Adaptive Tracking Control of Underactuated Quadrotor Unmanned Aerial Vehicles via Backstepping

Mu Huang, Bin Xian, Chen Diao, Kaiyan Yang, and Yu Feng

Abstract—This paper considers about the control problem for an underactuated quadrotor UAV system with model parameter uncertainty. Backstepping based techniques are utilized to design a nonlinear adaptive controller which can compensate for the mass uncertainty of the vehicle. Lyapunov based stability analysis shows that the proposed control design yields asymptotic tracking for the UAV’s motion in $x$, $y$, $z$ direction and the yaw rotation, while keep the stability of the closed loop dynamics of the quadrotor UAV. Numerical simulation results are provided to show the good tracking performance of proposed control laws.

I. INTRODUCTION

The automatic control of a quadrotor UAV is not a straight on mainly due to its underactuated properties [1]. The dynamic model of quadrotor UAV has six degree-of-freedom (DOF) with only four independent thrust forces generated by four rotors. It is difficult to control all these six outputs with only four control inputs. Moreover, uncertainties associate with dynamic model also bring more challenge for control design. Different strategies have been proposed to deal with uncertain quadrotor model, such as adaptive control, neural network based control, sliding mode control, $H_\infty$ control and so on. In [2], a direct adaptive control algorithm was designed for the tracking control of a quadrotor UAV’s roll, pitch, yaw angles, together with altitude while compensating for the model parameter uncertainties. A reference system corresponding to a virtual UAV which contains a third order oscillator was utilized to track the desired trajectory. In [3], a backstepping based approach was used for quadrotor UAV control, while two neural networks were used to approximate the uncertain aerodynamic components. By dividing the quadrotor’s dynamic model into an underactuated subsystem and a full actuated system, [4] designed a sliding mode controller for the underactuated system and a bounded PID controller for the full actuated system, these two controller drove the UAV to reach a desired position with a desired yaw angle while keep roll and pitch angles zero. More literature review for quadrotor UAV control can be found in [5].

One of the main parameter uncertainties associated with the quadrotor UAV’s dynamic model is the unknown mass parameter due to different payloads the UAV would take in different flight missions. It will be unpractical to measure the mass value of the vehicle together with its payload during each flight. In this paper, we propose a novel nonlinear adaptive control design without the knowledge of the quadrotor UAV’s mass parameter. Motivated by the techniques proposed in [6] and [7], we divide the quadrotor UAV’s dynamic model into four subsystem: one underactuated sub-systems, two full actuated sub-systems, and one subsystem about the dynamics of thrust forces. Backstepping approach together with parameter adaptive design are combined to design the adaptive controller. Projection operators are employed to ensure that some parameter estimates remained bounded to avoid singularity issues. Lyapunov based analysis is employed to prove that the proposed control design in this paper is able to drive the UAV to track time-varying desired trajectory of $x$–$y$–$z$ and yaw rotation under mild assumption about roll and pitch angles.

The rest of this paper is organized as follows. Section 2 presents the kinematic and dynamic model of quadrotor UAV system, and coordinate transfer. Problem statement together with adaptive control design are provided in Section 3. In Section 4, Lyapunov based analysis is utilized to prove the stability of closed loop system, and the convergence of tracking errors. Numerical simulation results are presented in Section 5 to testify the proposed controller. Finally, conclusion remarks were given in Section 6.

II. QUADROTOR UAV KINEMATIC AND DYNAMIC MODELS

The schematic of a quadrotor UAV is shown in Figure 1. It contains four rotors which can generate four identical thrust forces denoted by $f_i(t)$, $i=1,2,3,4$, respectively. More details about flight properties of a quadrotor UAV can be found in [8], [6], and [9].

A. Kinematic Model

Let $B$ denotes the body fixed frame attached to the quadrotor UAV, and $I$ denotes the inertial frame. The Euclidean position of the UAV with respect to $I$ is represented by $P(t) = [x(t) \ y(t) \ z(t)]^T \in \mathbb{R}^3$, the Euler angle of the UAV with respect to $I$ is represented by $\Theta(t) = [\phi(t) \ \theta(t) \ \psi(t)]^T \in \mathbb{R}^3$. The rotation matrix from $B$ to $I$ is represented by $R \in \mathbb{R}^{3\times3}$ as follows [8]

$$R = \begin{bmatrix} c_\phi c_\psi & s_\phi s_\psi c_\theta - s_\psi c_\theta & c_\phi s_\psi + s_\phi c_\psi c_\theta \\ s_\phi c_\psi & s_\phi s_\psi s_\theta + c_\psi c_\theta & c_\phi s_\psi c_\theta - s_\phi c_\psi c_\theta \\ -s_\psi & c_\psi s_\theta & c_\phi c_\theta \end{bmatrix}.$$  (1)

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The translational and rotational kinematic equations with respect to the inertial frame $I$ are given by
\[ \begin{align*}
\dot{P} &= Rv \\
\dot{\Theta} &= T \omega ,
\end{align*} \]  
(2)
where $v = [v_1 v_2 v_3]^T \in \mathbb{R}^3$ and $\omega = [\omega_1 \omega_2 \omega_3]^T \in \mathbb{R}^3$ denote the linear velocity and angular velocity of the UAV with respect to the inertial frame $I$ expressed in the body fixed frame $B$, the rotation velocity transfer matrix $T \in \mathbb{R}^{3 \times 3}$ is given by [6]
\[ T = \begin{bmatrix} 1 & 0 & -s_\theta \\
0 & c_\phi & c_\theta s_\phi \\
0 & -s_\phi & c_\phi c_\theta \end{bmatrix}, \]  
(3)
and $S(\cdot) \in \mathbb{R}^{3 \times 3}$ is a general form of some skew symmetric matrix defined as follows
\[ \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\
\xi_3 & 0 & \xi_1 \\
-\xi_2 & -\xi_1 & 0 \end{bmatrix} \forall \xi = \begin{bmatrix} \xi_1 \\
\xi_2 \\
\xi_3 \end{bmatrix}. \]  
(4)

B. Dynamic Model

The rigid body dynamics of the underactuated quadrotor UAV given in inertial frame $I$ can be described by the following equations [6], [8]
\[ \begin{align*}
mR \dot{P} + \dot{T} + mR^T \dot{P} + mR^T G &= F \\
JT \dot{\Theta} + J(\frac{\partial T}{\partial \phi} \dot{\phi} + \frac{\partial T}{\partial \theta} \dot{\theta}) + K_1 \dot{\Theta} + S(\dot{\Theta})JT \dot{\Theta} &= \tau ,
\end{align*} \]  
(5)
where $m \in \mathbb{R}$ represents the constant mass of the UAV, $J = \text{diag} \{ J_{11} J_{22} J_{33} \} \in \mathbb{R}^{3 \times 3}$ represents the constant moment of inertia of the UAV expressed in inertial frame $I$, $G = [0 \ 0 \ g]^T \in \mathbb{R}^3$ denotes the gravity vector ($g = 9.8 \text{m} \cdot \text{s}^{-2}$), $K_1 \in \mathbb{R}^{3 \times 3}$ and $K_r \in \mathbb{R}^{3 \times 3}$ represent constant, diagonal aerodynamic force factor matrices. The force vector $F = [F_x F_y F_z]^T \in \mathbb{R}^3$ and the torque vector $\tau = [\tau_1 \tau_2 \tau_3]^T \in \mathbb{R}^3$ are given by
\[ F = \begin{bmatrix} 0 & 0 & \sum_{i=1}^{4} f_i \end{bmatrix}^T \]  
(6)
and
\[ \tau = [d(f_2 - f_4) \ d(f_3 - f_1) \ c \sum_{i=1}^{4} (-1)^{i+1} f_i]^T \]  
(7)
respectively, with $d$ being the distance between the epicenter of the UAV and the rotor axes, $c$ being the drag factor, and $f_i(t)$ for $i = 1, 2, 3, 4$ being the thrust force generated by the corresponding rotor. The position $P(t)$, attitude angles $\Theta(t)$, linear velocity $v(t)$, and angular velocity $\omega(t)$ signals are assumed to be measurable for the following control development.

Assumption 1: It is assumed that pitch and roll angles satisfy the following inequalities
\[ -\frac{\pi}{2} < \phi(t) < \frac{\pi}{2} \quad -\frac{\pi}{2} < \theta(t) < \frac{\pi}{2} \]  
(8)
so that the inverse of matrix $T(\cdot)$ defined in (3) exists. This assumption is also utilized in [6] and [10].

C. Coordinate Transfer

Inspired by the control design proposed in [6], the following coordinate transfer are introduced
\[ \begin{align*}
x_1(t) &= [x(t) \ y(t) \ z(t) \ x_2(t) \ = \dot{x}_1(t) \\
x_3(t) &= [\phi(t) \ \theta(t) \ \psi(t) \ \dot{x}_3(t) \\
x_5(t) &= \psi(t), \ x_6(t) = \dot{\psi}(t) \\
x_7(t) &= \dot{z}(t), \ x_8(t) = \dot{x}_7(t) \\
x_9(t) &= [f_1(t) \ f_2(t) \ f_3(t) \ f_4(t) \ f_5(t)]^T .
\end{align*} \]  
(9)
For the purpose of subsequent control development, the dynamic equations in (5) can be divided into four subsystems $\Pi_1$, $\Pi_2$, $\Pi_3$, and $\Pi_4$, listed as follows
\[ \begin{align*}
\Pi_1 : \quad &\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = h_0(x_2, x_3, x_4, x_5, x_6, x_7, x_8) + \alpha_0(x_5, x_9) \beta_0(x_3) \\
\dot{x}_3 = x_4 \\
\dot{x}_4 = h_1(x_3, x_4, x_5, x_6, x_7, x_8, x_9) + \alpha_1(x_3) \beta_1(x_9)
\end{cases} \\
\Pi_2 : \quad &\begin{cases}
\dot{x}_5 = x_6 \\
\dot{x}_6 = h_0(x_2, x_5, x_6, x_7, x_8, x_9) + \frac{\cos\phi}{3} \cos\theta \beta_0(x_3) \\
\dot{x}_7 = x_8 \\
\dot{x}_8 = h_2(x_2, x_5, x_6, x_7, x_8) + \frac{1}{m} \cos\phi \cos\theta \beta_3(x_3)
\end{cases} \\
\Pi_3 : \quad &\begin{cases}
\dot{x}_9 = x_5 \\
\Pi_4 : \quad &\begin{cases}
\dot{x}_9 = u^*, \\
\end{cases}
\end{cases}
\end{align*} \]  
(10)
where auxiliary functions $h_0(\cdot) \in \mathbb{R}^2$, $h_1(\cdot) \in \mathbb{R}^2$, $h_2(\cdot) \in \mathbb{R}$, and $h_0(\cdot) \in \mathbb{R}$ are defined as follows
\[ \begin{align*}
[ &h_x \ h_y \ h_z]^T = -\frac{1}{m} RK_1 R^T \dot{\Phi} - G \\
[ &h_0 \ h_\psi \ h_\phi]^T = -(JT)^{-1} [J(\frac{\partial T}{\partial \phi} \phi + \frac{\partial T}{\partial \theta} \theta) \dot{\Theta} - \\
&K_1 T \dot{\Theta} - S(\dot{\Theta})JT \dot{\Theta}] + \\
&\begin{bmatrix} \frac{f_x}{3} \cos\phi \sin\theta \sum_{i=1}^{4} (-1)^{i+1} f_i \\
-\frac{f_y}{3} \sin\phi \sum_{i=1}^{4} (-1)^{i+1} f_i \\
\frac{d}{2} \sin\phi \sum_{i=1}^{4} (-1)^{i+1} f_i \end{bmatrix}
\end{align*} \]  
(11)
auxiliary functions $\alpha_0(\cdot), \alpha_1(\cdot) \in \mathbb{R}$ are defined as follows
\[ \begin{align*}
\alpha_0 &= \sum_{i=1}^{4} f_i \\
\alpha_1 &= \begin{bmatrix} \sin \psi & \cos \psi \\
-\cos \psi & \sin \psi \end{bmatrix}
\end{align*} \]  
(12)
auxiliary functions $\beta_0(\cdot) \in \mathbb{R}^2$, $\beta_1(\cdot) \in \mathbb{R}^2$, $\beta_2(\cdot) \in \mathbb{R}$, $\beta_3(\cdot) \in \mathbb{R}$ are defined as follows
\[ \begin{align*}
\beta_0 &= \begin{bmatrix} \sin \phi & \cos \phi & \sin \theta \\
\cos \phi & \sin \phi & \cos \theta \end{bmatrix} \\
\beta_1 &= \begin{bmatrix} d(f_2 - f_4) \\
(d(f_3 - f_1) \\
\end{bmatrix}
\end{align*} \]  
(13)
In (10), $\Pi_1$ is a underactuated subsystem related with UAV’s longitudinal and lateral motion.
\[
\begin{bmatrix}
  x(t) & y(t)
\end{bmatrix}^T,
\text{roll angle } \theta(t), \text{ and pitch angle } \phi(t). \text{ In (11), } \Pi_2 \text{ is a full actuated subsystem related with the dynamics of UAV’s yaw angle } \psi(t). \text{ In (12), } \Pi_3 \text{ is a full actuated subsystem related with the dynamics of UAV’s altitude } z(t). \text{ In (13), } \Pi_4 \text{ represents the dynamics of thrust forces.}
\]

III. ADAPTIVE TRACKING CONTROL DESIGN

A. Problem Statement

It is assumed that the value of UAV’s mass \( m \) is unknown for control design due to different payload in different flight operations. The objective is to design the four independent thrust force \( f_i(t), i = 1, 2, 3, 4 \), to ensure the quadrotor UAV’s position \( P(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}^T \) along with the yaw angle \( \psi(t) \), to track a desired position trajectory \( P_d(t) = \begin{bmatrix} x_d(t) & y_d(t) & z_d(t) \end{bmatrix}^T \in \mathbb{R}^3 \) and desired yaw angle trajectory \( \psi_d(t) \in \mathbb{R} \) despite the mass uncertainty, while ensuring the stability of the closed loop system.

Inspired by the backstepping design techniques proposed in [6], [11], and [12], the control design procedure in this paper are divided into the following 9 steps.

Step 1. The position tracking error for longitudinal and lateral motion \( x_1(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T \) is defined as
\[
\dot{x}_1(t) = x_{1d}(t) - x(t)
\]
with \( x_{1d}(t) = \begin{bmatrix} x_d(t) & y_d(t) \end{bmatrix}^T \in \mathbb{R}^2 \) denotes the desired trajectory in \( x \) direction and \( y \) direction. For the subsystem \( \Pi_1 \), we first consider about the virtual system \( \dot{z}_1 = u_1 \) where \( u_1(t) \in \mathbb{R}^2 \) is a virtual control input signal designed as follows
\[
u_1 = \Lambda_1 z_1 + \dot{x}_{1d}
\]
with \( \Lambda_1 = diag\{\lambda_{11}, \lambda_{12}\} \in \mathbb{R}^4 \) being a diagonal, positive definite gain matrix. The Lyapunov function candidate in this step is chosen as \( V_1 = \frac{1}{2} z_1^T z_1 \), and its time derivative along (21) is
\[
\dot{V}_1 = -z_1^T \Lambda_1 z_1 < 0.
\]

Step 2. In this step, a new auxiliary error signal is introduced as \( z_2(t) = u_1 - x_2 \), and the virtual system \( \dot{x}_2 = h_0(x_2, x_1, x_6, x_7, x_8) + \alpha_0(x_1, x_9)u_2 \) is considered where \( u_2(t) \in \mathbb{R}^2 \) is a virtual control input signal to be designed later. The time derivative of \( z_1(t) \) can be written as
\[
\dot{z}_1 = -\Lambda_1 z_1 + z_2
\]
To compensate the fact that the value of UAV’s mass \( m \) is unknown, (15) is rewritten as
\[
\begin{bmatrix}
  h_x & h_y & h_z
\end{bmatrix}^T = \begin{bmatrix}
  -\frac{1}{m} h_x & -\frac{1}{m} h_y & -\frac{1}{m} h_z - g
\end{bmatrix},
\]
where \( h_x(\cdot), h_y(\cdot), \text{ and } h_z(\cdot) \) do not contain the uncertain parameter \( m \). Let auxiliary function \( \tilde{h}_0(\cdot) \) and \( \tilde{\alpha}_0 \) be defined as
\[
\tilde{h}_0(\cdot) = \begin{bmatrix}
  \tilde{h}_x & \tilde{h}_y
\end{bmatrix}^T \text{ and } \tilde{\alpha}_0 = \frac{4}{9} \sum_{i=1}^{4} f_i \begin{bmatrix}
  \sin \psi & \cos \psi \\
  -\cos \psi & \sin \psi
\end{bmatrix}
\]
Now \( \dot{x}_2(t) \) can be rewritten as \( \dot{x}_2 = s_1 \dot{h}_0 + s_2 \dot{\alpha}_0 u_2 \) with \( s_1 = s_2 = \frac{1}{m} \). Note: since the total lift force of quadrotor UAV cannot be all zero, i.e., \( \sum_{i=1}^{4} f_i > 0 \) for all \( t > 0 \), the inverse of matrix \( \dot{\alpha}_0 \) will always exist. The time derivative of \( z_2(t) \) is now expressed as
\[
\dot{z}_2 = \dot{u}_1 - s_1 \dot{h}_0 - s_2 \dot{\alpha}_0 u_2.
\]
The adaptive virtual control input for (25) is designed as follows
\[
u_2 = \frac{1}{s_2} (\dot{\alpha}_0)^{-1} (-\dot{h}_0 \dot{s}_1 + \dot{A}_2 z_2 + \dot{u}_1 + z_1)
\]
where \( \Lambda_2 = diag\{\lambda_{21}, \lambda_{22}\} \in \mathbb{R}^{2 \times 2} \) is a diagonal, positive definite gain matrix. The adaptive estimate \( \dot{s}_1(t), \dot{s}_2(t) \in \mathbb{R} \) in (26) are designed as follows
\[
\dot{s}_1 = -\Gamma_1 (\dot{h}_0)^T z_2, \quad \dot{s}_2 = proj(\Gamma_2 z_1)
\]
with \( \Gamma_1, \Gamma_2 \in \mathbb{R} \) being some positive update gains, the auxiliary term \( \gamma_1(t) \) is defined as
\[
\gamma_1 = -\frac{1}{s_2} \left( -\dot{h}_0 \dot{s}_1 + \dot{A}_2 z_2 + \dot{u}_1 + z_1 \right),
\]
and the projection operator \( proj(\Gamma_2 z_1) \) is the same as the one in [13]. If the initial value of \( s_2(t) \) is set as \( s_2(0) < \Omega_x \), where \( \Omega_x \) is a pre-defined compact set, the projection operator \( proj(\Gamma_2 z_1) \) will ensure that \( \dot{s}_2(t) \) stay inside \( \Omega_x \) for all \( t > 0 \), so that \( \dot{s}_2(t) \) can always be above zero and remain bounded [12], [13], [14]. By substituting (26) into (25), the time derivative of \( z_2(t) \) becomes
\[
\dot{z}_2 = -\Lambda_2 z_2 - \dot{z}_1 \dot{h}_0 \dot{s}_1 - \frac{s_2}{s_2} (\dot{h}_0 \dot{s}_1 + \dot{A}_2 z_2 + \dot{u}_1 + z_1)
\]
where \( \dot{s}_1 = s_1 - \dot{s}_1 \) and \( \dot{s}_2 = s_2 - \dot{s}_2 \). The Lyapunov function candidate in step 2 is chosen as
\[
V_2 = V_1 + \frac{1}{2} z_2^T z_2 + \frac{1}{2} \Gamma_1^{-1} s_1^2 + \frac{1}{2} \Gamma_2^{-1} s_2^2.
\]
The projection operator in (27) will ensure that \( \dot{s}_2 (\gamma_1 - \Gamma_2^{-1} \dot{s}_2) < 0 \) [12], [13]; hence, (31) becomes
\[
\dot{V}_2 = -z_2^T \Lambda_2 z_2 = \frac{1}{2} \frac{z_2^T z_2}{s_2} - \frac{s_2}{s_2} \dot{h}_0 \dot{s}_1 + (A_2 z_2 + \dot{u}_1 + z_1).
\]

Step 3. A new auxiliary error signal is defined as \( z_3(t) = u_2 - \beta_0(x_3) \in \mathbb{R}^2 \) in the third step. The virtual system \( \dot{x}_3 = u_3 \) is considered with \( u_3(t) \in \mathbb{R}^2 \) being a virtual control input signal. The time derivative of \( z_3(t) \) can be expressed as
\[
\dot{z}_3 = \dot{u}_2 - H_0 u_3
\]
where the matrix \( H_0 \in \mathbb{R}^{4} \) is defined as
\[
H_0 = \frac{\partial \beta_0(x_3)}{\partial x_3} = \begin{bmatrix}
  \cos \phi & 0 \\
  -\sin \phi & \cos \phi \cos \theta
\end{bmatrix}
\]
It is not difficult to check that \( H_0^{-1} \) always exists due to Assumption 1. According to (10), the expression for the time derivative of \( x_2(t) \) can be expressed as \( \dot{x}_2 = s_0 \dot{h}_0 + s_0 \dot{\alpha}_0 \theta_0 \), hence, a new expression for \( \dot{z}_2(t) \) is obtained as
\[
\dot{z}_2 = \dot{h}_0 \dot{s}_1 - z_1 - \Lambda_2 z_2 - \frac{s_2}{s_2} \dot{h}_0 \dot{s}_1 + (A_2 z_2 + \dot{u}_1 + z_1) + s_3 \dot{s}_0^T z_3
\]
where \( s_3 = s_2 = \frac{1}{m} \). The adaptive virtual control input \( u_3(t) \) is designed as follows
\[
u_3 = H_0^{-1}(\dot{u}_2 + \Lambda_3 \dot{z}_3 + \hat{s}_3 \alpha_0^T z_2) \tag{36}\]
with the adaptive estimate \( \hat{s}_2(t) \in \mathbb{R} \) being designed as
\[
\hat{s}_3 = z_3^T \alpha_0^T z_2
\tag{37}
\]
and \( \Lambda_3 = \text{diag}(\lambda_3, \lambda_3) \in \mathbb{R}^{2 \times 2} \) being a positive definite gain matrix. The Lyapunov function candidate in the third step is chosen as
\[
V_3 = V_2 + \frac{1}{2} z_3^T z_3 + \frac{1}{2} \hat{s}_3^2 \tag{38}
\]
where \( \hat{s}_3 = s_3 - \hat{s}_3 \). The time derivative of \( V_3(t) \) along (35), (27), (33), (36), and (37) is
\[
\dot{V}_3 = -z_3^T \Lambda_3 \dot{z}_3 - z_3^T \Lambda_2 \dot{z}_3 + \hat{s}_3 \gamma_2 + z_3^T (\hat{s}_2 + \hat{s}_2^T \Lambda_3 \dot{z}_3) - s_3 \hat{s}_3 \\
\leq -z_3^T \Lambda_3 \dot{z}_3 - z_3^T \alpha_0^T \dot{z}_2 - z_3^T \Lambda_3 \dot{z}_3 \tag{39}
\]

**Step 4.** A new auxiliary error signal is defined as \( z_4(t) = u_3(t) - x_4(t) \in \mathbb{R}^2 \) in the fourth step. The virtual system \( \dot{x}_4 = h_1 + \alpha_1 u_4 \) is considered with \( u_4(t) \in \mathbb{R}^2 \) being a virtual control input signal. The time derivative of \( z_4(t) \) is
\[
\dot{z}_4 = \hat{u}_3 - h_1 - \alpha_1 u_4 \tag{40}
\]
Based on (10) and the definition of \( z_4(t) \), the time derivative of \( z_3(t) \) can be rewritten as
\[
\dot{z}_3 = -\Lambda_3 z_3 - \hat{s}_3 \alpha_0^T z_2 + H_0 z_4 \tag{41}
\]
by substituting (36). The virtual input signal \( u_4(t) \) is designed as follows
\[
u_4 = \alpha_1^{-1}(H_0^T z_3 + \dot{u}_3 - h_1 + \Lambda_4 z_4) \tag{42}
\]
where \( \Lambda_4 = \text{diag}(\lambda_4, \lambda_4) \in \mathbb{R}^{2 \times 2} \) is a positive definite gain matrix. Note: the matrix \( \alpha_1 \) will not be singular according to (17) and assumption 1. The Lyapunov function candidate in step 4 is chosen as \( V_4 = V_3 + \frac{1}{2} z_4^T z_4 \). The time derivative of \( V_4(t) \) along (41) and (40) can be obtained as
\[
\dot{V}_4 = -z_4^T \Lambda_4 \dot{z}_4 - z_4^T \alpha_0^T \dot{z}_2 - z_4^T \Lambda_4 \dot{z}_3 + \hat{s}_2 \gamma_4 \\
- \Gamma_2^{-1} \dot{\hat{s}}_2 + z_4^T (H_0^T z_3 + \dot{u}_3 - h_1 - \alpha_1 u_4) \\
\leq -z_4^T \Lambda_4 \dot{z}_4 - z_4^T \Lambda_4 \dot{z}_3 - z_4^T \Lambda_4 \dot{z}_3 - z_4^T \Lambda_4 \dot{z}_3 \leq 0 \tag{43}
\]
It can be seen that \( u_1(t), u_2(t), u_3(t), u_4(t) \) asymptotically stabilize the underactuated subsystem \( \Pi_1 \) and drive \([x(t) \ y(t)]^T \) to track the desired trajectory.

**Step 5.** The yaw angle tracking error signal is defined as follows \( \phi_5 = \psi - \psi_4 = x_5 - x_5d \) with \( x_5d(t) = \psi_4(t) \) denotes the desired yaw angle trajectory. For the full actuated subsystem \( \Pi_2 \) in (11), we first consider the following virtual system \( \dot{x}_5 = u_5 \), where \( u_5(t) \in \mathbb{R} \) is the virtual control input and designed as
\[
u_5 = \Lambda_5 z_5 + \dot{x}_5d \tag{44}
\]
with \( \Lambda_5 \in \mathbb{R} \) being some positive control gain. The Lyapunov function candidate in the fifth step is selected as \( V_5 = \frac{1}{2} z_5^2 \) and it is easy to check that its time derivative along (44) is
\[
\dot{V}_5 = -\Lambda_5 z_5^2 \leq 0. \tag{45}
\]

**Step 6.** A new auxiliary error signal in introduced in the sixth step as \( z_6(t) = u_5(t) - x_6(t) \in \mathbb{R} \). The virtual system \( \dot{x}_6 = h_2 + \frac{1}{m} \cos \theta \cos \theta u_6 M \) is considered with \( u_6(t) \in \mathbb{R} \) being an virtual control input signal designed as follows
\[
u_6 = \frac{1}{\cos \phi} I_{zz} \cos \theta (\hat{s}_5 + \hat{u}_5 - h_4 + \Lambda_6 z_6) \tag{46}
\]
where \( \Lambda_6 \in \mathbb{R} \) denotes a positive control gain. Based on the definition of \( z_6 \), the time derivative of \( z_5(t) \) can be expressed as \( \dot{z}_5 = -\Lambda_5 z_5 + z_6 \) where the first entry of (11) has been utilized. The Lyapunov function candidate in the sixth step is selected as \( V_6 = V_5 + \frac{1}{2} z_6^2 \). The time derivative of \( V_6(t) \) along \( z_5(t) \), \( x_6(t) \), and (46) is
\[
\dot{V}_6 = -\Lambda_5 z_5^2 - \Lambda_6 z_6^2 \leq 0. \tag{47}
\]
From (47), it can be checked that the virtual control input signals \( u_5(t) \) and \( u_6(t) \) can stabilize the subsystem \( \Pi_2 \) asymptotically, and drive the yaw angle \( \psi(t) \) to track the desired trajectory \( \psi_d(t) \).

**Step 7.** The altitude tracking error is defined as \( z_7 = z_d - z = x_{7d} - x_7 \) where \( x_{7d}(t) = z_6(t) \in \mathbb{R} \) represents the desired trajectory in \( z \) direction. For the subsystem \( \Pi_3 \), we first consider the virtual system \( \dot{x}_7 = u_7 \) where \( u_7(t) \in \mathbb{R} \) denotes the virtual control input signal designed as
\[
u_7 = \dot{x}_{7d} + \Lambda_7 z_7 \tag{48}
\]
with \( \Lambda_7 \in \mathbb{R} \) being a positive control gain. The Lyapunov function candidate in this step is chosen as follows \( V_7 = \frac{1}{2} z_7^2 \). The time derivative of \( V_7(t) \) along (48) is
\[
\dot{V}_7 = -\Lambda_7 z_7^2. \tag{49}
\]

**Step 8.** The virtual system \( \dot{x}_8 = h_z + \frac{1}{m} \cos \phi \cos \theta \cos \theta u_8 \) is considered where \( u_8(t) \in \mathbb{R} \) represents the virtual control input signal to be designed later. Since \( h_z = -\frac{1}{m} \dot{h}_z - g \), \( \dot{x}_8(t) \) can be rewritten as \( \dot{x}_8 = -s_4 \dot{h}_z - g + s_5 \cos \phi \cos \theta u_8 \) where \( s_4 = s_5 = m \), and \( \dot{h}_z(\cdot) \) is defined in (24) which does not contain the unknown parameter \( m \). The adaptive virtual control input \( u_8(t) \) can be designed as follows
\[
u_8 = \frac{1}{\hat{s}_5} \frac{1}{\cos \phi \cos \theta} (\Lambda_8 z_8 + \hat{s}_4 \dot{h}_z + g + z_7 + \hat{u}_7) \tag{50}
\]
where \( \Lambda_8 \in \mathbb{R} \) is a positive control gain, \( \hat{s}_4(t) \in \mathbb{R} \) and \( \hat{s}_5(t) \in \mathbb{R} \) denote adaptive estimates of unknown parameter \( s_4 \) and \( s_5 \). The update laws for \( \hat{s}_4(t) \), \( \hat{s}_5(t) \) are designed as
\[
\dot{\hat{s}}_4 = -\Gamma_4 \dot{h}_z \hat{s}_8 + \hat{s}_4 \Gamma_5 \gamma_2 \tag{51}
\]
where \( \Gamma_4, \Gamma_5 \in \mathbb{R} \) are some positive update gains. The auxiliary function \( \gamma_2(t) \) is defined as
\[
\gamma_2 = -z_8 \frac{1}{\hat{s}_5} (\Lambda_8 z_8 + \hat{s}_4 \dot{h}_z + g + z_7 + \hat{u}_7), \tag{52}
\]
\( \text{proj}(\cdot) \) denotes the same projection operator as the one in [13]. A auxiliary error signal in the eighth step is defined as \( z_8(t) = u_7(t) - x_8(t) \in \mathbb{R} \). The time derivative of \( z_8(t) \) along (50) is
\[
\dot{z}_8 = \tilde{s}_4 \dot{h}_2 - z_7 - \Lambda_8 z_8 - \frac{\tilde{s}_5}{\tilde{s}_5} (\Lambda_8 z_8 + \tilde{s}_4 \dot{h}_2 + g + z_7 + \dot{u}_7) \tag{53}
\]
where \( \tilde{s}_4(t) = s_4 - s_4(t) \) and \( \tilde{s}_5(t) = s_5 - \tilde{s}_5(t) \). Also, the time derivative of \( z_7(t) \) can be obtained as \( \dot{z}_7 = -\Lambda_7 z_7 + z_8 \). The Lyapunov function candidate in the eighth step is picked up as \( V_8 = V_7 + \frac{z_7^2}{2} + \frac{1}{\Gamma_3} z_8^2 + \frac{1}{\Gamma_3} z_5^2 \), and its time derivative along \( z_7(t) \) and (53) is
\[
\dot{V}_8 = -\Lambda_7 z_7^2 - \Lambda_8 z_8^2 + \tilde{s}_5 (\gamma_2 - \Gamma_5^{-1} \tilde{s}_5) \\
\leq -\Lambda_7 z_7^2 - \Lambda_8 z_8^2 \tag{54}
\]
where the projection operator ensures that \( \tilde{s}_5 (\gamma_2 - \Gamma_5^{-1} \tilde{s}_5) < 0 \). Virtual control inputs \( u_7(t) \) and \( u_8(t) \) can stabilize the subsystem \( \Pi_3 \) asymptotically, and drive \( z(t) \) to track the desired trajectory \( \dot{z}(t) \).

**Step 9.** We consider the subsystem \( \Pi_4 \) in (13). To design the real control input \( u^* \), an auxiliary error signal \( z_9 = [u_4 - \beta_1, u_6 - \beta_2, u_8 - \beta_3] \in \mathbb{R}^9 \) is introduced. It is easy to obtain that \( \beta_1 = u_4 - [I_2, 0_{2 \times 2}] z_9, \beta_2 = u_6 - [0, 0, 0, 1] z_9, \beta_3 = u_8 - [0, 0, 0, 1] z_9 \) where \( I_2 \) is the \( 2 \times 2 \) identity matrix, and \( 0_{2 \times 2} \) is the \( 2 \times 2 \) zero matrix. Taking the time derivative of \( z_9(t) \) along (10) and (42) yields \( \dot{z}_4 = -H_0(I \zeta - \Lambda_4 z_4 + \alpha_4 z_9) \), \( \alpha_3 = [0, 0, 0, \cos \theta] \in \mathbb{R}^{4 \times 1} \). The time derivative of \( z_6(t) \) along (11), and (46) is \( \dot{z}_6 = -z_5 - \alpha_2 z_6 + \alpha_2 z_9 \), \( \alpha_2 = [0, 0, 1, 1] \in \mathbb{R}^{4 \times 1} \). The time derivative of \( z_8(t) \) along (12) and (50) is
\[
\dot{z}_8 = -z_7 - \Lambda_8 z_8 + \tilde{s}_4 \dot{h}_2 - \frac{\tilde{s}_5}{\tilde{s}_5} (\Lambda_8 z_8 + \tilde{s}_4 \dot{h}_2 + g + z_7 + \dot{u}_7) + s_6 \alpha_3 \tilde{s}_5 z_9 \\
\tag{55}
\]
where \( \alpha_3 = [0, 0, \cos \theta \cos \phi] \in \mathbb{R}^{1 \times 4} \) and \( s_6 = \tilde{s}_5 - \frac{1}{m} \). The time derivative of \( z_9(t) \) can be expressed as
\[
z_9 = [u_4, u_6, u_8]^T - [H_1, H_2, H_2]^T \dot{x}_9 \tag{56}
\]
where \( H_1 = \frac{\partial \beta_1}{\partial \sigma_5} = [0, d, 0, d] \in \mathbb{R}^{2 \times 4}, H_2 = \frac{\partial \beta_2}{\partial \sigma_5} = [c, -c, c, -c] \in \mathbb{R}^{4 \times 4}, \) and \( H_3 = \frac{\partial \beta_3}{\partial \sigma_5} = [1, 1, 1, 1] \in \mathbb{R}^{4 \times 4} \). The control input \( u^* \) is designed as follows
\[
u^* = \left( H_1, H_2, H_2 \right)^T \left( \dot{u}_4, \dot{u}_6, \dot{u}_8 \right)^T + (\alpha_3^*)^T z_4 + (\alpha_3^*)^T \dot{z}_6 + s_6 (\beta_3^*)^T \dot{z}_8 + s_6 \gamma_3 \tag{57}
\]
where \( \Lambda_9 = diag(\lambda_{91}, \lambda_{92}, \lambda_{93}, \lambda_{94}) \in \mathbb{R}^{4 \times 4} \) is a positive definite gain matrix, \( \tilde{s}_6(t) \in \mathbb{R} \) represents the estimation of the unknown parameter \( s_6 \), and its update law is designed as
\[
\dot{s}_6 = s_6^2 (\alpha_3^*)^T z_8. \tag{58}
\]
The actual thrust force generated by four rotors can be obtained via
\[
\begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix}^T = \int_0^t u^*(\tau) d\tau \tag{59}
\]
then the total lift force \( F_z(t) \) and rotation torque \( \tau(t) \) in (5) can be calculated via (6) and (7) respectively.

**IV. Stability Analysis**

The stability of the closed-loop system is stated by the following theorem:

**Theorem IV.1.** Given the dynamic model in (5), the adaptive control law (21), (26), (36), (42), (44), (46), (50), (57), together with the adaptive parameter estimates in (27), (37), (51), and (58) ensure the boundedness of all system signals, and asymptotic translation tracking together with yaw angle tracking are achieved in the sense that
\[
\lim_{t \to \infty} z_1(t), z_2(t), z_3(t) = 0 \tag{60}
\]
where \( z_1(t), z_2(t), \) and \( z_3(t) \) are defined in step 1, step 5 and step 7 respectively, providing that assumption 1 is satisfied.

**Proof:** Define a composite Lyapunov function \( V(t) \) as follows
\[
V = V_4 + V_6 + V_8 + \frac{1}{2} z_9^2 + \frac{1}{2} z_6^2 \tag{61}
\]
where \( s_6 = s_6 - s_6 \) denotes the estimation error for the unknown parameter \( s_6 \) in (55), \( V_4, V_6, V_8 \) are defined in step4, step6 and step8 respectively. The time derivative of \( V(t) \) along (23), (35), (41), \( \dot{z}_4(t), \dot{z}_5(t), \dot{z}_6(t), \dot{z}_7(t), \dot{z}_8(t), \dot{z}_9(t) \), \( (55), (57), \) and (58) is
\[
\dot{V} \leq -z_1^2 \Lambda_1 z_1 - z_2^2 \Lambda_2 z_2 - z_3^2 \Lambda_3 z_3 - z_4^2 \Lambda_4 z_4 - \Lambda_5 z_5^2 - \Lambda_6 z_6^2 - \Lambda_7 z_7^2 - \Lambda_8 z_8^2 - \Lambda_9 z_9^2 < 0. \tag{62}
\]
Based on (61) and (62), one can show that \( z_i(t) \in \mathcal{L}_\infty \) for \( i = 1, 2, \ldots, 9 \), \( \dot{s}_i(t) \in \mathcal{L}_\infty \), thus by using standard signal chancing it can be shown that all the closed loop signals are bounded, and \( z_1(t), z_2(t), z_3(t) \in \mathcal{L}_2, z_4(t), z_5(t), z_6(t), z_7(t), z_8(t), z_9(t) \in \mathcal{L}_\infty \). Hence, Lemma A.31 in [16] can be utilized to prove the result in (60).

**V. Simulation Results**

In this section, the results of a numerical simulation are presented in order to demonstrate the performance of the proposed adaptive controller. The dynamic model in (5) is utilized with the same parameter values as the ones in [6]. The desired trajectory is set as \( x_d(t) = (1 - e^{-t^2}) \sin t \), \( y_d(t) = (1 - e^{-t^2}) \cos t \), \( z_d(t) = (1 - e^{-t^2}) \) m, \( \psi_d(t) = 30(1 - e^{-t^2}) \) deg. The initial state condition for UAV are set as all 0. The initial values for parameter estimates are set as \( \dot{s}_1(0) = \dot{s}_3(0) = \dot{s}_4(0) = \dot{s}_6(0) = 0 \), \( \dot{s}_1(0) = 0.5 \), \( \dot{c} = 10, \varepsilon = 0.2 \). Figure 2 shows tracking results in \( x, y, z \) direction and yaw rotation, Figure 3 shows propeller force \( f_1(t), f_2(t), f_3(t), f_4(t) \), it can be seen that they all have reasonable values. Parameter estimates for \( \dot{s}_1(t) \) to \( \dot{s}_6(t) \) are provided in figure 4.
VI. CONCLUSION

Backstepping techniques are employed in this paper to develop an new adaptive tracking controller for an underactuated quadrotor UAV system with unknown mass. Lyapnov based analysis is utilized to prove that an asymptotic tracking is achieved with some mild assumption of UAV’s roll and pitch angle. Numerical simulation results are provided to show the good performance of proposed design. Future work will focuses on adaptive control design for quadrotor UAV with unknown moment of inertial and aerodynamic force factors.

REFERENCES


