control problem using the passivity approach was proposed by [9], and the natural extension to tracking control was solved in [10] using the known passivity based PD+ controller. A similar control law was adopted for the spacecraft attitude control problem [11], where the authors claim global results. However, the problem of multiple equilibrium points was later identified, and in [12] the author presented an attitude error function choosing the closest equilibrium point at any time using a signum function. This was later developed in [13] where the closest equilibrium point was chosen a priori and kept throughout the maneuver. Another approach was presented by [14] using quaternion-based hybrid feedback where the choice of rotational direction is performed by a switching control law, which may lead to varying results when e.g. energy consumption is considered. The work of predicting preferable equilibrium point was pursued in [15] where parameters from the initial unit quaternion were analyzed showing promising results.

In this paper we derive a simple test to check which rotational direction for an attitude maneuver is preferable from a control input and final attitude error point of view. The test is based on optimal control of the linearized equations of motion with initial attitude and angular velocity as its input parameters. A passivity-based PD+ controller is derived and applied to the nonlinear dynamics, and utilized through simulations to show that the presented scheme is capable
of predicting the cheapest rotational direction in most cases even if it is based on linearization.

II. MATHEMATICAL BACKGROUND

In the following, we denote by $\dot{x}$ the time derivative of a vector $x$, i.e. $\dot{x} = dx/dt$, and moreover, $\ddot{x} = d^2x/dt^2$. The cross product operator $a \times b$ is denoted $\mathbf{S}(a)b$, $\omega_{a,b}^b$ is the angular velocity of frame $a$ relative frame $b$, expressed in frame $c$, $R^b_c$ is the rotation matrix from frame $a$ to frame $b$, and $\| \cdot \|$ denotes the 2-norm on $\mathbb{R}^n$. Reference frames are denoted by $F^i$, where the superscript denotes the frame. Especially two frames should be noted: $F^1$ is a fixed right handed orthonormal frame while $F^b$ is the body frame of the rigid body, located at its center of mass, and its basis vectors are aligned with the principle axis of inertia. When the context is sufficiently explicit, we may omit the arguments of a function, vector or matrix.

A. Quaternions

The attitude of a rigid body is often represented by a rotation matrix $R \in SO(3)$ fulfilling

$$SO(3) = \{ R \in \mathbb{R}^{3 \times 3} : R^T R = I, \det R = 1 \},$$

which is the special orthogonal group of order three, where $I$ denotes the identity matrix. A rotation matrix for a rotation $\theta$ about an arbitrary unit vector $k$ can be angle-axis parameterized as $[16]

$$R_{k,\theta} = I + \mathbf{S}(k) \sin \theta + \mathbf{S}^2(k)(1 - \cos \theta)$$

(2)

and coordinate transformation of a vector $r$ from frame $a$ to frame $b$ is written as $r^b = R^b_a r^a$. The rotation matrix in (2) can be expressed by an Euler parameter representation as

$$R = I + 2\eta \mathbf{S}(\epsilon) + 2\mathbf{S}^2(\epsilon),$$

(3)

where the matrix $\mathbf{S}(\cdot)$ is the cross product operator

$$\mathbf{S}(\epsilon) = \epsilon \times = \begin{bmatrix} 0 & -\epsilon_z & \epsilon_y \\ \epsilon_z & 0 & -\epsilon_x \\ -\epsilon_y & \epsilon_x & 0 \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{bmatrix}. \quad (4)$$

Quaternions are often used to parameterize members of $SO(3)$ where the unit quaternion is defined as $q = [\eta, \epsilon^T]^T \in S^3 = \{ x \in \mathbb{R}^4 : x^T x = 1 \}$, where $\eta = \cos(\theta/2) \in \mathbb{R}$ is the scalar part and $\epsilon = k \sin(\theta/2) \in \mathbb{R}^3$ is the vector part. The inverse rotation can be performed by using the inverse conjugated of $q$ as $\overline{q} = [\eta, -\epsilon^T]^T$. The set $S^3$ forms a group with quaternion multiplication, which is distributive and associative, but not commutative, and the quaternion product is defined as

$$q_1 \otimes q_2 = \begin{bmatrix} \eta_1 \eta_2 - \epsilon_1 \epsilon_2 \\ \eta_1 \epsilon_2 + \eta_2 \epsilon_1 + \mathbf{S}(\epsilon_1)\epsilon_2 \end{bmatrix}. \quad (5)$$

B. Kinematics and Dynamics

The time derivative of the rotation matrix is written as $[16]

$$\dot{R}^b_a = \mathbf{S}(\omega_{a,b}^b) R^b_a = R^b_a \mathbf{S}(\omega_{a,b}^b),$$

(6)

where $\omega_{a,b}^b \in \mathbb{R}^3$ is the angular velocity of a frame $F^b$ relative to a frame $F^a$, expressed in frame $F^a$. The kinematic differential equations may be expressed as

$$\dot{q} = T(q)\omega, \quad T(q) = \frac{1}{2} \begin{bmatrix} -\eta^T \\ \eta I + \mathbf{S}(\epsilon) \end{bmatrix} \in \mathbb{R}^{4 \times 3}. \quad (7)$$

The dynamical model of a rigid body can be described by a differential equation for angular velocity, and is deduced from Euler’s moment equation. This equation describes the relationship between applied torque and angular momentum on a rigid body as $[17]

$$J \ddot{\omega} = -S(\omega)J \omega + \tau,$$

(8)

where $\omega = \omega_{a,b}^b$ is the angular velocity of the body frame $F^b$ relative to an inertia frame $F^i$, expressed in the body frame, $\tau \in \mathbb{R}^3$ is the total torque working on the body frame, and $J \in \mathbb{R}^{3 \times 3} = \text{diag}\{ J_z, J_y, J_x \}$ is the inertia matrix. The torque working on the body is derived from two parameters, where $\tau_d$ is the disturbance torque, and actuator (control) torque $\tau_a$, such as $\tau = \tau_a + \tau_d$.

C. Optimal Control

To solve a continuous nonlinear optimal control problem, a set of known equations are solved for, e.g. $[18], [19]$. The system model can be written as

$$\dot{x} = f(x, u, t), \quad t \geq t_0,$$

(9)

and a performance index may be defined as

$$J(t_0) = \phi(x(T), T) + \int_{t_0}^T \mathcal{L}(x, u, t) dt.$$

(10)

A Hamiltonian is defined as

$$\mathcal{H}(x, u, t) = \mathcal{L}(x, u, t) + \lambda^T f(x, u, t),$$

(11)

while the costate equation is expressed as

$$-\dot{\lambda} = \frac{\partial \mathcal{H}}{\partial x} = \frac{\partial f^T}{\partial x} \lambda + \frac{\partial \mathcal{L}}{\partial x}, \quad t \leq T$$

(12)

and the stationary condition is defined as

$$0 = \frac{\partial \mathcal{H}}{\partial u} = \frac{\partial \mathcal{L}}{\partial u} + \frac{\partial f^T}{\partial u} \lambda. \quad (13)$$

As most nonlinear problems are impossible to solve in an analytical way we usually have two options: either linearize the system model, or use a numerical algorithm. Both methods have pros and cons, a linearized problem is fast and easy to solve but will not perform well for highly nonlinear equations, while a numerical algorithm for a large set of equations usually requires a lot of time to solve.

III. CART EXAMPLE

To clearly illustrate the main motivation of this paper, we present an example where a cart needs to choose between two ending points causing the cheapest travel. The dynamics of a cart moving in one direction without influence of friction is expressed as $F = ma$ where $F$ is the control force, $m$ is the masse of the cart, and $a$ is the acceleration. The velocity of the cart is expressed as the time derivative of the position, which gives us the state space representation

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = \frac{1}{m} u(t),$$

(14)
where $x_1(t)$ is the position and $x_2(t)$ the velocity of the cart, and $u(t)$ is the control signal. The initial states of the cart at $t_0$ are given as $x_1(t_0) = x_{10}$ and $x_2(t_0) = x_{20}$, and the final states at $t_f$ are $x_1(t_f) = x_{1f}$ and $x_2(t_f) = x_{2f}$. Assume that the cart starts at a position restricted by $-p \leq x_{10} \leq p$, where $p \in \mathbb{R}$ is an arbitrary distance. The problem is to find which way the cart should be moving, either towards $-p$ or $p$ to minimize the position error and energy consumption. This is done by considering two cost functionals, one for each ending point

$$J = \frac{1}{2} q(x_{1f} \pm p)^2 + \int_{t_0}^{t_f} \frac{1}{2} ru^2 dt,$$  \quad (15)

where $q$ and $r$ are weighting constants. The final value of the adjoint vector is

$$\lambda(t_f) = \begin{bmatrix} q(x_{1f} \pm p) \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1(t_f) \\ \lambda_2(t_f) \end{bmatrix},$$ \quad (16)

while the differential equation that describes the time history of the adjoint vector is according to (12)

$$\dot{\lambda}(t) = -\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_1(t) \end{bmatrix}.$$ \quad (17)

Solving (17) by integration from $t_f$, results in

$$\lambda_1(t) = \lambda_1(t_f) + \int_{t_f}^{t} (0) dt = q(x_{1f} \pm p)$$ \quad (18)

$$\lambda_2(t) = \lambda_2(t_f) - \int_{t_f}^{t} \lambda_1(t_f) dt = q(x_{1f} \pm p)(t_f - t),$$ \quad (19)

and then the control history is found by applying (13), such that

$$ru(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \begin{bmatrix} q(x_{1f} \pm p) \\ q(x_{2f} \pm p)(t_f - t) \end{bmatrix} = 0$$ \quad (20)

and solving (21) for the final position at $t_f$ yields

$$x_{1f} = \frac{x_{10} + x_{20}(t_f - t_0) \pm pT}{1 + T},$$ \quad (22)

where

$$T = \frac{q}{rm^2} \left[ -\frac{1}{3} t_f^3 + \frac{t_f^2 t_0}{2} - t_f t_0^2 + \frac{1}{3} t_0^3 \right].$$ \quad (23)

Define the position error as $\tilde{x}_e = | \pm p - x_{1f} |$, and assume that the cart stands still at $x_{10} = 0$ where $t_0 = 0$ with $q = r = m = 1$, $p = 100$ and $t_f = 10$. This gives an error of 0.3009 for both $\pm p$, in other words it is indifferent which way the cart moves. If the cart stands still at $x_{10} = 0$ the error for $-p$ is 0.3310 while for $p$ it is 0.2708, which means that the closest point is the optimal one. If the cart now starts at the exact same spot but has an initial velocity of $x_{20} = -3$ we get an error of 0.2407 for $-p$ while 0.3611 for $p$. Hence, because of the initial velocity the farthest point is the optimal one.

### IV. CHOICE OF EQUILIBRIUM

In this section we present a scheme to decide which rotational direction gives the cheapest rotation where control input and final attitude error are considered, within a given time frame, similar to the cart example given in Section III.

#### A. Problem Formulation

The control problem is to design a controller that makes the state $q(t)$ converge towards the generated reference specified as $q_d$ satisfying the kinematic equation

$$\dot{q}_d = T(q_d) \omega_d.$$ \quad (24)

The error quaternion $\tilde{q} = [\tilde{\eta}, \tilde{\epsilon}^T]^T$ is found by applying the quaternion product

$$\tilde{q} = q \otimes \tilde{q}_d = \begin{bmatrix} \eta_d \epsilon - \eta_d^2 - S(\epsilon) \epsilon_d \\ \eta_d \epsilon + \delta \epsilon_d \end{bmatrix},$$ \quad (25)

and the error dynamic can according to [12] be expressed as

$$\dot{\tilde{q}} = \frac{1}{2} T(\tilde{q}) \tilde{\omega}.$$ \quad (26)

It should be noted that the quaternion representation is an inherent redundant representation, giving two different equilibria in the closed-loop system, namely $\tilde{q}_{+} = [\pm 1, 0^T]^T$, which both represent the same physical representation, however one is rotated $2\pi$ relative to the other about an arbitrary axis. Often is $\tilde{q}_{+} = [1, 0^T]^T$, while for the negative equilibrium point $\tilde{q}_{-} = [-1, 0^T]^T$, which can be viewed upon as a coordinate transformation to ensure that the system is driven towards origo during stability analysis.

As the obvious choice of equilibrium point would be to define two cost functions $J_1 = e_{r+}^T Q e_{r+} + \int_{t_0}^{t_f} \tau_a^T R \tau_a dt$ and $J_2 = e_{r-}^T Q e_{r-} + \int_{t_0}^{t_f} \tau_a^T R \tau_a dt$, where $Q \in \mathbb{R}^{4 \times 4} = \text{diag}(Q_1, Q_2, Q_3, Q_4)$ and $R \in \mathbb{R}^{7 \times 7} = \text{diag}(R_1, \ldots, R_7)$, and then choose the equilibrium point requiring the cheapest rotation, such that $J = \min\{J_1, J_2\}$, this would require us to solve two optimal control problems which is far too computational consuming for most tasks such as spacecraft attitude maneuvers. Therefore we have to apply some simplification which makes it possible to solve the problem in a more convenient way.

#### B. Linearization

First we start by linearizing the equations of motion for a rigid body according to [20]

$$\dot{x}(t) \approx A x(t) + B \tau_a(t),$$ \quad (27)

where $x_1(t) = \dot{q}(t)$, $x_2(t) = \dot{\omega}(t)$, $x(t) = [x_1(t)^T, x_2(t)^T]^T$, $A = \frac{\partial f(x, \tau_a)}{\partial x}$, $B = \frac{\partial f(x, \tau_a)}{\partial \tau_a}$ \quad (28)
around the equilibrium points \((q, \omega) = (\pm 1, 0, 0)\). By differentiation of (7) and (8) we obtain

\[
\dot{x} = \pm \frac{1}{2} \begin{bmatrix} 0_{4 \times 4} & 0_{3 \times 4} & 0_{3 \times 3} \\ 0_{3 \times 4} & I_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} x + \begin{bmatrix} 0_{4 \times 7} & 0_{4 \times 4} \end{bmatrix} \tau_a. \quad (29)
\]

C. Optimization

Using an attitude error vector defined as \(e_{q \pm} = [1 \mp \bar{\eta}, \tilde{e}^T]^T\) that means we evaluate both the positive and negative rotation around the rotational axis. Instead we rotate by \(\theta\) for \(e_{q +}\) according to \(\hat{\eta} = \cos(\theta/2)\) and \(\tilde{e} = k \sin(\theta/2)\) and for the opposite rotation we rotate by \(\phi = \theta - 2\pi\) for \(e_{q -}\) yielding \(-\tilde{q}\) and thus an error vector defined as \(e_{q \pm} = [1 - \bar{\eta}, \pm \tilde{e}]^T\). This is done because linearization causes the first row of A to disappear which results in loss of \(\bar{\eta}\) during optimization. A cost function is defined as

\[
J = \frac{1}{2} e_{q \pm}^T Q e_{q \pm} + \frac{1}{2} \int_{t_0}^{t_f} \tau_a^T R \tau_a dt, \quad (30)
\]

the final value of the adjoint vector is found to be

\[
\lambda(t_f) = \begin{bmatrix} Q e_{q \pm}(t_f) \\ 0 \end{bmatrix}, \quad (31)
\]

and solving the adjoint vector according to (12) yields

\[
\lambda_1(t) = Q e_{q \pm}(t_f), \quad \lambda_2(t) = \frac{1}{2} Q \hat{e}_f(t_f - t), \quad (32)
\]

where \(\hat{Q} = \text{diag}\{Q_2, Q_3, Q_4\}\) and \(\tilde{e}_f = \hat{e}(t_f)\). The control history is found by applying (13) resulting in

\[
R \tau_a + B^T \left[ \frac{1}{2} Q e_{q \pm}(t_f - t) \right] = 0 \quad (33)
\]

\[
\tau_a = -\frac{1}{2} R^{-1} \begin{bmatrix} 0 \\ J^{-1} Q \hat{e}_f \end{bmatrix}(t_f - t), \quad (34)
\]

which leads to the state equation

\[
\dot{x}(t) = Ax - \frac{1}{2} B R^{-1} \begin{bmatrix} 0 \\ J^{-1} Q \hat{e}_f \end{bmatrix}(t_f - t). \quad (35)
\]

Solving (35) yields

\[
x_2(t) = x_2(t_0) - \frac{1}{2} J^{-1} J^{-1} \hat{e}_f \left[ t_f(t_f - t) - \frac{1}{2}(t_f^2 - t_0^2) \right] \quad (36)
\]

where \(\hat{R} = \text{diag}\{R_5, R_6, R_7\}\), and

\[
x_1(t) = x_1(t_0) \pm \frac{1}{2} \int_{t_0}^{t_f} x_2(t) dt \quad (37)
\]

As we are interested to find out which equilibrium point we want the dynamical system to converge to resulting in the smallest attitude error, we consider \(\hat{e}_f\) since \(\bar{\eta}\) is lost, evaluated at \(t = t_f\) yielding

\[
\hat{e}_f \pm = H^{-1} \left[ \hat{e}_0 \pm \frac{1}{2} \hat{\omega}_0 \tau_f \right], \quad (38)
\]

where \(\hat{e}_0 = e(t_0)\) and \(\hat{\omega}_0 = \omega(t_0)\), and

\[
H = I \pm \frac{1}{6} J^{-1} J^{-1} J^{-1} Q \tau_f^3. \quad (39)
\]

The weighting matrices \(Q\) and \(\hat{R}\) and time constant \(t_f\) should be selected such that \(\text{rank}(H) = 3\). We summarize the result in the following theorem.

**Theorem 4.1:** To find the cheapest rotational direction for the system (7), (8), according to the cost function defined in (30) we choose the equilibrium point satisfying either \(e_{q +}\) if \(\tilde{\eta} \geq 0\) or \(e_{q -}\) if \(\tilde{\eta} < 0\) for \(\omega_0 = 0\), or \(e_{q +}\) if \(\|\tilde{e}_f\| \leq \|\tilde{e}_f - \|H\|\) or \(e_{q -}\) if \(\|\tilde{e}_f\| < \|\tilde{e}_f - \|H\|\) for \(\omega_0 \neq 0\), according to (38) and (39), where \(e_{q \pm}\) denotes positive and negative equilibrium point respectively.

**Remark 4.1:** The time constant \(t_f\) can be considered as a weighting variable between initial attitude and initial angular velocity. A larger time constant tends to utilize the initial angular velocity.

**Remark 4.2:** As our result is achieved by performing linearization of the system equations the resulting equilibrium point will not necessarily be the optimal one. Nevertheless, we observe that the preferable equilibrium is chosen in most cases when simulations of the full non-linear model are performed, and solutions in the boundary area \(||\tilde{e}_f|| \approx \|\tilde{e}_f - \|H\||\) tends to provide little fuel saving or none at all if either point is chosen compared to the other.

**Remark 4.3:** It is assumed that the rigid body has available torque to stop the rotation at the chosen equilibrium point given the initial angular velocity.

V. CONTROLLER DESIGN

For control of the rigid body attitude, we incorporate a model-dependent tracking control law as in [11]. The reason for this is that even if the controller is tracking a given reference, we argue that the choice of equilibrium point can be done based on the errors between initial conditions for dynamics and reference. For the control law it is assumed that we have available information of its attitude \(q\), angular velocity \(\omega\), inertia matrix \(J\) and that all perturbations are known and accounted for. In the following it is assumed that the equilibrium point is chosen according to \(e_{q \pm} = [1 \mp \bar{\eta}, \tilde{e}]^T\) is either the positive or negative equilibrium point and doesn’t change during the maneuver. We define desired attitude \(q_d(t)\), desired angular velocity \(\omega_d(t)\) and desired angular acceleration \(\dot{\omega}_d(t)\) which are all assumed to be bounded functions. The control law is expressed as

\[
\tau_a = J \dot{\omega}_d - S(J \omega) \omega_d - k_\omega T_e e_\omega - k_e e_\omega - \tau_d \quad (40)
\]

where \(k_\omega > 0\) and \(k_e > 0\) are feedback gain scalars, \(e_\omega = \omega - \omega_d\) are the angular velocity error and in accordance with general kinematic relations

\[
\dot{e}_\omega = T_e(\varepsilon_{q \pm}) e_\omega, \quad T_e(\varepsilon_{q \pm}) = \frac{1}{2} \left[ \hat{\eta} I + S(\hat{\varepsilon}) \right]. \quad (41)
\]

By insertion of (40) into (8) we obtain the closed-loop rotational dynamics

\[
J \dot{e}_\omega + (k_\omega - S(J \omega)) e_\omega + k_e T_e e_\omega = 0, \quad (42)
\]

by applying the property \(S(\omega)J \omega = -S(J \omega)\omega\). A radial unbounded, positive definite Lyapunov function candidate is defined as

\[
V = \frac{1}{2} (e_\omega^T J e_\omega + e_\omega^T k_e e_\omega) > 0, \quad \forall e_\omega \neq 0, \quad e_q \neq 0, \quad (43)
\]
and by differentiating of (43) and inserting (42) and (41) we obtain
\[
\dot{V} = e_q^\top k_q T_e e_\omega + e_q^\top (S(J_\omega) - k_\omega) e_\omega - e_q^\top k_q T_e^\top e_q ,
\] (44)
where the first part of the second term in equation (44) is zero because \(S(J_\omega)\) is a skew-symmetric matrix, which leads to
\[
\dot{V} = -e_q^\top k_q e_\omega .
\] (45)
Hence, from standard Lyapunov theorems (cf. [21]), it follows that the equilibrium point \((e_q, e_\omega) = (0, 0)\) is uniformly asymptotically stable (UAS). By introducing an auxiliary function
\[
W = e_q^\top T_e^2 k_q^2 T_e^\top e_q ,
\] (46)
we obtain a negative definite derivative
\[
\dot{W} = -e_q^\top T_e^2 k_q^2 T_e^\top e_q
\] (47)
in the set \(E : \{\dot{V} = 0\} = \{e_\omega = 0\}\). By employing Lyapunov arguments we find that the closed-loop system in equation (42) is uniformly asymptotically stable (UAS) in the equilibrium point \((e_q, e_\omega) = (0, 0)\).

We are not able to achieve global representation since the term global refers to the whole state space \(\mathbb{R}^n\) according to [22], which is not the case due to the redundancy in the quaternion representation.

**Remark 5.1:** For the proof it is assumed that the scalar parameter of the quaternion does not change sign, so that \(\text{sgn}(\tilde{\eta}(t_0)) = \text{sgn}(\tilde{\eta}(t))\) to ensure that \(\tilde{\eta} \rightarrow 1\) for \(e_q^+\) and \(\tilde{\eta} \rightarrow -1\) for \(e_q^-\) respectively. As the choice of equilibrium point according to theorem 4.1 opens for a change in the sign of \(\tilde{\eta}\) during a maneuver the mentioned assumption does not hold, but as remarked in [13] it is imposed for technical reasons to obtain negative definite bounds on Lyapunov function derivatives, and violation of the assumption doesn’t seemingly cause any trouble from a practical point of view.

VI. SIMULATIONS

In the following, the performance of our equilibrium test is illustrated by presenting simulation results of a rigid body with arbitrary initial values, where power consumption are compared for both equilibrium points. The simulations were performed in Simulink using a fixed sample-time Runge-Kutta ODE4 solver, with sample time of \(1 \times 10^{-2}\) s. The rigid body inertia matrix was set to \(J = \text{diag}\{4.35, 4.33, 3.664\}\) kgm\(^2\), the controller gains \(k_q = 1, k_\omega = 2\), and all disturbances were omitted to better illustrate the purpose of our contribution. With these initial conditions a maneuver is typically performed within \(30\) s so \(t_f = 30\) s.

All simulation results are summarized in Table I where we have presented the initial attitude as Euler angles (in degrees to give a better understanding of the physical orientation) and initial angular velocities. The total power consumption is calculated using \(P_{f\pm} = \int_{t_0}^{t_f} \tau_{\pm}^\top \tau_{\pm} dt\) for choosing the positive and negative equilibrium point, and \(|\epsilon_{f\pm}|\) is found according to (38). We do not present simulation results without initial velocity because the result is obvious, shortest rotation gives cheapest maneuver.

The first simulation is presented in Figure 1 where the quantity of initial rotational error is equal for both equilibrium points, but with a small initial velocity which favor the negative equilibrium point. Choosing the optimal equilibrium gives 14.88 percent energy saving which is a considerable amount especially for applications such as spacecraft maneuvers where the fuel cost is extremely high.

The second simulation is presented in Figure 2 and shows a maneuver where the positive equilibrium have the shortest path and the initial velocity is about another axis. The result shows that the positive equilibrium also is the optimal one, which should be of no surprise.

In the third simulation we choose an arbitrary attitude and angular velocity where it apparently is hard to tell which equilibrium is the optimal one. Even though the path length for the positive equilibrium is the greater and the angular velocity is higher during all parts of the simulation, it is confirmed that the positive equilibrium is the optimal equilibrium and result in 3.12 percent energy saving, as can be seen in Figure 3.

The choice of optimal equilibrium tends to favor the positive equilibrium point when small initial angular velocities are considered combined with a low value for \(t_f\), even if the

<table>
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<th>Initial values and results</th>
<th>Sim. 1</th>
<th>Sim. 2</th>
<th>Sim. 3</th>
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<tbody>
<tr>
<td>(\phi) [deg]</td>
<td>180</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>(\theta) [deg]</td>
<td>0</td>
<td>90</td>
<td>-130</td>
</tr>
<tr>
<td>(\psi) [deg]</td>
<td>0</td>
<td>0</td>
<td>150</td>
</tr>
<tr>
<td>(\omega_x) [rad/s]</td>
<td>0.01</td>
<td>0.01</td>
<td>-0.01</td>
</tr>
<tr>
<td>(\omega_y) [rad/s]</td>
<td>0</td>
<td>0</td>
<td>0.04</td>
</tr>
<tr>
<td>(\omega_z) [rad/s]</td>
<td>0</td>
<td>0</td>
<td>0.02</td>
</tr>
<tr>
<td>(</td>
<td></td>
<td>\epsilon_{f+}</td>
<td></td>
</tr>
<tr>
<td>(</td>
<td></td>
<td>\epsilon_{f-}</td>
<td></td>
</tr>
<tr>
<td>(P_{f+})</td>
<td>0.3358</td>
<td>0.1402</td>
<td>0.3015</td>
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<tr>
<td>(P_{f-})</td>
<td>0.2923</td>
<td>0.2221</td>
<td>0.3109</td>
</tr>
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</table>

Fig. 1. Comparison in power consumption between choosing positive and negative equilibrium during an attitude maneuver where rotational error is equal but with initial angular velocity error of \(\omega = [0.01, 0, 0]^\top\).
path length is greater. An initial value of \( \omega = [10^{-4}, 0, 0]^\top \) using initial attitude as in the first simulation will favor the positive equilibrium even if the negative is the optimal, but simulation result shows that the energy consumption for both points are 0.3138 and 0.3134 for positive and negative equilibrium point respectively which means that only 0.13 percent differs. By increasing to \( t_f = 300 \) s moves the switch of equilibrium point down to about \( \omega = [10^{-8}, 0, 0]^\top \) before the non-optimal is chosen, which will result in theoretical difference in energy consumption.

VII. CONCLUSIONS

In this paper we have presented a new scheme for choosing the optimal equilibrium point for rigid bodies working on the rotational sphere where both initial attitude and angular velocity are considered, based on optimization of final attitude error and energy consumption. The derivation was performed using a linearization technique to ensure an analytical solution, and simulation results showed good results concerning energy consumption during attitude maneuvers.

VIII. ACKNOWLEDGMENTS

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REFERENCES