Gain scheduling versus robust control of LPV systems: the output feedback case

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Abstract—We compare robust and gain scheduling control for continuous-time LPV systems. In the literature it has been shown that some duality properties as well as a separation principle hold for gain-scheduling control. This means that an LPV plant can be stabilized by a gain scheduling control law if and only if a proper gain scheduling LPV state observer and LPV stabilizer can be designed. Moreover observer and state-feedback designs are dual problems.

In this paper we show, by means of counterexamples that, in robust design neither of the above properties hold. Precisely, it is shown that, if the parameter values are not measurable on-line, then there exist robust stabilizable plants for which no state detection is possible and that there exist plants which are stabilizable via state feedback whose dual system does not admit robust observers. In passing we solve an open question. In the state-feedback design case it was previously shown the equivalence between robust and gain-scheduling control for polytopic systems, namely that state-feedback gain-scheduling stabilizability implies state-feedback robust stabilizability. We prove that this is not true anymore in the output feedback case.

The discrete-time case will be briefly discussed at the end.

Index Terms—LPV systems, robust control, gain-scheduling control, separation principle, duality.

I. INTRODUCTION

Linear parameter-varying (LPV) systems are a useful generalization of linear time-invariant (LTI) systems because they provide the natural setting in the adaptive (gain-scheduling) control of linear plants whose parameters vary in time and because many nonlinear plants can conveniently be embedded into a linear differential inclusion [9], Sect. 4.3; and [19] [14]). In fact, recent surveys have pointed out the importance given to the LPV framework in the modern control system literature [21][18]. In this paper we consider the specific aspect of comparing the gain scheduling and the robust control for LPV systems. Roughly the former is the case in which the controller is “informed” on-line of the actual values of the parameters. Basically our aim is to show, by counterexamples, that all the properties which hold for gain scheduling controllers, namely duality and separation principles, fail in the robust control case.

The stabilization problem for LPV systems is essentially based on the Lyapunov theory, as it has been shown that the stability or stabilizability of an LPV system is equivalent to the existence of a Lyapunov norm [16] [11] [10] [4], even if not a quadratic one, in general. The theory of robust control (see for instance [15][13][12]) and gain scheduling control (see [19][1][2]) have been often based on the theory of quadratic functions. Here we are not considering any specific tool but we examine these problem in general.

In the gain scheduling control of LPV systems some recent results have been achieved such as duality and separation principle. The main results of this paper are counterexample which show that these properties do not hold in the robust control case. Precisely we show the following.

- There is no separation principle in robust control design of LPV.
- No duality results hold in robust control design of LPV.
- In the output feedback case there is no equivalence, differently from the state feedback case, between robust and gain scheduling stabilization.

II. PRELIMINARY DEFINITIONS

Consider a strictly-proper LPV plant described by

\[
\begin{align*}
\dot{x}(t) &= A(w(t)) x(t) + B(w(t)) u(t), \\
y(t) &= C(w(t)) x(t),
\end{align*}
\]

where \(x(t) \in \mathbb{R}^n\), \(u(t) \in \mathbb{R}^m\), \(y(t) \in \mathbb{R}^p\), and \(w(t) \in \mathcal{W}\), with \(\mathcal{W}\) compact.

In this paper we consider the stabilization problem for the above class of systems and we focus on a specific problem, namely the relation between the two situations in which the controller relies on parameter measurement or not.

Following the standard nomenclature we define

- **Gain-Scheduling (GS) Control**: a control which measures the parameters on-line;
- **Robust (R) Control**: a control which does not measure the parameters on-line;

The two situations correspond to the case in which the dashed link in Fig. 1 is active and, respectively, inactive.

Clearly there is an intermediate situation in which the controller can measure part of the parameters. This situation will not be considered in view of the negative nature of the results of this paper. We will also distinguish the situations in which the controller is linear and non-linear, although the paper is mainly concerned with the linear compensator case.

In the sequel we will consider, as a special case of interest, the polytopic class of systems, according to the following definition
Definition 2.1: System (1) is said a polytopic LPV system (PLPV) if
\[
\begin{bmatrix}
A(w) & B(w) \\
C(w) & 0
\end{bmatrix}
= \sum_{i=1}^{s}
\begin{bmatrix}
A_i & B_i \\
C_i & 0
\end{bmatrix}
w_i
\tag{2}
\]
with
\[w_i \geq 0, \quad \sum_{i=1}^{s} w_i = 1,\]

More in general, if the matrix family in (1) is convex (not necessarily polytopic) then we say that the system is a convex LPV (CLPV) system.

Definition 2.2: The system
\[
\begin{bmatrix}
A(w) & B(w) \\
C(w) & 0
\end{bmatrix}
\begin{bmatrix}
A(w) & C(w) \\
B(w) & 0
\end{bmatrix}^T
\tag{3}
\]
or
\[
\dot{x}(t) = A^T(w)x(t) + B^T(w)\eta(t),
\eta(t) = C^T(w)x(t),
\tag{4}
\]
is said to be the dual of (1).

Given a system of the form (1) we are interested in its stabilization by means of linear compensators of the form
\[
\begin{align*}
\dot{z}(t) &= F(w(t))z(t) + G(w(t))\eta(t), \\
u(t) &= H(w(t))z(t) + K(w(t))\eta(t)
\end{align*}
\tag{5}
\]
According to the previous definitions, we say that the controller is robustly stabilizing if \(F, G, H, K\) do not depend on \(w\) and ensure closed-loop stability for any \(w(t)\).

III. A BRIEF REVIEW OF EXISTING RESULTS

A. Duality and separation principle for gain-scheduling control design

The following result holds [8].

Theorem 3.1: System (1) can be stabilized by a control of the form (5) if and only if the two following conditions are satisfied

- the system is stabilizable via state feedback by a compensator of the form
  \[
  \begin{align*}
  \dot{z}(t) &= F_{SF}(w(t))z(t) + G_{SF}(w(t))x(t), \\
u(t) &= H_{SF}(w(t))z(t) + K_{SF}(w(t))x(t)
  \end{align*}
  \tag{6,7}
  \]

next theorem shows that state feedback control and observer design are dual problems.

Theorem 3.2: System (1) can be stabilized by state feedback if and only if its dual admits a generalized observer. A formal proof of the above theorem is provided in [3], in the case of constant \(B\) and \(C\). The result can be easily extended to the general case of time-varying \(B(w)\) and \(C(w)\) in view of the results in [8].

B. State feedback case

For the state feedback case, several interesting results can be proved. Perhaps the most relevant for our investigation is the following, which holds for convex LPV systems.

Theorem 3.3: [6] If a convex LPV system is gain scheduling stabilizable via state feedback then it is robustly stabilizable.

Briefly, the "parameter measurement" link in Fig. 1 is useless as far as the state feedback stabilization of a CLPV system is concerned. The author is referred to [7] for a more detailed discussion on this issue. The question if this property is true for CLPV systems in the output feedback case is open and will be addressed later.

The next result concerns the linearity of the compensator.

Theorem 3.4: If a convex LPV system is gain scheduling stabilizable then it is always possible to find a linear GS compensator of the form (6) (7).

Note that, in general, this property is not true for robust control [5] where nonlinear compensators can outperform linear ones.
Proposition 3.1: There exist convex LPV systems which are robustly stabilizable but which do not admit linear compensators. We remind that this negative result was already evidenced in other contexts such as in quadratic stabilization [17] and $l_1$ theory [20].

IV. MAIN RESULTS

This section we show that none of the “nice properties” concerning the GS stabilization problem for LPV systems are valid in the robust stabilization case.

Note that to be meaningful, all the provided counterexamples are of the polytopic type.

A. There is no separation principle in robust control design of LPV

We show the above claim by a very simple counterexample (which is taken and suitably extended from [5]). The PLPV system

\[ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = 0 \]

This can be shown by means of the Lyapunov function

\[ V(x) = \frac{1}{2} x^T P x \]

whose derivative, along the system trajectory, is

\[ \dot{V}(x) = \dot{\lambda} x^T P x \]

with \( \lambda \) an arbitrary positive diagonal matrix.

Though it is not always decreasing, stabilizability can be proved by Krasowski arguments.

However, no device can asymptotically reconstruct the state with arbitrary \( u(\cdot) \). For instance take \( u \equiv 0 \), and assume that the trajectory is the unit circle. Whenever \( x_2 \) is in the strip \( \xi \leq |x_2| \leq 1/\xi \), one might assume that \( \gamma(t) = 1/|x_2(t)| \) so that \( y(t) = \pm 1 \), say no information about the state is available. On the other hand, the state can rotate with different speeds depending on \( \omega(t) \). Roughly, when the state enters the “blind strip” of Figure 3 no detection is possible and a persistent error is unavoidable. It is easy to change this example by perturbing the \( A_{22} \) coefficient with a positive small term \( \epsilon \) to show that the error might diverge.

B. In robust design state feedback and state detection are not dual

The lack of duality was pointed out in [3] for discrete–time systems. Here we show a continuous–time case by adapting the example proposed in [5]. The system is

\[ \dot{x}_1 = \omega x_2 + u(t) \]
\[ \dot{x}_2 = -\omega x_1 + \beta u(t) \]

and

\[ \beta \leq \rho \quad \text{and} \quad \mu \leq \omega \leq 1/\mu. \]

This system can be stabilized by the GS feedback

\[ u(t) = -\kappa \frac{x_1 + \beta x_2}{1 + \kappa} \]

Consider the same Lyapunov function as before \( (x_1^2 + x_2^2)/2 \) to get

\[ \dot{\Psi}(x_1, x_2) = -\kappa \frac{x_1 + \beta x_2}{1 + \kappa} \]

Again this function is only non–positive for arbitrary \( \kappa > 0 \). However, if \( \kappa \) is small the system has a “rotating behavior” so that the state periodically reaches the region in which

\[ [x_1 + \beta x_2]^2 \neq 0 \]

in which the Lyapunov derivative is necessarily negative so that \( x(t) \to 0 \).

We let the reader note that, in view of theorem 3.3, the existence of a GS stabilizing feedback implies that the system is also robustly stabilizable. Unfortunately, as it will be shown next, the dual system admits a GS observer but not a robust observer. Consider now the dual problem

\[ \dot{x}_1 = -\omega x_2 \]
\[ \dot{x}_2 = \omega x_1 \]
\[ y = x_1 + \gamma x_2 \]

and

\[ |\gamma| \leq \rho \quad \text{and} \quad \mu \leq \omega \leq 1/\mu. \]

For this system the following GS Luenberger observer

\[ \dot{x} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} x + \begin{bmatrix} -\kappa \\ -\kappa \gamma \end{bmatrix} y - \begin{bmatrix} 1 & \gamma(t) \end{bmatrix} \]

provides an asymptotic estimate of the true state \( x \) when \( \kappa > 0 \) (this can be seen by using the same Lyapunov arguments used for the GS state feedback). If no parameter measurements are available, say in the robust case, no asymptotic estimator can be found. Indeed, in the double sector (see Fig. 4)

\[ S_{\rho} = \{ (x_1, x_2) : |x_1| \leq \rho |x_2| \} \]

there always exists an admissible value of \( \gamma(t) \) such that \( y(t) = x_1(t) + \gamma x_2(t) = 0 \) precisely

\[ \gamma(t) = -\frac{x_1(t)}{x_2(t)} \]

Now, if the above is the actual value of \( \gamma(t) \) whenever \( x(t) \in S_{\rho} \), then no output information is available. On the other hand, \( x(t) \) rotates at a speed depending on \( \omega \), so that, again, no state detection is possible.
Remark 4.1: It is not possible to provide examples of CLPV systems which are robustly detectable whose dual is not state feedback stabilizable. Indeed robust detection implies GS detection which implies GS state–feedback stabilizability, hence robust stabilizability, of the dual.

C. Robust and gain scheduling design are not equivalent for output feedback

Unlike the state feedback case, we can show that GS–

compensators provide advantages over the R-compensators for output feedback. We initially consider a system which

is not convex so that, in principle, it cannot serve as a counterexample but gives an intuitive explanation of the idea. This system will be “merged” in a convex differential inclusion. Consider the LPV system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
y_1 \\
y_2
\end{bmatrix} =
\begin{bmatrix}
\zeta 
& \cos(\beta) & -\sin(\beta) \\
\sin(\beta) & \cos(\beta) & 0 \\
\eta & \cos(\gamma) & -\sin(\gamma) \\
\sin(\gamma) & \cos(\gamma) & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
x_1 \\
x_2
\end{bmatrix}
\]

with

\[
1 \leq \zeta(t) \leq 1/\sigma, \quad 1 \leq \eta(t) \leq 1/\nu,
\]

where \(0 < \sigma, \nu < 1\) and with

\[
|\beta| \leq \rho \quad |\gamma| \leq \theta
\]

where the angles bounds are

\[
\frac{\pi}{4} \leq \rho, \theta \leq \frac{\pi}{2}
\]

We let the reader note that the output \(y = C(\gamma, \eta)x\) is equal to \(x\) rotated by an angle \(\gamma\) and scaled by \(\eta\), so that the knowledge of \(\gamma\) and \(\eta\) allows to determine \(x\) from output measurement. Similarly, matrix \(B(\beta, \zeta)\) rotates and scales the forcing term by an angle \(\beta\) and \(\zeta\) and the knowledge of such two parameters is equivalent to have a full control action on the system. Clearly, the situation changes completely if \(\gamma\) and \(\beta\) are unknown.

To recast the above system in a CPLV system, we absorb the \(B\) and \(C\) matrices in a polytopic system. This can be done by noting that \(B\) has the form

\[
B = \begin{bmatrix}
b_1 \\
b_2 \\
b_2 \\
b_1
\end{bmatrix}
\]

and vector \(b = [b_1 b_2]^T\) is in the “bow” (see Fig. 5)

\[
B = \{b = [b_1 b_2]^T : \quad 1 \leq \|b\| \leq 1/\sigma, \quad |\arg(b)| \leq \rho\}
\]

where the complex interpretation \(\arg(b) = \arg(b_1 + jb_2)\) is adopted. The bow can in turn be absorbed in the convex hull of the five points \(p_i, i = 1, \ldots, 5\) (see Fig. 5), i.e. \(b \in P\) where

\[
P = \text{conv}\left\{\sigma \begin{bmatrix}
\cos(\rho) \\
\sin(\rho)
\end{bmatrix}, \frac{1}{\sigma} \begin{bmatrix}
\cos(\rho) \\
\sin(\rho)
\end{bmatrix}, \sigma \begin{bmatrix}
\cos(\rho) \\
-\sin(\rho)
\end{bmatrix}, \frac{1}{\sigma} \begin{bmatrix}
\cos(\rho) \\
-\sin(\rho)
\end{bmatrix}, \begin{bmatrix}
\frac{1}{\sigma \cos(\rho)}
\end{bmatrix}\right\}
\]

Thus our actual example is a polytopic LPV system in the form

\[
\dot{x} = Bu, \quad y = Cx
\]

where \(x, u, z \in \mathbb{R}^2\), \(B\) is an in (10) where \(b \in P\) and where \(C\) has the same form

\[
C = \begin{bmatrix}
c_1 \\
c_2 \\
c_1
\end{bmatrix}
\]

with \(c = [c_1 c_2]^T\) in a similar polygon.

To show that the system is GS stabilizable it is sufficient to take the following GS control

\[
\begin{bmatrix}
u_1 \\
u_2 \\
y_1 \\
y_2
\end{bmatrix} = -\kappa \begin{bmatrix}
c_1 \\
c_2 \\
-c_2 \\
c_1
\end{bmatrix} \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
\]

so that the closed–loop system becomes

\[
\frac{d}{dt} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = -\kappa [c_1^2 + c_2^2] \begin{bmatrix}
1 & -b_1 \\
b_1 & b_2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

The derivative of the Lyapunov function \(\frac{1}{2}\|x\|^2\) along the closed–loop system trajectories is

\[
\dot{\Psi}(x_1, x_2) = -\kappa [c_1^2 + c_2^2] [b_1 (x_1^2 + x_2^2)]
\]

which is clearly negative definite for \(\kappa > 0\), since the terms \([c_1^2 + c_2^2]\) and \(b_1\) are positive and lower bounded.

It takes a bit more to show the following

Proposition 4.1: There is no robust stabilizer for the considered polytopic plant.

To prove the proposition we need to show that if there is no information for the compensator about the parameters...
b_1 \ b_2 and c_1 \ c_2$, there is no way to drive the state to 0. To simplify our task we show that no stabilization is possible even under additional information for the compensator and additional restrictions for the parameters.

**Assumption 1:** Parameters $b$ and $c$ belong to the arcs

$$B = \{ b = [b_1 \ b_2]^T : \ ||b|| = 1 \ \ |\arg(b)| \leq \rho \}$$

(note that this is a single dimensional arc) and

$$C = \{ c = [c_1 \ c_2]^T : \ 1 \leq ||c|| \leq 1/\nu, \ |\arg(c)| \leq \theta \}.$$  

The idea is the following: given the current state $x(t)$ the corresponding output $y = C(t)x$ is obtained by a rotation and amplification of it. The maximal rotation is $\theta$ and the maximal amplification is $1/\nu$.

Assume now that $x(0)$ is given. It can be immediately seen that the condition $y(t) = C(t)x(t) = x(0)$ can be satisfied with a suitable $C$ as long as $x(t)$ forms an angle with $x(0)$ smaller that $\theta$ 

$$|\arg(x(t), x(0))| \leq \theta$$

and as long as the condition

$$\nu ||x(0)|| \leq ||x(t)|| \leq ||x(0)||,$$

If $x(0) = [1 \ 0]^T$, the output can be kept fixed to $y(t) = [1 \ 0]^T$ as long as $x(t)$ remains in the set depicted in Fig. 6. If we adopt a complex plane representation, $z = b_1 + j b_2$, this set is just the inverse set of $C$. So let us call it $C^{-1}$. As long as the state is in this sector $C(t)$ must be such that the true state is “hidden” since $y(t) = x(0)$ (Fig. 6). We provide

- once $x(t_1)$ is reached, $x(t_1)$ is revealed to the compensator and the parameters in $C(t)$ provide the constant output $y(t) = x(t_1)$ as long as this is possible and so on ...

The value $t_1$ is the first value for which $x(t)$ leaves the bow $C^{-1}$. Since in this interval the control can work only open–loop, $u(t)$ must be chosen as a function of $x(0)$. By integration we get

$$x(t_1) = x(0) + \int_{0}^{t_1} B(t) u(t) dt$$

If the system is stabilizable, $x(t)$ must converge to 0 and hence it must leave $C^{-1}$. Assume that $B(t) = I$ on the interval $[0, t_1]$.

$$x(t_1) = x(0) + \int_{0}^{t_1} u(t) dt = x(0) + U_{t_1}$$

(11) where $U_{t_1} = \int_{0}^{t_1} u(t) dt$ is taken open–loop. Since $x(t_1)$ must be on the boundary of $C^{-1}$, then for sufficiently small $\nu$ and for $\rho \simeq \pi/2$ one of the following conditions hold true

- **C1** $||U_{t_1}|| \geq 1 - \nu$: this is necessary if the state leaves $C^{-1}$ crossing the boundary either in the segments $M-Q$ or $P-N$ or through the arc $Q-P$ (see Fig. 6);
- **C2** $||x(t_1)|| = 1$: this is true if the state leaves $C^{-1}$ through the arc $M-N$ (see Fig. 6);
- **C3** $||x(t)||$ is increasing and state leaves immediately $C^{-1}$, namely $||x(t_1)|| > 1$ for positive $t_1$ (see Fig. 6);

Assume that the first condition holds. Then, let $\tau$ be the first time in the interval $[0, t_1]$ for which

$$||U_{\tau}|| = ||\int_{0}^{\tau} u(t) dt|| = 1 - \nu$$

(which correspond to the fact that $x(\tau)$ is on the dashed arc in Fig. 7). Now, assume that, instead of $B(t) = I$ we take $B(t) = B(0)$. This can rotate the vector $U_{\tau}$ at will of an angle up to $\theta$. Figure 7 represents the situation. In particular

![Fig. 6. The hidden state](image)

![Fig. 7. The hidden state](image)
of the rotated trajectory can intersect the boundary of $C^{-1}$ unless on the external arc $M – N$. Then we are able to update $t_1$ by setting it to the first time instant for which the rotated trajectory intersects the boundary on the external arc, namely the first value in the interval $[0, \tau]$ for which $\|x(t)\| = 1$. Then we claim that, no matter how the control is taken, for suitable $B(0)$ there exists $t_1 > 0$ such that

$$\|x(t_1)\| = \|x(0)\| = 1, \quad \tau > 0. \quad (12)$$

Therefore, the case $C1$ can be reduced to case $C2$. Case $C3$, which means that the control is self–killing and keeps $\|x(t)\|$ increasing or non–decreasing (so that $y(t) = x(t)$) there is nothing to prove since condition (12) is obviously true for some $t_1 > 0$.

To proceed we have to note that the system is rotation–invariant for its equations are invariant with respect to rotations (applied to all vectors $x, y, u$) $\dot{x} = R(\varphi)x \quad \dot{u} = R(\varphi)u \quad \dot{y} = R(\varphi)y$ where

$$R(\varphi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

Once the new state $\|x(t_1)\| = 1$ is reached, we can apply a suitable rotation to take the new state on

$$\dot{x}(t_1) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$

Proceeding in the same way, we are able to recursively show that for proper choices of the matrix $B(t)$ we find an increasing sequence $0 < t_1 < t_1 < \ldots t_k < \ldots$ for which

$$\|x(t_{k+1})\| \geq \|x(t_k)\| = 1. \quad (13)$$

**Remark 4.2:** In the discrete–time case finding counterexamples is even easier. Gain scheduling and robust control are not equivalent even for state feedback [6]. As a consequence, it is possible to find out examples of systems which are robustly state–feedback stabilizable whose dual is not detectable and vice versa. The reader is referred to [3] for details.

V. OPEN QUESTIONS

There are still open questions which we have been unable to answer at the moment of writing this paper. For instance it is not clear if, in the gain–scheduling output–feedback case, nonlinear controllers can outperform linear controllers. This problem is related to the gain–scheduling state estimation. It is still unclear if nonlinear gain–scheduling state observer can do better than linear gain–scheduling observers.

We have been not able, so far, to construct a SISO counterexample in which we show that gain scheduling are better that robust controllers. We conjecture that

$$\dot{x}_1 = \omega x_2 + u(t)$$
$$\dot{x}_2 = -\omega x_1 + \beta u(t)$$
$$y = x_1 + \gamma x_2(t)$$

with large variations on $\beta, \gamma$ and $\omega > 0$ could serve as counterexample.

A further question is that the definition of detectability, namely the possibility of assuring $\dot{x}(t) = x(t) \to 0$ for all $u(\cdot)$ is quite strong. Can it be relaxed to find some duality?

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