Randomized Algorithms for Uncertain Complex Dynamical Systems Design

Chenxi Lin, Thordur Runolfsson

Abstract—In this paper we consider the problem of optimal design of an uncertain discrete time dynamical system. We consider two types of performance criteria, corresponding to an apriori equilibrium measure and asymptotic output measure, respectively, that result in two different optimal design methods. However, these two measures are difficult to obtain analytically for most uncertain complex dynamical systems. In order to derive the optimal controller numerically, we apply randomized algorithms for average performance synthesis to approximate the optimal solution. Results from statistical learning theory, providing the relationship between the sample complexity and the approximation error, show that the obtained design methodology is an efficient algorithm for uncertain system design.

I. INTRODUCTION

In this paper we consider optimal design of complex dynamical systems that are subject to uncertainty. Many pessimistic results on the complexity-theoretic barriers of classical robust control problem have stimulated research in the direction of finding alternative solution [1]. In this paper we consider the problem of optimal design of uncertain systems by a probabilistic robust design approach using two types of design criteria. In particular we assume that the system is described by a discrete time dynamical system

\[ x_{i+1} = T(x_i, \theta, \xi), \]

\[ y_i = f(x_i), \]

where \( \xi \) represents the uncertainty in the system which is assumed to be a random parameter with distribution \( \rho \) and \( \theta \) is a vector of design parameters that can either be physical parameters and/or controller parameters. Frequently the design objective is to minimize a cost function \( g(\pi) \) over all admissible \( \theta \in \Theta \), where \( \pi \) is the stationary output for the system (i.e. ignoring the effect of dynamics, see e.g. [2]). In this case the resulting optimization problem is

\[ \min_{\theta \in \Theta} g(\pi) \quad \text{s.t.} \quad \pi = T(\tau, \theta, \xi), \quad f(\pi). \]  

(1)

If the parameter \( \xi \) is random the steady state value \( \tau \) is random as well and consequently the objective function \( g(\pi) \) is random. The probabilistic robust design approach is to replace \( g(\pi) \) with the average cost with respect to some probability measure defined for the output. In [3], an apriori design method is discussed using a-priori equilibrium measure \( \pi_0^\theta \) resulting in the optimal design problem

\[ \min_{\theta \in \Theta} \int_{\mathbb{R}} g(z) \pi_0^\theta (dz) \]  

(2)

II. MATHEMATICAL SETUP

We consider a dynamical system whose evolution is determined by the random difference equation

\[ x_{i+1} = T(x_i, \xi) \]

\[ y_i = f(x_i), \]

where \( x_i \in G \subseteq \mathbb{R}^m \) is the system state, \( \xi \in H \subset \mathbb{R}^n \) is a random parameter defined on some probability space \((\Omega, F, P)\) and \( y_i \) is a scalar valued output, \( i \in \mathbb{Z} \). Denote \( T_\xi(x) = T_\xi \circ \cdots \circ T_\xi \) where \( T_\xi(x) = T(x, \xi) \).

Definition 1: Let \( \mathcal{M}(G) \) be the vector space of real valued measure on \( G \). For a fixed value of \( \xi \) the Perron-Frobenius (P-F) operator \( P_\xi : \mathcal{M}(G) \to \mathcal{M}(G) \) corresponding to the dynamical system \( T_\xi : G \to G \) is defined as

\[ P_\xi \mu(B) = \int_{T_\xi^{-1}(B)} d\mu = \mu \left( T_\xi^{-1}(B) \right) \]

Definition 2: A measure \( \mu_\xi \in \mathcal{M}(G) \) is said to be a \( T_\xi \) invariant measure if

\[ \mu_\xi(B) = P_\xi \mu_\xi(B) = \mu_\xi \left( T_\xi^{-1}(B) \right) \]

The objective of the method is to select the design parameters so as to minimize the cost function based on the a-priori uncertainty information due to initial conditions and parameter uncertainty. However, for some complex dynamical systems that have uncertain behavior generated by the dynamics of the system itself, e.g. chaotic motion, the above method fails. In the same paper [3], an alternative design method using an asymptotic output measure \( \pi_0^\theta \) is discussed to solve this problem with the optimal design criteria

\[ \min_{\theta \in \Theta} \int_{\mathbb{R}} g(z) \pi_0^\theta (dz) \]  

(3)

This criteria accounts for both uncertainty due to initial conditions and parameters as well as uncertainty induced by the system dynamics.

Obviously, in order to obtain the optimal controller, we have to compute these two types of measures. Unfortunately, both the a-priori equilibrium measure and the asymptotic output measure are difficult to obtain in a closed analytic form for complex dynamical systems. Therefore, there is a strong need for an effective computation method for uncertain system design. In this paper, we employ an efficient approach called randomized algorithms for average performance synthesis based on statistical learning theory to solve this problem [4]. Furthermore, we obtain the numerical form of optimal controller for the two design methods.
for all sets $B \in \mathcal{B}_G$ (the set of Borel sets on $G$).

We remark that the P-F operator characterizes the evolution of the distribution of the state $x_t$, i.e. if the initial state $x_0$ has distribution $\nu \in \mathcal{M}(G)$ the distribution of $x_t$ on set $B$ is $P_t^\xi \nu(B)$ and the invariant measure $\mu_\xi(B) = \lim_{t \to \infty} P_t^\xi \nu(B)$.

Let $\mathbf{P} = G \times H$ be the state-uncertainty product space and endow it with the product $\sigma$-algebra $\mathcal{P}$ in the usual way, i.e. if $\mathcal{B}_G$ is the Borel $\sigma$-algebra on $G$ and $\mathcal{F}$ is a $\sigma$-algebra on $H^Z$, then $\mathcal{P} = \mathcal{B}_G \times \mathcal{F}$.

**Definition 3**: A probability measure $\eta$ on $\mathcal{P}$ is called an input measure.

We are interested in the question of how the uncertainty in the "output" of the process depend on the input measure. For the output defined by $f : G \to \mathbb{R}$, the "initial" uncertainty is described by a probability measure $\varpi_i$ on $\mathbb{R}$ (endowed with the Borel $\sigma$-algebra $\mathcal{B}$) defined by

$$\varpi_i(E) = \eta(f^{-1}(E)),$$

where $E \in \mathcal{B}$. This measure evolves in time, becoming

$$\varpi^n(E) = \eta(f \circ T_n^{-1}(E)) = \eta((T_n^\xi)^{-1} f^{-1}(E)) = P_n^\xi \eta(f^{-1}(E)).$$

We call $\varpi^n$ an output measure. It describes the uncertainty of the system output at the $n$-th step of the process given the input measure $\eta$.

Frequently we are mostly interested in the long term behavior of the solution of the system. In this case the uncertainty in the system output is best studied in terms of the uncertainty in the asymptotic properties of the system. In particular, define the time-average

$$f^*(x, \xi) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T(x_i, \xi))$$

and the asymptotic output measure $\varpi_\alpha$

$$\varpi_\alpha(E) = \lim_{n \to \infty} \varpi^n(E) = \eta((f^*)^{-1}(E)).$$

Return now back to the uncertain system (4) with assumption that $\xi = c$ where $c$ represents a (certain) parameter. In other words, there is no parametric uncertainty and the input measure $\eta$ is replaced by uncertainty distribution $\nu$ for the initial conditions of the states. Then we can rewrite the asymptotic output measure as

$$\varpi_\alpha(E|\xi = c) = \lim_{n \to \infty} P_n^\xi \nu(f^{-1}(E)) = \mu_\xi(f^{-1}(E))$$

Therefore, if the random parameters in the dynamical systems are fixed, the asymptotic output measure defined on the output space can be characterized by the invariant measure of system states.

### III. Uncertain System Design

In many practical problems the system performance is subject to uncertainties that can be attributed to inaccuracies in physical parameters, lack of knowledge about physical characteristics as well as modeling inaccuracies and approximation errors. Many pessimistic results on the complexity-theoretic barriers of classical robust control problem have stimulated research in the direction of finding alternative solution [1]. One effective solution is to shift the meaning of robustness from its usual deterministic sense to a probabilistic one [1]. This shift in meaning implies a statistical description of the uncertainty, opposed to a purely unknown-but-bounded one. It has been shown that the probabilistic approach presents itself as a natural tool to deal with the random character of uncertainties affecting control systems [5]. In this section we discuss the incorporation of the uncertainty measure defined in the previous section into the probabilistic robust design approach.

Consider the dynamical system (4) that now depends on a control input $u_i$. We also assume that the parametric uncertainty $\xi$ is a random parameter with distribution $\rho$. Thus the system is now

$$x_{i+1} = T(x_i, u_i, \xi)$$

$$y_i = f(x_i)$$

Furthermore, we assume the control input $u_i$ has the form $u_i = u(x_i, \theta)$ where the control parameter $\theta \in \Theta$, the design parameter space. We denote the class of all such control laws by $\mathcal{U}(\theta)$. We assume that there exists a nontrivial compact subset $\Theta \subset \Theta$ and a subset $\mathcal{U}(\theta) \subset \mathcal{U}(\theta)$ that has the property that any control law $u \in \mathcal{U}(\theta)$ with $\theta \in \Theta$ stabilizes the closed loop system for all initial states and all parameters $\xi \in H$. In other words, we focus our attention on stabilizing state feedback control laws. For $u \in \mathcal{U}(\theta)$, $\theta \in \Theta$ the dynamical system equation can be rewritten as

$$x_{i+1} = \hat{T}(x_i, \theta, \xi)$$

$$y_i = f(x_i)$$

Assume that the objective of the design for system (9) is to minimize the steady state performance function (1) over all admissible $\theta \in \Theta$. In the design literature $y$ is frequently evaluated at an equilibrium point of the system. Since the parameter $\xi$ is random the steady state value $\bar{x}$ is random as well and consequently the objective function $g(\bar{y})$ is random. Following [6], [7] we use an average performance as the objective function to be minimized. Thus, we replace $g(\bar{y})$ with the average cost resulting in the optimal design problem

$$\min_{\theta \in \Theta} E^\rho[g(\bar{y})] \quad \text{s.t.} \quad \bar{y} = \hat{T}(\bar{x}, \theta, \xi), \quad \bar{y} = f(\bar{x})$$

where $E^\rho[\cdot]$ is the expectation with respect to the distribution $\rho$. If we define a a-priori equilibrium measure $\varpi_{\rho}$ by

$$\varpi_{\rho}(E) = \rho(\{\xi \in H | f(x) \in E, x = T(x, \theta, \xi)\})$$

then (10) can be rewritten as (2). In the remainder of the paper we refer to (2) as the a-priori design criteria.

We note that if for some values of $\xi$ and $\theta$ the equilibrium is unstable then the system will never reach the steady state and consequently the optimization problem formulated above
does not make sense. Furthermore, the system may exhibit uncertain behavior that is “generated” by the dynamics of the system itself. All uncertainty effects can be correctly accounted for by reformulating the design problem utilizing the asymptotic output measure corresponding to the time-averages of the output $y$. Indeed, formulating the optimal design problem as (3) captures all uncertainty effects through the definition of asymptotic output measure $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f \left( \hat{T}(x_i, \theta, \xi) \right)$ where
\begin{equation}
\hat{f}_{\theta}^*(x, \xi) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f \left( \hat{T}(x_i, \theta, \xi) \right)
\end{equation}
and $\eta$ is the input measure. We call (3) the a-posteriori design criteria.

We remark that both the a-priori design (2) as well as a-posteriori design (3) are in general difficult to compute analytically, since the a-priori equilibrium measure $\hat{w}_n^\theta$ and asymptotic output measure $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f \left( \hat{T}(x_i, \theta, \xi) \right)$ cannot generally be characterized analytically for complex systems. Consequently, since it is difficult to derive the performance function analytically it is even harder to obtain the optimal controllers that minimize the performance functions. Therefore, we need an effective computational method for the optimal design of the uncertain system. In the following sections, we discuss an efficient approach called randomized algorithms based on statistical learning theory to solve this problem [4].

IV. RANDOMIZED ALGORITHMS FOR AVERAGE PERFORMANCE SYNTHESIS

In the previous section we observed that assessing probabilistic robustness of a given uncertain system does not result in simplification of the computational complexity of the design problem, since both the a-priori equilibrium measure and asymptotic output measure are difficult to obtain analytically. To address this issue randomization is used: the performance function is estimated by randomly sampling the uncertainties, and tail inequalities are used to bound the estimation error. The algorithms obtained, usually called randomized algorithms (RA), often have low complexity and the resulting robustness bounds are generally less conservative than classical ones [8]. Let $\xi \in H$ represent the random uncertainty acting on the system and let $\rho$ be the distribution of $\xi$. The controller is represented by a vector $\theta \in \Theta$, where $\Theta \subseteq \mathbb{R}^{n\alpha}$ is the set of allowable design parameters. Furthermore, let $J : H \times \Theta \rightarrow [0, 1]$ be a performance function for the uncertain system. The following definition of RA for average performance synthesis is from [4].

**Definition 4:** Let $\epsilon \in (0, 1), \delta \in (0, 1)$ be given probability levels. Let
\begin{equation}
\phi(\theta) = \mathbb{E}^\rho[J(\xi, \theta)]
\end{equation}
denote the average performance (with respect to $\rho$) of the controlled plant, and
\begin{equation}
\phi^* = \min_{\theta \in \Theta} \phi(\theta)
\end{equation}
denote the optimal achievable average performance. A randomized average synthesis algorithm should return with probability $1 - \delta$ a design vector $\hat{\theta}_N \in \Theta$, such that
\begin{equation}
\hat{\phi}(\hat{\theta}_N) - \phi^* \leq \epsilon
\end{equation}
The controller parameter $\hat{\theta}_N$ is constructed based on a finite number $N$ of random samples of $\xi$.

We note that the computation of the expected value $\phi(\theta) = \mathbb{E}^\rho[J(\xi, \theta)]$ is generally difficult. Therefore, we approximate the expectation $\phi(\theta)$ by its empirical version. That is, $N$ samples $\xi_1, ..., \xi_N$ are collected and the empirical mean is obtained as
\begin{equation}
\hat{\phi}_N(\theta) = \frac{1}{N} \sum_{i=1}^{N} J(\xi_i, \theta).
\end{equation}
Obviously, the approximation error will be affected by sample size $N$. Then a sub-optimal solution $\hat{\theta}_N \in \Theta$ that minimizes the empirical mean (approximate expectation) $\hat{\phi}_N(\theta)$ need to be found. In principle the minimization of $\hat{\phi}_N(\theta)$ over $\theta \in \Theta$ can be performed by any numerical optimization method. A viable approach, that works even in non-convex situations, would be to use a random search algorithm for the determination of a probable minimum of $\hat{\phi}_N(\theta)$. In order to apply random search algorithm to find the optimal controller, we introduce an artificial probability distribution $\pi$ over the set $\Theta$ of controller parameters (sometimes based on some a-priori estimates about the optimal controller parameters).

Correspondingly, the performance function $\hat{\phi}_N(\theta_k)$ is evaluated for each sample $\theta_k, k = 1, ..., M$ and the empirical optimal controller is obtained as
\begin{equation}
\hat{\theta}_{NM} = \arg\min_{k=1,\ldots,M} \hat{\phi}_N(\theta_k).
\end{equation}

**Theorem 1:** [4]Given $\epsilon_1, \epsilon_2, \delta \in (0, 1)$, let $M \geq \frac{\ln 2}{\ln \frac{\epsilon_1}{\epsilon_2}}$ and $N \geq \frac{\ln \frac{4M}{\epsilon_1}}{\epsilon_2}$. Then with confidence $1 - \delta$, it holds that
\begin{equation}
\Pr \left\{ \phi(\theta) < \hat{\phi}_N(\hat{\phi}_{NM}) - \epsilon_1 \right\} \leq \epsilon_2
\end{equation}
Now, we present the randomized algorithms for average performance synthesis based on Theorem 1 as follow [4]:

**Algorithm 1:** (RA for average performance synthesis) Let $J : H \times \Theta \rightarrow [0, 1]$. Assume $\theta$ and $\xi$ are random with distribution $\pi$ and $\rho$. Given $\epsilon_1, \epsilon_2, \delta \in (0, 1)$, this RA returns with probability at least $1 - \delta$ a design vector $\hat{\theta}_{NM}$ such that (14) holds.
1) Determine $M = \overline{M}(\epsilon_2, \delta)$ and $N = \overline{N}(\epsilon_1, \delta, M)$ according to Theorem 1;
2) Draw $M$ iid samples $\theta_1, ..., \theta_M$ from $\pi$;
3) Draw $N$ iid samples $\xi_1, ..., \xi_N$ from $\rho$;
4) Return the empirical controller
\begin{equation}
\hat{\theta}_{NM} = \arg\min_{k=1,\ldots,M} \frac{1}{N} \sum_{i=1}^{N} J(\xi_i, \theta_k).
\end{equation}
V. Optimal Design of Uncertain Systems

In this section we apply the algorithm presented in the previous section to both the a-priori and a-posteriori design criteria for optimal control design for an uncertain system.

We assume that all performance functions are normalized to satisfy the requirement in Algorithm 1.

A. The a-priori design criteria

As we discuss in earlier, for the a-priori design criteria, by comparing (10) and (13) we get the performance function

\[ J_a(\xi, \theta) = g(\mathbf{y}_0, \xi) \]

where \( \mathbf{y}_0, \xi \) is the stationary output of the system equations (9) determined by \( \theta \) and \( \xi \). In applying Algorithm 1 we sample the parameter and controller spaces to obtain \( \xi_i \) and \( \theta_k \) for \( i = 1, \ldots, N \), \( k = 1, \ldots, M \). Thus, we obtain

\[ J_a(\xi_i, \theta_k) = g(\mathbf{y}_{k,i}) \]

where \( \mathbf{y}_{k,i} \) is the output solution of steady state equations for \( \theta_k, \xi_i \), that is,

\[ \mathbf{y} = T(\mathbf{y}, \theta_k, \xi_i) \]

\[ \mathbf{y}_{k,i} = f(\mathbf{x}) \]  

(17)

If \( \mathbf{y}_{k,i} \) is not unique, then the a-priori equilibrium measure is only determined by the system steady state equations, and has nothing to do with system dynamics, \( \omega_\theta^a \) is a Dirac delta measure concentrated at every stationary output with equal strengths, i.e.

\[ J_a(\xi_i, \theta_k) = \frac{1}{n(\mathbf{y}_{k,i})} \sum_{j=1}^{n(\mathbf{y}_{k,i})} g(\mathbf{y}_{k,i,j}) \]

where \( n(\mathbf{y}_{k,i}) \) is the number of distinct solution \( \mathbf{y}_{k,i} \) for equation (17). Furthermore, by Algorithm 1 we obtain the a-priori design controller,

\[ \hat{\theta}^a_{NM} = \arg \min_{k=1, \ldots, M} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{n(\mathbf{y}_{k,i})} \frac{g(\mathbf{y}_{k,i,j})}{n(\mathbf{y}_{k,i})} \]  

(18)

B. The a-posteriori design criteria

In order to apply the randomized design algorithm we rewrite the a-posteriori design criteria (3) as follows.

**Proposition 2**: The a-posteriori design criteria satisfies

\[ \int \mathbf{R} g(z) \omega_\theta^a(\mathbf{z}) \rho(\mathbf{z}) \]

\[ = \int H J_a(\xi, \theta) \rho(\mathbf{z}) \]

where the performance function \( J_a(\xi, \theta) \) is given by

\[ J_a(\xi, \theta) = \int \mathbf{R} g(z) \nu(x \in G| f^a_\theta (x, \xi)) \in \mathbf{z}) \]

**Proof**: The proof is omitted to save space and is available from the authors.

Now that we have rewritten the design criteria in the standard form (13) we proceed to apply the randomized algorithm for optimal design. We sample parameter and controller space to obtain \( J_a(\xi_i, \theta_k) \),

\[ J_a(\xi_i, \theta_k) = \int \mathbf{R} g(z) \nu(x \in G| f^a_\theta (x, \xi_i)) \in \mathbf{z}) \]

(19)

From the definition of the asymptotic output measure \( \omega_\theta^a \) we have

\[ \omega_\theta^a(\mathbf{z}) = \nu(x \in G| f^a_\theta (x, \xi_i)) \in \mathbf{z}) \]

Also, by (7), we have

\[ \omega_\theta^a(\mathbf{z}) = \mu_\xi^a(\mathbf{z}) \]

where \( \mu_\xi^a \) is the invariant measure of the dynamical system for given \( \xi_i \) and \( \theta_k \). Thus, we can rewrite (19) with respect to invariant measure as

\[ J_a(\xi_i, \theta_k) = \int \mathbf{R} g(z) \mu_\xi^a(\mathbf{z}) \]

\[ = \int G g(\mathbf{z}) \mu_\xi^a(\mathbf{z}) \]

By (16), the formula for a-posteriori design controller is

\[ \hat{\theta}^a_{NM} = \arg \min_{k=1, \ldots, M} \frac{1}{N} \sum_{i=1}^{N} \int G g(\mathbf{z}) \mu_\xi^a(\mathbf{z}) \]

(20)

C. Computation of invariant measure

A common approach for the computation of the invariant measures \( \mu_\xi^a \) is to use a Monte Carlo type method that samples the distribution of the initial state and then simulates the system until it reaches stationarity (steady state). Unfortunately, one needs many simulation runs to achieve reasonable accuracy and for complex dynamic systems the computational effort may be excessive. Here, we adopt an alternative approach for the computation of the invariant measure that is discussed in [9]. In particular, we note that under the appropriate conditions the system has an ergodic invariant measure that is characterized as a fixed point of the Perron Frobenius operator \( P_\xi \) for \( T_\xi \). The computational approach relies on the discretization of the P-F operator.

In order to obtain a finite-dimensional (discrete) approximation of the P-F operator, we consider a finite partition of the state space \( G \), denoted as \( B_1, B_2, \ldots, B_L \), where \( B_i \cap B_j = \phi \) and \( \cup B_i = G \). Corresponding to each partition element we associate a positive number \( \mu_j \in [0, 1] \) with \( \sum_{j=1}^{B_L} \mu_j = 1 \), i.e., \( \mu = (\mu_1, \ldots, \mu_L) \in \mathbb{R}^L \) is a probability vector. Define a probability measure on \( G \) as

\[ \pi(dx) = \sum_{i=1}^{L} \pi_i \chi_{B_i}(x) \frac{m(dx)}{m(B_i)} \]

(21)

where \( m \) is the Lebesgue measure and \( \chi_{B_i} \) is the indicator function for \( B_i \). Then, the action of the Perron Frobenius operator \( P_\xi \) on \( \pi \) on the element \( B_i \) is

\[ P_\xi \pi(B_j) = \sum_{i=1}^{L} \pi_i \pi(T_\xi^{-1}(B_j) \cap B_i) \]

(22)

where the \( L \times L \) matrix with entries \( \pi_{ij}(\xi) \) is a stochastic transition matrix. In [9], we have shown that the operator
\( \mathcal{P}(\xi) \) is a "good" approximation of \( P_\xi \) and the invariant measure for \( P_\xi \) can be approximated by a measure \( \pi \) defined by (21) where the coefficients of \( \mathcal{P} \) are invariant for \( \mathcal{P}(\xi) \), i.e. satisfy \( \mathcal{P}(\xi) = \pi(\xi) \), where \( \pi = \pi(\xi) \).

Having obtained the approximate invariant measure \( \pi_\xi \) we replace the original formulation for the a-posteriori design criteria (20) by the following approximate formulation defined on the partition of the state space \( G \). With respect to the approximate invariant measure \( \pi_{0_k} \), the approximate optimization problem for (20) becomes,

\[
\hat{\theta}_{NM} = \arg \min_{\theta_k} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{L} g \circ f(m_{ij}) \cdot \pi_{0_k}(j) \tag{23}
\]

We note that here \( L \) is the number of boxes in the partition of the state space \( G \), \( m_{ij} \) is the central point of the \( j \)-th box and \( \pi_{0_k} \) is the approximate invariant measure calculated by the corresponding transition matrix \( \mathcal{P}(\theta_k, \xi) \) defined in (22).

Note that the principal factor affecting the computational complexity in the calculation of the approximate invariant measure is the discretization level on the state space. In [9], an efficient adaptive subdivision algorithm for state space partition is proposed by incorporating information about the invariant measure.

VI. EXAMPLE

Consider an optimal design problem for a continuous stirred tank reactor (CSTR) system. In [10], the kinetic equations of the reaction can be reduced to a dimensionless second-order ordinary differential equation which we represent in the form as (8)

\[
\begin{align*}
\dot{x}_1 &= -x_1 + u \cdot \exp \{ x_2 \} \\
\dot{x}_2 &= -(1 + \beta)x_2 + Bu \cdot \exp \{ x_2 \}
\end{align*}
\]

where \( x_1 \in [0, 1], x_2 \in [0, 10] \) are the system states denoting the reactant concentration and temperature respectively, \( u \) is the control input for the reactant concentration. The constant \( B \in [5, 9] \) is defined by the dimensionless adiabatic temperature rise and related to the inverse of the reactant feed temperature which, in turn, is viewed as a random parameter. Finally, the constant \( \beta = 0.5 \) is a dimensionless heat transfer coefficient. It is well known that the norm of the steady state \( ||x_{s}(B, u)||_2 \) has an interpretation as the productivity of the reaction process in the system, and maximization of this productivity in a stable steady state by an appropriate control method is an important practical goal. Here, we consider the simple state feedback control strategy,

\[
u = Da \cdot (1 - x_1)
\]

where \( Da \) is the Damkohler number is identified with the inverse of the input flow rate and is considered here as a control parameter. Then, we modify the dynamical system (24) by adding the control parameter \( Da \) and an output equation related to the objective function,

\[
\begin{align*}
\dot{x}_1 &= -x_1 + Da(1 - x_1) \exp \{ x_2 \} \\
\dot{x}_2 &= -(1 + \beta)x_2 + BDa(1 - x_1) \exp \{ x_2 \} \\
y &= \sqrt{x_1^2 + x_2^2}
\end{align*}
\]

It can be shown that steady state \( x_s = (x_{1s}, x_{2s})^T \) of (25) satisfies

\[
\begin{align*}
Da &= (x_{1s}/(1 - x_{1s})) \exp \{-B/(1 + \beta)\}x_{1s} \\
x_{2s} &= (B/(1 + \beta))x_{1s}
\end{align*}
\]

Therefore,

\[
\mathcal{g} = ||x_s||_2 = x_{1s}(1 + B^2/(1 + \beta)^2)^{1/2}
\]

which results, for all values of the control parameter \( Da \), in the S-shape steady state characteristic shown in Figure 1, referred to as a system with Arrhenius dynamics. The negative slope part of the S-shape curve represents a set of unstable steady states.

In practice, the upper stable branch of the curve is not acceptable because of technological reasons. Hence, our performance function is chosen to maximize the norm of steady states bounded to some acceptable range. Specifically, the resulting optimization problem is

\[
\min_{\delta \in S} \mathcal{g}(\mathcal{g}) = \min_{Da \in S} \frac{1}{\chi_{E}||x_s||_2}
\]

where \( \chi_{E} \) is the indicator function for some acceptable system operation range \( E \) determined by reaction device parameters. Due to the random parameter \( B \), the steady states are random also. The problem can be considered as an uncertain system design problem of the type discussed in the previous sections. Below we will apply both the a-priori design method and the a-posteriori design method. In our design, we assume initial states have uniform distribution in \([0, 1] \times [0, 10] \) and the control parameter \( Da \) is uniform on \([0.06, 0.12] \) and choose \( E = [0, 2] \). The discretization step is \( 10^{-4} \).
samples of the uncertainty, we solve the steady state equation (26) and find the corresponding average optimal performance value is

\[ \bar{\phi}_N(Da^*) = \frac{1}{N} \sum_{i=1}^{N} J_a(\xi_i, Da^*) \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{L} \frac{1}{\chi_E(\bar{\gamma}_{ij})} \cdot n(\bar{\gamma}_{ij}) = 2.1872 \]

The stationary distribution of system output \( ||x_s||_2 \) colored by a-priori equilibrium measure is shown in Figure 2.

2) a-posteriori design method:

Setting the same probability levels as in the a-priori design method results in the same number for samples \( M = 51 \) and \( N = 496 \). For every sample of \( Da \) and \( B \), we use the steady state equation (26) and find the optimal controller parameter by (18) as \( Da^* = 0.0784 \). The corresponding average optimal performance value is

\[ \hat{\phi}_N^a(Da^*) = \frac{1}{N} \sum_{i=1}^{N} J_a(\xi_i, Da^*) \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{L} \frac{1}{\chi_E(\gamma_{ij})} \cdot n(\gamma_{ij}) = 2.8409 \]

The stationary distribution of output with respect to asymptotic output measure is shown in Figure 3.

Although the numerical calculations show that \( \hat{\phi}_N^a(Da^*) < \hat{\phi}_N(Da^*) \), the a-priori equilibrium measure \( \overline{\omega}^a \) does not discriminate between stable and unstable points and as result the optimal calculation includes values evaluated at some unstable steady states as shown in Figure 2. Obviously, unstable points are not allowed as part of the optimal solution. Consequently, since the support of asymptotic output measure \( \overline{\omega}^a \) is only on the stable branches of the steady states characteristic and a-posteriori design method does not include any such unstable points we conclude that it still yield a better result. Moreover, the a-posteriori design method is more robust than a-priori design method since the norm of steady states stays in the accessible range for a larger range of values of the uncertain parameter.

REFERENCES


