Observer-based output feedback linear control applied to a denitrification reactor

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Abstract—In this paper an observer-based output dynamic feedback controller for a denitrification biofilter is proposed. It is based on a linear reduced-order model obtained from the original parabolic distributed parameter system using modal analysis through the proper orthogonal decomposition method. A state feedback control law and an associated Luenberger observer are computed by solving two $H_2$ optimization problems under LMI constraints to reject external disturbances and to stabilize the closed-loop dynamics.

I. INTRODUCTION

The state variables in many biological and chemical processes are distributed in both time and space so they have to be described by means of partial differential equations (PDE). It should be highlighted that the control design in this kind of systems is more difficult than in systems depending only on time. Indeed, the research concerning such bioprocesses described by PDEs is mainly focused in modelling and simulation issues without taking, very often, control and observation issues into account. However, a few control strategies for distributed parameter biosystems have been proposed in the past years, for instance: in [4] a variable structure control is applied to fixed bed reactors, in [1] the authors deal with the linear boundary control problem in an anaerobic digestion process, in [5] a sliding model control strategy is evaluated on a denitrifying biofilter, adaptive control schemes are also applied to nonlinear distributed parameter bioreactors [9].

The aim of this work is the design of an observer-based output feedback controller to maintain the nitrogen concentration at the output of a denitrification reactor, represented by a parabolic PDE system, below the European norm by reducing the energy of a disturbance at the reactor input. An additional ethanol supply source may be used as a control input action or at least to ensure a sufficiently high ratio $C/N$ such that the biofilm does not become the limiting source for the growth. Denitrification is a two-stage reaction performed in anaerobic conditions. The first stage is the nitritation which transforms nitrate ($\mathrm{NO}_3$) into nitrite ($\mathrm{NO}_2$) while the second phase transforms nitrite into gaseous nitrogen ($\mathrm{N}_2$). The same micro-organism population (bacteria) is involved in both stages, with ethanol as co-substrate. This biomass accumulates on the solid media surface due to filtration of bacteria present in the feeding water (if any) and to net growth. Thus, the biomass forms a biofilm around the filter particles, which thickens with time. One can then consider that all the biomass is fixed and does not move along the reactor. On the contrary, the soluble compounds (nitrate, nitrite and ethanol) are transported along the biofilter.

The dynamics of the biofilter can be deduced from mass balance for the four different components, considering the following assumptions:

- The detachment of biofilm and particles retained by filtration is neglected;
- Once the biofilm reaches a critical 'per unit' surface thickness, the deeper part of the biofilm is considered as inactivated, and a maximum active biomass concentration $X_{\text{amax}}$ is reached [6]. Then, after some transition period, the growth of micro-organisms just balances the death and inactivation process;
- The decay of biomass is neglected, hidden in the notion of maximum active biomass concentration;
- Radial dispersion is negligible. Axial dispersion obeys Fick’s diffusion law.

Remark 1: Diffusion phenomenon has often been neglected to simplify the PDE system [6], [10]. It is however a key parameter for two main reasons. First, if the boundary layer is destroyed by surface irregularities of the biofilm,
substrate diffusion in the biofilm is strongly influenced by fluid turbulence. The gradient of concentration inside the biofilm is reduced [13]. Secondly, neglecting the diffusion phenomenon results in hyperbolic PDEs, instead of parabolic PDEs, which have distinctive properties and are usually more difficult to solve numerically.

The denitrification reactor is then modeled by the following parabolic PDE system:

\[
\frac{\partial x_1(z, t)}{\partial t} = D_f \frac{\partial^2 x_1(z, t)}{\partial z^2} - \frac{v}{\epsilon} \frac{\partial x_1(z, t)}{\partial z} - \frac{1 - Y_{h_1}}{1.14Y_{h_1}} \mu_1(x_1, x_3)x_4(z, t) - \frac{1 - Y_{h_2}}{1.14Y_{h_2}} \mu_2(x_2, x_3)x_4(z, t)
\]

\[
\frac{\partial x_2(z, t)}{\partial t} = D_f \frac{\partial^2 x_2(z, t)}{\partial z^2} - \frac{v}{\epsilon} \frac{\partial x_2(z, t)}{\partial z} + \frac{1 - Y_{h_1}}{1.14Y_{h_1}} \mu_1(x_1, x_3)x_4(z, t) - \frac{1 - Y_{h_2}}{1.71Y_{h_2}} \mu_2(x_2, x_3)x_4(z, t)
\]

\[
\frac{\partial x_3(z, t)}{\partial t} = D_f \frac{\partial^2 x_3(z, t)}{\partial z^2} - \frac{v}{\epsilon} \frac{\partial x_3(z, t)}{\partial z} - \frac{1}{Y_{h_1}} \mu_1(x_1, x_3)x_4(z, t) - \frac{1}{Y_{h_2}} \mu_2(x_2, x_3)x_4(z, t)
\]

\[
\frac{\partial x_4(z, t)}{\partial t} = \left( \frac{\mu_1(x_1, x_3)x_4(z, t) + \mu_2(x_2, x_3)x_4(z, t)}{X_{a, max}} \right) \left( 1 - \frac{x_4(z, t)}{X_{a, max}} \right)
\]

for \(0 < z \leq L\), where \(z\) is the axial space variable, \(x_1(z, t), x_2(z, t), x_3(z, t)\) and \(x_4(z, t)\) represent the nitrate \((g[N]/m^3)\), nitrite \((g[N]/m^3)\), ethanol \((g[DCO]/m^3)\) and active biomass concentrations, respectively. \(D_f\), \(v\), \(Y_{h_1}\), \(Y_{h_2}\), \(\mu_1\) and \(\mu_2\) represent, respectively, diffusivity, the flow velocity \(m/h\) (computed as the ratio between the feeding rate \((m^3/h)\) at reactor input and the biofilter transverse surface \((m^2)\)), micro-organisms yield coefficients and population specific rates which transform nitrate into nitrite, then nitrite into gas nitrogen \((1/h)\).

The nitrate and nitrite growth specific rates are described by the Monod law with two substrate limitations:

\[
\mu_1(x_1, x_3) = \eta_g \mu_{1, max} \frac{x_1}{x_1 + K_{NO_3}x_3 + K_C}
\]

\[
\mu_2(x_2, x_3) = \eta_g \mu_{2, max} \frac{x_2}{x_2 + K_{NO_2}x_3 + K_C}
\]

where \(\eta_g\), \(\mu_{1, max}\), \(\mu_{2, max}\), \(K_{NO_3}\), \(K_{NO_2}\) and \(K_C\) are the correction factor for the anaerobic growth, the maximum specific growth rates of biomass on nitrate and nitrite and the affinity constants with respect to nitrate, nitrite and ethanol, respectively.

Let us consider \(x_{1, in}(t), x_{2, in}(t), x_{3, in}(t)\) and \(x_{4, in}(t)\) representing the nitrate, nitrite, ethanol and active biomass at reactor input, respectively. Associated to the dynamic equation for the denitrification process, appropriate initial and boundary conditions are given by:

- Initial spacial profile at \(t = 0\) for \(0 \leq z \leq L\):
  \[x_1(z, t = 0) = x_{1, 0}(z) = 16.93 g[N]/m^3\]  \[(5)\]
  \[x_2(z, t = 0) = x_{2, 0}(z) = 0 g[N]/m^3\]  \[(6)\]
  \[x_3(z, t = 0) = x_{3, 0}(z) = 101.5 g[COD]/m^3\]  \[(7)\]
  \[x_4(z, t = 0) = x_{4, 0}(z) = X_{a, max} g[COD]/m^3\]  \[(8)\]

- Robin boundary conditions at \(z = 0\) (input) for \(t > 0\):
  \[
  \frac{\partial x_1}{\partial z} = \frac{v}{\epsilon D_f} (x_1 - x_{1, in})
  \]
  \[(9)\]
  \[
  \frac{\partial x_2}{\partial z} = \frac{v}{\epsilon D_f} (x_2 - x_{2, in})
  \]
  \[(10)\]
  \[
  \frac{\partial x_3}{\partial z} = \frac{v}{\epsilon D_f} (x_3 - x_{3, in})
  \]
  \[(11)\]

- Neumann boundary conditions at \(z = L\) (output) for \(t > 0\): \(\frac{\partial x_1}{\partial z} = 0, \frac{\partial x_2}{\partial z} = 0\) and \(\frac{\partial x_3}{\partial z} = 0\).

Remark 2: Initial conditions express that biomass and substrate are homogeneously distributed along the biofilter. Substrate concentrations are set equal to the corresponding influent concentrations \([6]\).

For the numerical application, the following nominal values borrowed from \([6]\) and \([12]\) are considered:

\[Y_{h_1} = 0.56\]
\[Y_{h_2} = 0.54\]
\[\mu_{1, max} = 0.361/h\]
\[\mu_{2, max} = 0.321/h\]
\[K_{NO_3} = 1.5 g[N]/m^3\]
\[K_{NO_2} = 1.0 g[N]/m^3\]
\[K_C = 40 g[COD]/m^3\]
\[X_{a, max} = 800 g[COD]/m^3\]
\[n_g = 0.8\]
\[\epsilon = 0.52\]
\[x_{1, in}(t) = 16.93 g[N]/m^3\]
\[x_{2, in}(t) = 0 g[N]/m^3\]
\[x_{3, in}(t) = 101.5 g[COD]/m^3\]
\[x_{4, in}(t) = 0 g[COD]/m^3\]
\[D_f = 0.4756 m^2/h\]
\[v = 4 m/h\]
\[L = 2.1 m\]

It has been previously shown in \([6]\) that, except during the initial colonization step, the biomass concentration remains almost constant at \(X_{a, max}\), even after washing out, so its influence on the system can be neglected. Therefore, for the sake of simplicity, the system (1)-(4) is reduced by taking only the first three PDEs and by considering \(X_a(z, t) = X_{a, max}\).

The reduced system (1)-(3) can be rewritten in matrix form as:
\[
\frac{\partial x}{\partial t} = A_1 \frac{\partial^2 x}{\partial z^2} + A_2 \frac{\partial x}{\partial z} + H(x) \tag{12}
\]
where \(x = [x_1 \ x_2 \ x_3]^T\) is the state vector, \(A_1, A_2 \in \mathbb{R}^{3 \times 3}\) are diagonal matrices whose elements are \(D_f\) and \(v/\epsilon\) respectively, and \(H(x) \in \mathbb{R}^3\) is a vector of non-linear functions.

### III. PDE SYSTEM SOLUTION

#### A. Method of lines

Basically the method of lines proceeds in two steps. In the first step, the spatial domain is discretized into a number of points in order to approximate the continuous spatial differential operators by discrete operators. As a result, a system of ODEs is obtained. Such ODE system is integrated, in the second step, in order to obtain the numerical solution of the PDE. In this work, the finite element method (FEM) is employed in the first step of the method of lines. Let us discretize the spatial domain into \(N\) points, after the application of the FEM, the following matrices are obtained (see [17] for details): \(M_{Mi} \in \mathbb{R}^{N \times N}\), \(M_{Ti} \in \mathbb{R}^{N \times N}\), \(D_f \in \mathbb{R}^{N \times N}\) and \(M_{Bi} \in \mathbb{R}^{N \times N}\) which allow us to approximate the spatial derivatives as:

\[
\frac{\partial}{\partial z} = M_{Mi}^{-1} M_{Ti} \tag{13}
\]

\[
\frac{\partial^2}{\partial z^2} = -M_{Mi}^{-1} (D_f + M_{Bi}) \tag{13}
\]

Using these matrices and after some algebra, the following semi-discrete ODE system is obtained:

\[
\frac{dx_{\xi i}}{dt} = -M_{Mi}^{-1} (D_f M_{Di} + \frac{v}{\epsilon} (M_{Ti} + M_{Bi})) x_{\xi i} + M_{Mi}^{-1} g_i(v, x_{i,in}) + h_{\xi i}(x) \tag{13}
\]

where vectors \(x_{\xi i} \in \mathbb{R}^N\) with \(i = 1, 2, 3\) represent the discretized versions of the states variables. \(g_i(v, x_{i,in}) \in \mathbb{R}^N\) collects the non-homogeneous part of the boundary conditions and \(h_{\xi i}(x)\) corresponds to the discrete version of the nonlinear term \(H(x)\). The resulting system of semi-discrete ODEs is integrated in time to compute the final solution.

#### B. Modal analysis

A convenient and useful form of analysis of second-order equations is through modal decomposition. This form of analysis is possible when second order equations like (12) have a spatial operator which can be made self-adjoint and which have a real discrete spectrum of eigenvalues [14].

Let us consider the state variable \(x_i(\xi, t) : \mathbb{R}^N \times T \rightarrow \mathbb{R}\), where \(T\) being the semi-open time interval \([0, \infty)\), expanded in a truncated Fourier series, so that:

\[
x_i(\xi, t) \approx \sum_{k=1}^{N_m} \phi_k^i(\xi) m_k^i(t) \tag{14}
\]

The sets \(\{\phi_k^i\}_{k=1}^{N_m}\) and \(\{m_k^i\}_{k=1}^{N_m}\) must be determined in order to obtain the solution \(x_i\).

The set \(\{\phi_k^i\}_{k=1}^{N_m}\) can be computed off-line using the proper orthogonal decomposition method (POD) [16], [3]. It has shown to be very efficient, especially in PDE systems with non-homogeneous boundary conditions and convection term [11]. In the POD technique, the set of basis functions are computed as those which minimize the distance between a set of measurements of the system (snapshots) and the subspace built with such basis. The solution of this optimization leads to the following eigenvalue problem:

\[
\phi_i(\xi) = \lambda_i \int _z \mathcal{K}_i(\xi, \xi^T) \phi_i(\xi^T) d\xi^T \tag{15}
\]

where the kernel \(\mathcal{K}_i\) may be defined, in its discrete version, as:

\[
\mathcal{K}_i = \frac{1}{k} \sum_{j=1}^{k} \phi_i^j \phi_i^j \tag{16}
\]

where \(\mathcal{X}_i \in \mathbb{R}^N\) are measurements of the original state variable \(x_i(\xi, t)\) at a finite number \(N\) of spatial points and at a specific time \(t\).

The FEM mass matrix \(M_{Mi}\) can be also employed for approximating spatial integrals by algebraic operations (see [17] for details). In this way, in order to compute a matrix \(\Phi_i = [\phi_1^i \ \phi_2^i \ \ldots \ \phi_N^i] \in \mathbb{R}^{N \times N_m}\), the eigenproblem (15) may be approximated as:

\[
\Phi_i = \Lambda_i \mathcal{K}_i M_{Mi} \Phi_i \tag{17}
\]

with \(\mathcal{K}_i\) constructed as in (16). It is important to point out that the set of eigenvectors forms a complete orthonormal basis set on a Hilbert space [11]. The eigenvectors contain the spatial information of the solution (14). In order to be able to reconstruct the solution, the modes \(m_i\) (time information) have to be computed. To that purpose, let us define the spatial projection operator as \(P_i = \Phi_i^T M_{Mi}\). Applying this operator to (13) the following equation is obtained:

\[
\frac{dm_i(t)}{dt} = -\Phi_i^T (D_f M_{Di} + \frac{v}{\epsilon} (M_{Ti} + M_{Bi})) \Phi_i m_i(t) + P_i \left( M_{Mi}^{-1} g_i(v, x_{i,in}) + h_{\xi i}(x) \right) \tag{18}
\]

where \(m_i = [m_1^i \ m_2^i \ \ldots \ m_N^i]^T \in \mathbb{R}^{N_m}\) are the time dependent modes of the discretized state variable \(x_{\xi i} (i = 1, 2, 3)\). The state variable is then reconstructed using (14) as:

\[
x_{\xi i}(t) = \Phi_i m_i(t) \tag{19}
\]

### IV. STATE-SPACE REPRESENTATION

From (18), a nonlinear state-space model describing the evolution of the modes obtained from the original PDE system (12) may be stated as:

\[
\dot{m}(t) = F_m(m, u, w) ; \quad m(0) = m_0 \tag{20}
\]
where \( m \in \mathbb{R}^{3N_m} \) is a vector constructed as:

\[
\begin{bmatrix}
m_1^1 & m_1^2 & m_1^3 & \cdots & m_1^{N_m} \\
m_2^1 & m_2^2 & m_2^3 & \cdots & m_2^{N_m} \\
m_3^1 & m_3^2 & m_3^3 & \cdots & m_3^{N_m}
\end{bmatrix}^T
\]

with \( N_m \leq N \), \( u \in \mathbb{R}^{N_u} \) is the controlled input vector, \( w \in \mathbb{R}^{N_w} \) is a disturbance vector and \( F_\xi \in \mathbb{R}^{3N_m} \) is a vector of functions constructed as:

\[
F_\xi(m,u,w) = \left[ f_{\xi_1}^1 f_{\xi_1}^2 f_{\xi_1}^{N_m} f_{\xi_2}^1 f_{\xi_2}^2 f_{\xi_2}^{N_m} f_{\xi_3}^1 f_{\xi_3}^2 f_{\xi_3}^{N_m} \right]^T
\]

System (20) is linearized by using a Taylor series expansion developed around an operating point \((m^*, u^*, w^*)\) such that \( F_\xi(m^*, u^*, w^*) = 0 \). The following local linear version of (20) is then obtained:

\[
\dot{\hat{m}}(t) = A\hat{m}(t) + B_u \hat{u}(t) + B_w \hat{w}(t)
\]  

with \( \hat{m}(t) = m(t) - m^* \), \( \hat{u}(t) = u(t) - u^* \) and \( \hat{w}(t) = w(t) - w^* \). The matrices \( A \in \mathbb{R}^{3N_m \times 3N_m} \), \( B_u \in \mathbb{R}^{3N_m \times N_u} \) and \( B_w \in \mathbb{R}^{3N_m \times N_w} \) are the Jacobian matrices \( J_{F_\xi}(m,u,w)(m_1^1, m_1^2, m_1^3, \ldots, m_1^{N_m}, m_2^1, m_2^2, m_2^3, \ldots, m_2^{N_m}, m_3^1, m_3^2, m_3^3, \ldots, m_3^{N_m}) \) and \( J_{F_\xi}(m,u,w)(u_1^1, u_2^1, \ldots, u_1^{N_u}, u_2^1, u_2^2, \ldots, u_2^{N_u}) \) respectively, evaluated at the operating point.

V. OBSERVER-BASED OUTPUT FEEDBACK CONTROLLER

A. Controlled output

The control problem can be set as follows: maintain the nitrate and the nitrite concentrations at the reactor outlet lower than the European norm \((5.65 mg[N]/m^3)\) while limiting as much as possible the activity of the control input. Therefore, the controlled output is represented as:

\[
z(t) = C_z \bar{x}_\xi(t) + D_z \hat{u}(t)
\]

where \( \bar{x}_\xi = x_\xi - x_\xi^* \in \mathbb{R}^{3N} \), \( C_z \in \mathbb{R}^{1 \times 3N} \), \( D_z \in \mathbb{R}^{1 \times N_u} \) and \( z \in \mathbb{R} \).

From (19) we can construct a matrix \( \Phi_T \in \mathbb{R}^{3N_m \times 3N_m} \) to approximate \( \bar{x}_\xi(t) \) as:

\[
\bar{x}_\xi(t) = \Phi_T \hat{m}(t)
\]

Substituting (23) into (22), the following representation for the controlled output is obtained:

\[
z(t) = C_{zm} \bar{m}(t) + D_z \hat{u}(t)
\]

with \( C_{zm} = C_z \Phi_T \in \mathbb{R}^{1 \times 3N_m} \).

B. State feedback controller

Different control design strategies may be applied in order to find a gain \( K \) so that the control law \( \bar{u} = K \bar{m} \) guarantees a given closed-loop performance. In the current case, we are interested in the minimization of the effect of the disturbance \( w \) on the control output \( z \) and more specifically to minimize the \( H_2 \) norm of the transfer function \( T_{wu} \) from \( w \) to \( z \). This may be done by solving the following convex optimization \( H_2 \) problem [15], [2]:

\[
\begin{align*}
\min_{w_1, w_2, W_3} & \text{trace}(W_3) \\
\text{subject to} & \\
AW_1 + W_1 A^T + B_u W_2 + W_2^T B_u^T + B_w B_w^T & \leq 0 \\
W_1 & > 0 \\
W_1 C_{zm}^T W_3 + W_2^T D_z^T W_3 & \geq 0
\end{align*}
\]

where \( W_1 \in \mathbb{R}^{3N_m \times 3N_m} \), \( W_2 \in \mathbb{R}^{1 \times 3N_m} \) and \( W_3 \in \mathbb{R} \). The controller gain is calculated as \( K = W_2 W_1^{-1} \) and \( \|T_{wu}\|_2 = \text{trace}(W_3) \). Additional LMI constraints may be added to improve, for instance, some pole placement constraint to the closed-loop system, see section (V-D).

C. Measured output

In real applications it is however difficult to measure all the system states, either because there does not exist the sensor for online measurement of some state variables or because it would be too expensive. This is typically the case when the state is distributed in space. It is more realistic to take only a reduced number of measurements as long as the observability of the system is satisfied. In this way, the measured output is represented as:

\[
y(t) = C_y \bar{y}_\xi(t)
\]

where \( C_y \in \mathbb{R}^{p \times 3N} \) and \( y \in \mathbb{R}^p \).

Substituting (23) into (26), the following representation for the measured output is obtained:

\[
y(t) = C_{ym} \bar{m}(t)
\]

where \( C_{ym} = C_y \Phi_T \in \mathbb{R}^{p \times 3N_m} \).

D. Full order observer

The state feedback controller design presented in section (V-B) may be complemented, according to the Separation Principle, with an observer in order to compensate for the absence of measurements of a part of the state vector. Let the classical Luenberger observer with estimated modes vector \( \hat{m} \) be given by:

\[
\dot{\hat{m}}(t) = A \hat{m}(t) + B_u \hat{u}(t) + \left( \Gamma y(t) - \hat{y}(t) \right)
\]

with the estimated measured output \( \hat{y} \) given by:

\[
\hat{y}(t) = C_{ym} \hat{m}(t)
\]

The observer design objective may be set as the minimization of the effect of the disturbance \( w \) on the estimation error \( e \), that is, as the minimization of the \( H_2 \) norm of the transfer function \( T_{we} \) from \( w \) to \( e \). Moreover, an additional requirement is to impose that the observer dynamics \( (A - \Gamma C_{ym}) \) are faster than those ones of the state feedback \( (A + B_u K) \). This corresponds to add a pole placement constraint in some given region typically formed, as shown in figure 1, by the intersection of an area limited by a vertical strip and a disk.
centered at the origin. This may be done by solving the following convex optimization $H_2$ problem [15], [7]:

$$\min_{W_1, W_2, W_3} \text{trace}(W_3)$$

subject to

$$W_1 > 0$$

$$\left[ \begin{array}{cccc}
W_1 A + A^T W_1 + 2W_1 - W_2 C y_m - C y_m^T W_2 & B_w^T W_1 \\
-W_1 A - W_2 C y_m & -r W_1 \\
A^T W_1 - C y_m^T W_2 & -r W_1 \\
W_1 & I & I & W_3
\end{array} \right] \leq 0$$

(30)

where $W_1 \in \mathbb{R}^{3N_m \times 3N_m}$, $W_2 \in \mathbb{R}^{3N_m \times p}$, $W_3 \in \mathbb{R}^{3N_m \times 3N_m}$, $l$ is the distance between the origin and the vertical strip and $r$ is the radius of the disk. The observer gain is calculated as $\Gamma = W_1^{-1} W_2$ and $\|T_{wc}\|_2 = \text{trace}(W_3)$.

![Observer stability region](image1)

**Fig. 1.** Observer stability region

**E. Output feedback controller**

At this stage a state feedback controller and a full order observer have been developed by using the modes of the parabolic PDE system (1)-(4). The overall control law is the aggregation of the state feedback control with the state observer, that is:

$$\tilde{u}(t) = K \hat{m}(t)$$

(31)

$$\dot{\hat{m}}(t) = (A + B_w K - \Gamma C y_m) \hat{m}(t) + \Gamma y(t)$$

(32)

**VI. SIMULATIONS AND RESULTS**

In order to implement the control strategy developed, the control variable can be either the flow rate $v(t)$ or the ethanol concentration $x_{3, in}(t)$ at the reactor input [6]. In the current study we select the flow rate as control input $u(t) = v(t)$, with $N_u = 1$. The nitrate concentration at the reactor input (the boundary condition) is considered as a disturbance $w(t) = x_{1, in}(t)$ whose value may vary between $14.93 g[N]/m^3$ and $18.93 g[N]/m^3$ around a nominal regime determined by $x_{1, in} = 16.93 g[N]/m^3$, with $N_w = 1$. Therefore, the disturbance domain is determined by:

$$D = \{ \tilde{w} \in \mathbb{R} : \tilde{w}^T \tilde{w} \leq \gamma_0^2, \gamma_0 = 2 \}$$

(33)

The control objective is to maintain the nitrate and nitrite concentrations below the value stated in the European norm while limiting as much as possible the activity of the control input. In this way, $C_z$ and $D_z$ are proposed as:

$$C_z = [0 0 0 \ldots 1 1 0] ; \quad D_z = 0.5$$

Finally, in order to build the linear state space model (21), it has been verified that, as expected, the number of discretization points $N$ has no significant influence on the number of modes which will be kept after reduction ($N_m$) neither on their values. In the present case, twenty one discretization points have been used ($N = 21$).

Figure 2 shows the evolution of the first five modes related to nitrate. Note that, see equation (14), the larger the value of $k$ used, the faster the convergence of $m^k(t)$ to zero. In fact, for $k > 5$ the modes evolve so fast to zero that their contribution to the solution may be neglected. A reduced order model with five modes is then considered as a good approximation to the real solution. The same behaviour can be observed about the modes associated with the nitrite and the ethanol. Therefore, $N_m = 5$ is considered in order to compute both the control law gain $K$ and the observer gain $\Gamma$.

![Nitrate modes](image2)

**Fig. 2.** Nitrate modes

After several observability tests, it has been verified that the pair $(A, C y_m)$ is observable when considering six measurement points of the nitrate and the nitrite along the reactor ($p = 12$). Furthermore, the values of $l = 5$ and $r = 170$ were used in order to place the poles into the stability region showed in figure 1.

In order to test the control scheme, the process is represented by a full FEM simulator.

The open loop response of the system (1)-(4) is shown in figure 4, facing the disturbance input $x_{1, in}$, shown in figure 3, given as a composed sine signal to simulate inlet variations. It was mentioned that the control objective is to maintain nitrate and nitrite concentrations at reactor output below than European norm. It is suggested here as objective to remain below $4mg[N]/m^3$. As shown in figure 4, the system output in open loop (blue line) violates the constraint above stated (red line).
The closed loop response in the presence of the disturbance from figure 3 is shown in figure 5 and the evolution of controlled input is shown in figure 6. It is clear that the output in closed-loop (blue line) is correctly maintained below $4mg[N]/m^3$ (red line).

VII. CONCLUSIONS

In this work, an early lumping approach was used to design an observer-based output feedback controller which maintains correctly the nitrogen concentration at the output of a denitrification reactor, described by a parabolic PDE system, below a pre-established reference. The PDE control problem was first transformed into an ODE one. The high order ODE system obtained was reduced by keeping only its most important modes. Then, the reduced order ODE system was linearized around an operating point. In this way, the advantage of both $H_2$ control and the LMI framework were well used to design a dynamic control law which reduces the energy of a disturbance at the reactor input. The proposed technique may be used for any bioprocesses involving time and space evolution, as soon as the PDE system is parabolic, that is, is involving diffusion phenomena.

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